Emergent spin

Michael Creutz

BNL

Introduction

- quantum mechanics + relativity $\Longleftrightarrow$ spin statistics connection
  - fermions have half integer spin

On a lattice

- no relativity
- can consider spinless fermions hopping around

If low energy excitations have a relativistic spectrum

- spin must “emerge” from the dynamics

Lattice examples

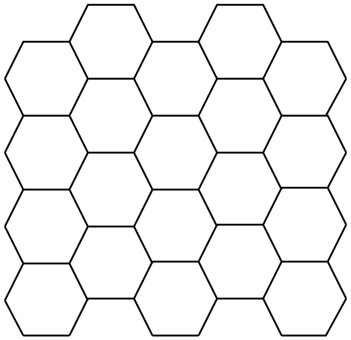
- graphene $D = 2$ (Mecklenburg and Regen, PRL)
- staggered fermions $D = 2, 3, 4$

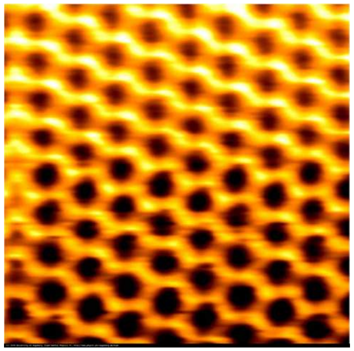
In the continuum

- fermions can appear as topological objects
  - bosonization
  - Skyrmions

Graphene

A two dimensional hexagonal planar structure of carbon atoms





- A. H. Castro Neto et al., RMP 81,109 [arXiv:0709.1163]
- http://online.kitp.ucsb.edu/online/bblunch/castroneto/

Held together by strong "sigma" bonds, sp^2

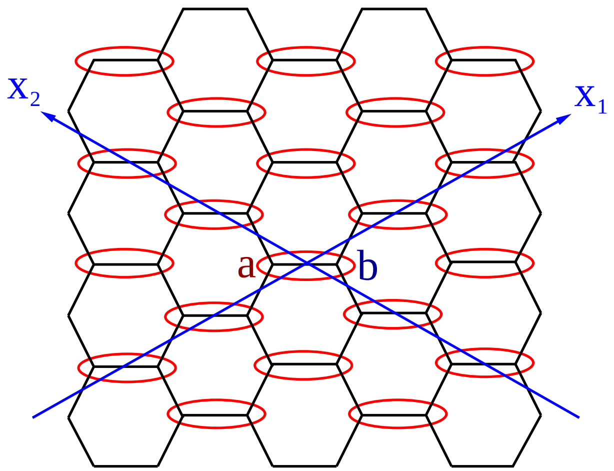
One "pi" electron per site can hop around

Consider only nearest neighbor hopping in the pi system

- tight binding approximation

Fortuitous choice of coordinates helps solve

- Hopping on a hexagonal lattice



Form horizontal bonds into ‘‘sites’’ involving two types of atom

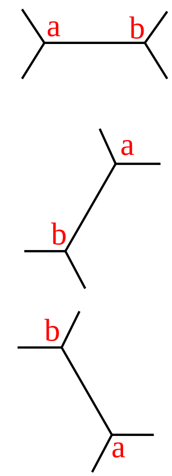
- ‘‘a’’ on the left end of a horizontal bond
- ‘‘b’’ on the right end

Label ‘‘sites’’ with non-orthogonal coordinates x_1 and x_2

- axes at 30 degrees from horizontal

Hamiltonian

$$H = K \sum_{x_1,x_2} a^\dagger_{x_1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1,x_2}$$
$$+ a^\dagger_{x_1+1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1+1,x_2}$$
$$+ a^\dagger_{x_1,x_2} b_{x_1,x_2+1} + b^\dagger_{x_1,x_2+1} a_{x_1,x_2}$$



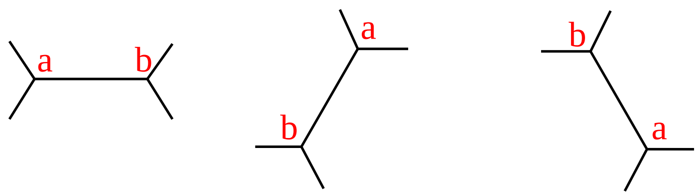
Go to momentum (reciprocal) space

- $a_{x_1,x_2} = \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \; e^{ip_1 x_1} \; e^{ip_2 x_2} \; \tilde{a}_{p_1,p_2}.$
- $-\pi < p_\mu \leq \pi$

