Entanglement of indistinguishable particles

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Outline

F.B., R. Floreanini, U. Marzolino: Ann. Phys. **325** (2010)
F.B., R. Floreanini, U. Marzolino: JPB **44** (2011)
G. Argentieri, F.B., R. Floreanini, U. Marzolino: IJQI **9** (2011)
F.B., R. Floreanini, U. Marzolino: Ann. Phys. **327** (2012)
F.B., R. Floreanini, U. Marzolino: PRA **85** (2012)

Entanglement of Identical Particles

From particle entanglement to mode entanglement

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Identical versus Indistinguishable qubits

- Single qubit states out of the vacuum $|0\rangle$:

$$a^\dagger|0\rangle = |1\rangle \ , \quad b^\dagger|0\rangle = |2\rangle$$

- Two qubits: $\mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \ni |i,j\rangle$, $i,j = 1,2$
- Two 2-mode Bosons: Hilbert space $\mathbb{H}^{(2)}_{symm} = \mathbb{C}^3$

$$|1,1\rangle = \frac{(a^\dagger)^2}{\sqrt{2}}|0\rangle \ , \ |2,2\rangle = \frac{(b^\dagger)^2}{\sqrt{2}}|0\rangle$$

$$\frac{|1,2\rangle + |2,1\rangle}{\sqrt{2}} = a^\dagger b^\dagger|0\rangle$$

Spatial modes

- $a^\dagger |0\rangle = |1\rangle$: one Boson in the left well
- $b^\dagger |0\rangle = |2\rangle$: one Boson in the right well

Figure: Double-Well Potential

Figure: Left and Right localized states

N 2-mode Bosons: pseudo-spins

- Total Boson number: $a^\dagger a + b^\dagger b = N$

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$$J_x = \frac{a^\dagger\, b + a\, b^\dagger}{2}\ , \quad J_y = \frac{a\, b^\dagger - a^\dagger\, b}{2i}\ , \quad J_z = \frac{a^\dagger\, a - b^\dagger\, b}{2}$$

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- N **standard qubits**:

$$\vec{J} = \sum_{k=1}^{N} \vec{j}_k \quad \Longrightarrow \quad \text{Spin Squeezing Inequalities (SSI)}$$

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- N 2-**mode Bosons**: single particle angular moment not accessible
- Mode-separable states may violate **some SSI**

Mode richer structure

From Spatial modes to Energy modes

- Bogolubov transformation:

$$c^\dagger = \frac{a^\dagger + b^\dagger}{\sqrt{2}}\ , \quad d^\dagger = \frac{a^\dagger - b^\dagger}{\sqrt{2}}$$

- Single particle energy eigenstates

$$|g\rangle = d^\dagger|0\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}} \qquad |e\rangle = c^\dagger|0\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$$

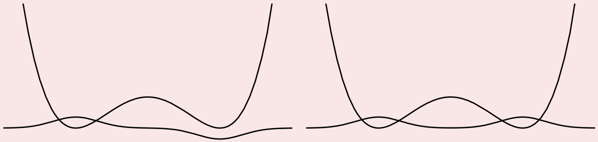

Figure: Ground and first excited states

N distinguishable qubits: separability

N distinguishable qubits: natural tensor product structure

- of single particle Hilbert spaces: $\bigotimes_{j=1}^{N} \mathbb{C}^2$
- of single particle algebras: $\bigotimes_{j=1}^{N} M_2(\mathbb{C})$
- **fully separable states**: $\rho = \sum_k \lambda_k \bigotimes_{j=1}^{N} |\Psi_j^k\rangle\langle\Psi_j^k|$

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Local operators: tensor products of single qubit operators

- Total spin operator

$$\vec{J} = \sum_{j=1}^{N} \frac{\vec{\sigma}_j}{2}\ ,\ \ \vec{\sigma}_j = (\sigma_{jx}, \sigma_{jy}, \sigma_{jz})$$

- All rotations are local:

$$\mathrm{e}^{i\theta\vec{J}\cdot\vec{n}} = \bigotimes_{j=1}^{N} \mathrm{e}^{\frac{i}{2}\,\theta\,\vec{\sigma}_j\cdot\vec{n}} = \mathrm{e}^{\frac{i}{2}\,\theta\,\vec{\sigma}_1\cdot\vec{n}} \otimes \mathrm{e}^{\frac{i}{2}\,\theta\,\vec{\sigma}_2\cdot\vec{n}} \otimes \cdots \otimes \mathrm{e}^{\frac{i}{2}\,\theta\,\vec{\sigma}_N\cdot\vec{n}}$$