Hamiltonian breaks into two by two blocks

$$H = K \int_{-\pi}^{\pi} \frac{dp_1}{2\pi}\frac{dp_2}{2\pi} \left(\tilde a^\dagger_{p_1,p_2} \quad \tilde b^\dagger_{p_1,p_2} \right) \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \begin{pmatrix} \tilde a_{p_1,p_2} \\ \tilde b_{p_1,p_2} \end{pmatrix}$$

- where $z = 1 + e^{-ip_1} + e^{+ip_2}$



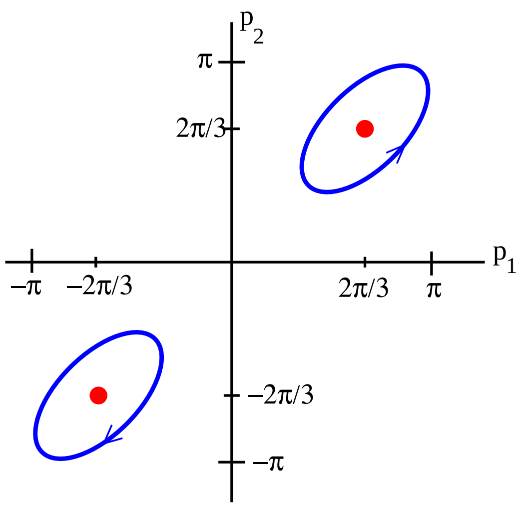
Spinor $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ “emerges”

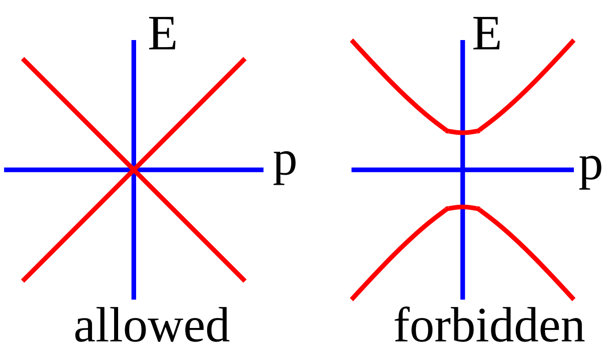
- $\tilde{H}(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$
- $z = \ 1 \ + e^{-ip_1} \ + e^{+ip_2}$

- eigenstates are two component spinors
  - $\psi = \frac{1}{\sqrt{2|z|}} \begin{pmatrix} \sqrt{z} \\ \pm\sqrt{z^*} \end{pmatrix}$
  - $E(p_1, p_2) = \pm K|z|$
- energy vanishes when $|z|$ does
- exactly two points $p_1 = p_2 = \pm 2\pi/3$

Topology and spin

- contour of constant energy near a zero point
- phase of z wraps around unit circle
- cannot collapse contour without going to $|z| = 0$



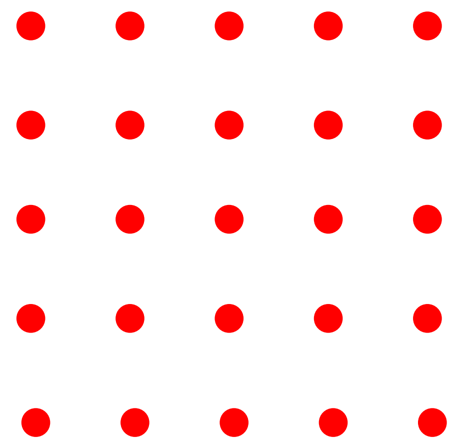


$\psi \to -\psi$ on 2π rotation of momentum around a zero

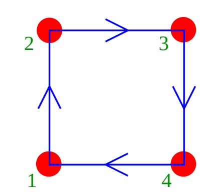
- half integer spin emerges!