Separability: identical Bosons

No natural tensor product structure

- Only projections onto symmetrized vector states are allowed
- 2 two-mode Bosonic vector states: $|1,1\rangle \;, \quad |2,2\rangle \;, \; \dfrac{|1,2\rangle + |2,1\rangle}{\sqrt{2}}$

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Bosonic states are not obtainable by symmetrizing density matrices of distinguishable qubits

symmetric states $\rho \otimes \rho$ not allowed in general:

$$\langle asym|\rho \otimes \rho|asym\rangle = \mathrm{Det}(\rho) \neq 0 \ , \qquad |asym\rangle = \frac{|1,2\rangle - |2,1\rangle}{\sqrt{2}}$$

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How to proceed?

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How to proceed? Associate locality with commutativity in a second quantized context

Zanardi: PRA **65** (2002), Narnhofer: PLA **310** (2004)
Barnum et al., PRL **92** (2004)

Locality: commuting sub-algebras

Commuting sub-algebras

- qubits: local (single-particle) algebras commute

$$\Big[A \otimes 1\,,\, 1 \otimes B\Big] = 0\ , \quad A \otimes B \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$$

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- 2-mode Bosons: single particle Hilbert space $\mathbb{C}^2 \ni \{|1\rangle, |2\rangle\}$
- creation and annihilation operators: $a, a^\dagger; b, b^\dagger$

$$[a\,,\, a^\dagger] = [b\,,\, b^\dagger] = 1\ , \quad [a\,,\, b] = 0$$

- **Commuting sub-algebras**:

$$\mathcal{A} = \{a, a^\dagger\}\ ,\ \mathcal{B} = \{b, b^\dagger\}\ , \quad [\mathcal{A}\ ,\ \mathcal{B}] = 0$$

Indistinguishable particles: local operators

Definition

An observable X is $(\mathcal{A},\mathcal{B})$ local iff $X = A\,B \qquad A \in \mathcal{A}\ , \quad B \in \mathcal{B}$

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Example

- **pseudo angular momentum operators**:

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- $(\mathcal{A},\mathcal{B})$-non-local rotations:

$$\mathrm{e}^{i\,\theta\,J_x} = \mathrm{e}^{i\,\frac{\theta}{2}\,(a^\dagger\, b + a\, b^\dagger)}\,, \quad \mathrm{e}^{i\,\theta\,J_y} = \mathrm{e}^{\frac{\theta}{2}\,(a\, b^\dagger - a^\dagger\, b)}$$

- $(\mathcal{A},\mathcal{B})$-local rotation:

$$\mathrm{e}^{i\,\theta\,J_z} = \mathrm{e}^{i\,\frac{\theta}{2}\,a^\dagger\, a}\,\mathrm{e}^{-i\,\frac{\theta}{2}\,b^\dagger\, b}$$

Bogolubov transformations

$$c^\dagger = \frac{a^\dagger + b^\dagger}{\sqrt{2}}\ , \quad d^\dagger = \frac{a^\dagger - b^\dagger}{\sqrt{2}}$$

turns $(\mathcal{A}, \mathcal{B})$-local rotations into $(\mathcal{C}, \mathcal{D})$-**non-local** rotations

$$\mathrm{e}^{i\,\theta\,J_z} = \mathrm{e}^{i\,\frac{\theta}{2}\,(c^\dagger\,d + d^\dagger\,c)}$$

Separability of Bosonic states

Definition: Bosonic separable states

States ρ on the 2-mode Boson algebra are $(\mathcal{A}, \mathcal{B})$-separable **iff**

$$\mathrm{Tr}(\rho\, AB) = \sum_i p_i \left(\mathrm{Tr}(\rho_i^{(a)}\, A)\right) \left(\mathrm{Tr}(\rho_i^{(b)}\, B)\right) ,$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $\rho_i^{(a,b)}$ states on the 2-mode Boson algebra

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Example

- Fock number states:

$$a^\dagger a|n_a, n_b\rangle = n_a\,|n_a, n_b\rangle \ , \ b^\dagger b|n_a, n_b\rangle = n_b\,|n_a, n_b\rangle$$

- $A \in \mathcal{A}$, $B \in \mathcal{B}$:

$$\langle n_a, n_b|\, A\, B\,|n_a, n_b\rangle = \langle n_a, n_b|\, A\,|n_a, n_b\rangle\,\langle n_a, n_b|\, B\,|n_a, n_b\rangle$$

Theorem

ρ is a $(\mathcal{A},\mathcal{B})$ separable state for N 2-mode Bosons **iff**

$$\rho^{sep}_{\mathcal{A},\mathcal{B}} = \sum_{k=0}^{N} p_k \, |k, N-k\rangle_{\mathcal{A},\mathcal{B}}\langle k, N-k|$$

(F.B., R. Floreanini, U. Marzolino: Ann. Phys. 325 (2010))

Example

- Fock number states:

$$|k, N-k\rangle_{\mathcal{A},\mathcal{B}} = \frac{(a^\dagger)^k\,(b^\dagger)^{(N-k)}}{\sqrt{k!(N-k)!}}|0\rangle$$

$(\mathcal{A}, \mathcal{B})$-local and separable

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$(\mathcal{A},\mathcal{B})$-local and separable

- Bogulobov rotate the nodes:

$$|k, N-k\rangle_{\mathcal{A},\mathcal{B}} = \left(\frac{1}{\sqrt{2}}\right)^N \frac{\left(c^\dagger - d^\dagger\right)^k \left(c^\dagger + d^\dagger\right)^{N-k}}{\sqrt{k!(N-k)!}}|0\rangle$$

$(\mathcal{C},\mathcal{D})$ non-local and entangled

Negativity and Entanglement Witnessing

- **Partial transposition** on the first mode

$$\rho = \sum_{k,\ell=0}^{N} \rho_{k\ell} |k, N-k\rangle\langle \ell, N-\ell| \mapsto \rho^{T_1} = \sum_{k,\ell=0}^{N} \rho_{k\ell} |\ell, N-k\rangle\langle k, N-\ell|$$

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- $\mathcal{N}(\rho) = \sum_{k\neq\ell=0}^{N} \left|\rho_{k\ell}\right| = 0$ **iff** $\rho_{k\ell} = 0$ **for all** $k \neq \ell$

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An exhaustive entanglement witness for two-mode Bosons

$\mathcal{N}(\rho) = \mathbf{0}$ **iff** ρ is $(\mathcal{A}, \mathcal{B})$-separable

Spin Squeezing Inequalities

SSI: N qubits vs N 2-mode Bosons

- Standard qubit entanglement condition Toth et al. PRA 79 (2009)

$$\langle J_{\vec{n}_1}^2 \rangle + \langle J_{\vec{n}_1}^2 \rangle - \frac{N}{2} - (N-1)\Delta^2 J_{\vec{n}_3} > 0$$

satisfied by $(\mathcal{A}, \mathcal{B})$-separable states

$$\rho = \sum_{k=0}^{N} p_k \, |k, N-k\rangle_{(\mathcal{A},\mathcal{B})} \langle k, N-k|$$

for suitable distributions p_k.

- Standard qubit entanglement condition Korbicz et al. PRL 95 (2005)

$$N\Delta^2 J_{\vec{n}} + \langle J_{\vec{n}} \rangle^2 < \frac{N^2}{4}$$

satisfied by $|k, N-k\rangle_{(\mathcal{A},\mathcal{B})}$, $0 < k < N$, $\vec{n} = \hat{z}$

Quantum metrology
(Giovannetti et al. Science 306 (2004), Phys. Rev. Lett. 96 (2006))

Measuring rotation angles on N qubits

- Rotate an N-qubit density matrix: $\rho \mapsto \rho_\theta$

$$\rho_\theta = \mathrm{e}^{-i\theta J_{\vec{n}_1}}\, \rho\, \mathrm{e}^{i\theta J_{\vec{n}_1}}$$

$$\vec{J} = \sum_{j=1}^{N} \frac{\vec{\sigma}_j}{2}\ , \quad \vec{\sigma}_j = (\sigma_{jx}, \sigma_{jy}, \sigma_{jz})\ ,\ J_{\vec{n}_1} = \vec{n}_1 \cdot \vec{J}$$

- Measure $J_{\vec{n}_2}$, $\vec{n}_2 \perp \vec{n}_1$ and estimate θ with sensitivity $\delta\theta$