Hopping in a magnetic field

- two dimensional lattice

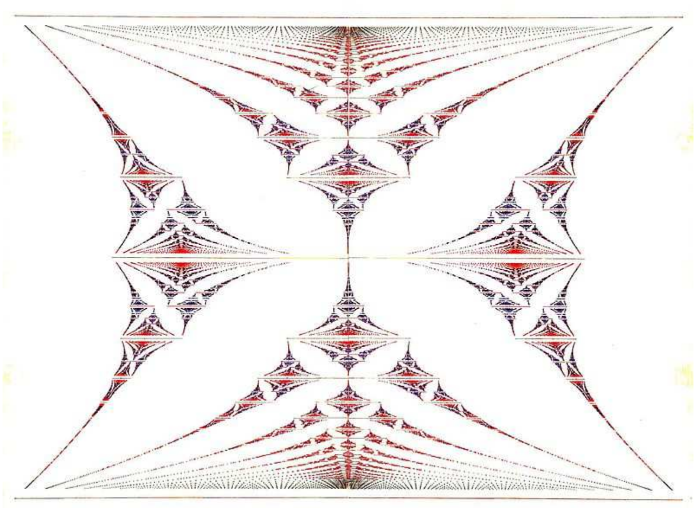


- $H = K \sum_{\{i,j\}} a_i^\dagger Z_{ij} a_j + a_j^\dagger Z_{ij}^\dagger a_i$
- $Z = e^{i\phi} \in U(1)$
- constant $B = \mathrm{Re} \prod_P Z$



General B gives a fractal structure

- Hofstadter's "butterfly"



Douglas Hofstadter

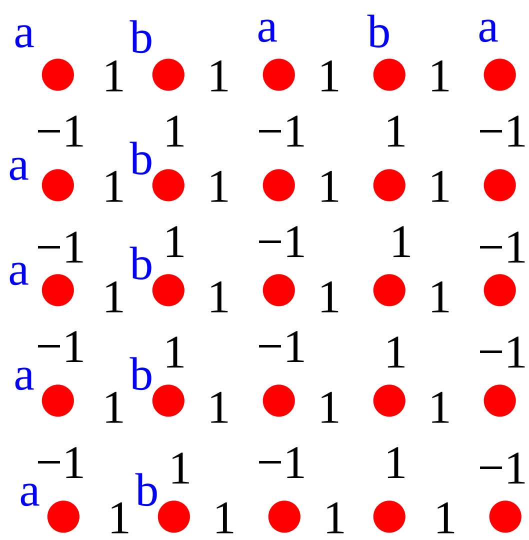
- $B = p/q$ gives q bands

Concentrate on $B = 1/2$

- every plaquette gives $Z_P = -1$
- two bands, touching at two "Dirac cones"

Choose a convenient gauge

- $Z_x = 1, Z_y = (-1)^x$



- two types of site
  - a on top of negative y bonds
  - b on top of positive y bonds

Periodicity

- by 2 in x direction
- by 1 in y direction

Momentum space

- $\psi(2n, m) = e^{2ipn+iqm}\ \psi(0,0)$
- $\psi(2n+1, m) = e^{2ipn+iqm}\ \psi(1,0)$
- two component "base spinor" $\Psi = \begin{pmatrix} \psi(0,0) \\ \psi(1,0) \end{pmatrix}$

- $0 \le q < 2\pi$
- $0 \le p < \pi$

"Half" sized Brilloin zone

A two by two Hamiltonian matrix

- $H = K \begin{pmatrix} 2\cos(q) & 1+e^{2ip} \\ 1+e^{-2ip} & 2\cos(q) \end{pmatrix}$
- $E = \pm 2K\sqrt{\cos^2(p) + \cos^2(q)}$

- zeros at $(p,q) = (\pi/2, \pi/2)$ and $(p,q) = (\pi/2, 3\pi/2)$
- two Dirac cones as with graphene

This is a rewriting of staggered fermions

Three dimensions

Thread half integer magnetic flux through every plaquette

- convenient gauge gives 4 types of site
  - $Z_x = 1,\ Z_y = (-1)^x,\ Z_z = (-1)^{x+y}$
- translate to a four component base spinor
  - $\begin{pmatrix} \psi(0,0,0) \\ \psi(1,0,0) \\ \psi(0,1,0) \\ \psi(1,1,0) \end{pmatrix}$
  - $E = 2K\sqrt{\cos^2(p_x) + \cos^2(p_y) + \cos^2(p_z)}$
  - p_x and p_y (not p_z) restricted to "half" zones $0 \le p < \pi$

Equivalent to 3D staggered Hamiltonian

Chiral symmetry

Hamiltonian anticommutes with $(-1)^{x+y+z} \sim$ "γ_5"

- two dirac cones have opposite chirality
- a two "flavor" theory with one exact chiral symmetry
  - actually a "flavored" chiral symmetry
  - consistent with anomaly

Adding non-Abelian gauge fields

Spinless fermion coupled to two gauge fields

- $SU(N)$ of color; U_{ij} on links
- auxiliary Z_2; Z_{ij} on links
- fermion hop picks up product $U_{ij}Z_{ij}$

Two gauge couplings

- $\beta \sim \frac{1}{g^2}$ for the color group
- β_z for the Z_2

Take the limit $\beta_z \to -\infty$

- forces each Z_2 plaquette to -1

This is staggered Hamiltonian lattice gauge theory

Periodicity of Brilloin zone implies two cones

- “fermion doubling” -- chiral symmetry is flavored

$SU(2)$ color

- $-1 \in SU(2)$
- can absorb phases in the group links
- flipping sign of the gauge coupling $\beta \to -\beta$

$SU(2)$ Hamiltonian lattice gauge theory of spinless fermions

- at negative β is equivalent to staggered fermions!