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- Measure $J_{\vec{n}_2}$, $\vec{n}_2 \perp \vec{n}_1$ and estimate θ with sensitivity $\delta\theta$
- **Shot Noise**: $\delta^2\theta = \frac{1}{N}$
- **Sub-shot Noise**: $\delta^2\theta < \frac{1}{N}$
- **Heisenberg Limit**: $\delta^2\theta = \frac{1}{N^2}$

Quantum Estimation and Fisher Information

Quantum Fisher information (Paris, I.J.Q.I. 7 (2009))

- Rotation: $\rho \mapsto \rho_\theta = \mathrm{e}^{-i\theta J_{\vec{n}}}\,\rho\,\mathrm{e}^{i\theta J_{\vec{n}}}$
- **Unbiased quantum estimator** $\hat{E}$: $\delta^2_\rho\theta = \mathrm{Tr}\Big(\rho\,(\hat{E}-\theta)^2\Big)$
- **Quantum Fisher Information**:

$$F[\rho, J_{\vec{n}}] = \mathrm{Tr}\Big(\rho\,L^2\Big)\ , \quad \partial_\theta\,\rho_\theta\Big|_{\theta=0} = \frac{1}{2}\Big(\rho\,L + L\,\rho\Big) = i\Big[J_{\vec{n}}\,,\,\rho\Big]$$

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- Quantum Cramer-Rao bound: $\delta^2_\rho\theta \geq \dfrac{1}{F[\rho, J_{\vec{n}}]}$

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- pure states: $F[\Psi, J_{\vec{n}}] = 4\,\Delta^2_\Psi\, J_{\vec{n}}$

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- pure states: $F[\Psi, J_{\vec{n}}] = 4\, \Delta^2_\Psi\, J_{\vec{n}}$
- convexity: $\rho = \sum_j \lambda_j |\Psi_j\rangle\langle\Psi_j|$:

$$F[\rho, J_{\vec{n}}] \leq \sum_j \lambda_j\, F[\Psi_j, J_{\vec{n}}]$$

QFI and Metrology

QFI and separability: N standard qubits

- **Fully separable N-qubit states**:

$$\rho_{sep}^{N} = \sum_{k} \lambda_k \, |\Psi_k^{\otimes N}\rangle\langle\Psi_k^{\otimes N}| \ , \qquad |\Psi_k^{\otimes N}\rangle = \bigotimes_{j=1}^{N} |\psi_j^k\rangle$$

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- From convexity (Pezzé, Smerzi, PRL (102)):

$$F[\rho^N_{sep}, J_{\vec{n}}] \leq \sum_j \lambda_j \, F[\Psi_j^{\otimes N}, J_{\vec{n}}] = 4 \sum_j \lambda_j \, \Delta^2_{\Psi_j^{\otimes N}} J_{\vec{n}}$$

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- **Fully separable vector state** $|\Psi_{FS}\rangle$: $\Delta^2_{\Psi_{FS}} J_{\vec{n}} \leq \frac{\mathbf{N}}{\mathbf{4}}$
- **Fully separable mixed state** ρ^N_{FS}:

$$F[\rho^N_{FS}, J_{\vec{n}}] \leq \mathbf{N}$$

Entanglement necessary to beat the shot-noise limit

- Quantum Cramer-Rao bound

$$\delta^2_\rho\theta \geq \frac{1}{F[\rho, J_{\vec{n}}]}$$

- In order to have

$$\frac{1}{F[\rho, J_{\vec{n}}]} \leq \delta^2_\rho\theta < \frac{1}{N}$$

necessarily

$$F[\rho, J_{\vec{n}}] > N$$

Cold atom interferometry

Mach-Zehnder interferometry with ultracold atoms
C. Gross *et al.*, **M.F. Riedel** *et al.*: *Nature* **464 (2010)**

- N ultracold atoms trapped by a double-well potential as **pseudo qubits** via the Schwinger representation

$$J_x = \frac{a^\dagger\, b + a\, b^\dagger}{2}\ , \quad J_y = \frac{a\, b^\dagger - a^\dagger\, b}{2i}\ , \quad J_z = \frac{a^\dagger\, a - b^\dagger\, b}{2}$$

- **If** they were **standard quits** the locality of rotations **would require entangled input states**
- However, trapped cold atoms are Bosons:
  - single angular momenta not accessible
  - rotations not necessarily local