Not true for $SU(3)$ since -1 not in the group

Four dimensions

Staggered formulated as above brings in one extra doubling

- effective 8 component spinor
- four “tastes”
- 3d Hamiltonian has only two tastes

Can we insert the 3D Hamiltonian in a 4D path integral

- to get a two taste formulation of staggered?

Fermionic path integrals

- connect Hamiltonian and Euclidian path integral formulations
- consider a product of normal ordered operators

$$\mathrm{Tr}\left(: f_1(a^\dagger, a) :: f_2(a^\dagger, a) : \ldots : f_n(a^\dagger, a) :\right)$$
$$= \int (d\psi d\psi^*) f_1(\psi_1^*, \psi_1) \ldots f_n(\psi_n^*, \psi_n)\ e^{\sum_j \psi_j^*(\psi_j - \psi_{j-1})}$$

- where
  - ψ_n and ψ_n^* are independent Grassmann variables
  - $\psi_0 = -\psi_n$ antiperiodic boundaries
- note the asymmetric discrete derivative

Applied to $\mathrm{Tr}\left(: e^{-\beta H/N_t} :\right)^{N_t}$

- using above 3D Hamiltonian gives a Fermion action
- minimally doubled with two “tastes”
  - minimal number required for chiral symmetry

But:

- different time treatment breaks $4d$ hypercubic symmetry
- contains both Hermitean and anti-Hermitean parts

All known 4d minimally doubled chiral formulations

- break hypercubic symmetry
- is there a theorem?

Gauge fields and topology

Index theorem: $\nu = n_+ - n_-$

- $n_\pm$ zero modes of Dirac operator of $\pm$ chirality
- ν topological index of the gauge field
- formally $\mathrm{Tr}\,\gamma_5 = \nu$

Singlet chiral symmetry is anomalous

- $\psi \to e^{i\gamma_5\theta}\psi$
- changes integration measure $d\psi \to e^{i\nu\theta}d\psi$

$m\,\overline{\psi}\psi \to m\,\overline{\psi}e^{i\gamma_5\theta}\psi$

- an inequivalent theory; "theta vacuum"
- violates CP symmetry

Consider any lattice Dirac operator D

- assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$
  - all operators in practice satisfy this (except twisted mass)

Divide D into hermitean and antihermitean parts

- $D = K + M$
- $$K = (D - D^\dagger)/2$$
  $$M = (D + D^\dagger)/2$$

Then

- $[K, \gamma_5]_+ = 0$
- $[M, \gamma_5]_- = 0$

$M \to e^{i\theta\gamma_5} M$ an exact symmetry of the determinant

- Where is the anomaly?

Earlier constructions solve this with doublers

- half use γ_5 and half $-\gamma_5$
- the naive chiral symmetry is actually flavored

How about Wilson fermions?

- doublers given masses of order the cutoff
- the rotation $M \to e^{i\theta\gamma_5} M$ also rotates their phases

Physical Θ is a relative angle

- independently rotate the fermion mass and the Wilson term
  - Seiler and Stamatescu

The overlap operator

- eigenvalues on a circle
- zero eigenmodes have heavy counterpart
- rotation of Hermitean part rotates heavy mode as well
- anomaly brings in $\hat\gamma_5$: $D\gamma_5 = -\hat\gamma_5 D$
  - $\nu = \mathrm{Tr}(\gamma_5 + \hat\gamma_5)/2$

Message for continuum QCD:

- physical Θ can be moved around
- placed on any one flavor at will

Θ can be entirely moved into the top quark phase

- top quark properties relevant to low energy physics!
- decoupling theorems don't apply non-perturbatively

Summary

Excitations on a spinless Dirac sea

- can carry spin
- required by relativistic form of spectrum

Close connections with chiral symmetry

- mass topologically protected
- doublers required
- entwined with the CP violating parameter Θ