Beating the shot-noise limit with Bosons

Quantum Fisher Information

- $(\mathcal{A},\mathcal{B})$-separable Fock states: $|k, N-k\rangle_{\mathcal{A},\mathcal{B}}$: $\vec{n} = (n_x, n_y, 0)$,

$$\begin{aligned} F\Big[|k, N-k\rangle_{\mathcal{A},\mathcal{B}}, J_{\vec{n}}\Big] &= 4\,\Delta^2_{|k,N-k\rangle_{\mathcal{A},\mathcal{B}}} J_{\vec{n}} \\ &= N\,(2\,k+1) - 2\,k^2 > N \qquad \forall\, k \neq 0, N \end{aligned}$$

Beating the shot-noise limit with Bosons

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- Getting close to the Heisenberg limit $F[\rho, J_{\vec{n}}] = N^2$:

$$|N/2, N/2\rangle_{\mathcal{A},\mathcal{B}} \Longrightarrow F\Big[|N/2, N/2\rangle_{\mathcal{A},\mathcal{B}}, J_{\vec{n}}\Big] = \frac{N^2}{2} + N$$

(F.B., R. Floreanini, U. Marzolino: J. Phys. B **44** (2011))

Non-locality from the apparatus

Sub-shot noise with $(\mathcal{A}, \mathcal{B})$-separable states: How?

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The interferometer is $(\mathcal{A},\mathcal{B})$-non-local

- Take $\vec{n}=(0,1,0)$: $\rho \mapsto \rho_\theta = \exp(i\,\theta\,J_y)\,\rho\,\exp(-i\,\theta\,J_y)$

$$J_y = \frac{a\,b^\dagger - a^\dagger\,b}{2i} \Longrightarrow \exp(i\,\theta\,J_y) \qquad (\mathcal{A},\mathcal{B}) - \text{non-local}$$

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- Theorem
  If the state $(\mathcal{A},\mathcal{B})$-separable and the apparatus $(\mathcal{A},\mathcal{B})$-local then

$$F\left[\rho^{sep}_{(\mathcal{A},\mathcal{B})}\,,\,J_\mathcal{A} + J_\mathcal{B}\right] = 0$$

FB, D. Braun: submitted to PRA

Entangling noise

$(\mathcal{A},\mathcal{B})$-dephasing noise

- Lindblad master equation:

$$\partial_t \rho(t) = \gamma\Big(J_z\, \rho(t)\, J_z \;-\; \frac{1}{2}\Big\{ J_z^2\,,\, \rho(t)\Big\}\Big)$$

- Solution: **mixture** of $(\mathcal{A},\mathcal{B})$-local operations

$$\rho(t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}u\, \mathrm{e}^{-u^2/4}\, \mathrm{e}^{-i\sqrt{t\gamma/2}\,u\, J_z}\, \rho\, \mathrm{e}^{+i\sqrt{t\gamma/2}\,u\, J_z}$$

- Exponential decay of $(\mathcal{A},\mathcal{B})$-entanglement:

$$\mathcal{N}_{\mathcal{A},\mathcal{B}}(\rho(t)) = \sum_{k\neq\ell} \mathrm{e}^{-t\gamma(k-\ell)^2/2}\, |\rho_{k\ell}| \leq \mathrm{e}^{-t\gamma/2}\, \mathcal{N}_{\mathcal{A},\mathcal{B}}(\rho)$$

$(\mathcal{C},\mathcal{D})$-entangling noise

- Initial $(\mathcal{C},\mathcal{D})$-separable state: $\frac{(c^\dagger)^N}{\sqrt{N!}}|0\rangle = |N,0\rangle_{\mathcal{C},\mathcal{D}}$
- State at time $t > 0$:

$$\rho(t) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{d}u\, \mathrm{e}^{-u^2/4}\, \mathrm{e}^{-i\sqrt{t\gamma/2}\,u\,J_z}|N,0\rangle_{\mathcal{C},\mathcal{D}}\langle N,0|\mathrm{e}^{i\sqrt{t\gamma/2}\,u\,J_z}$$

$$\mathrm{e}^{-i\sqrt{t\gamma/2}\,u\,J_z}|N,0\rangle_{\mathcal{C},\mathcal{D}} = \frac{1}{\sqrt{N!}}\left(\sqrt{\xi_t}c^\dagger + i\sqrt{(1-\xi_t)}d^\dagger\right)^N|0\rangle$$

$$\xi_t = \cos^2\left(u\sqrt{\frac{t\gamma}{2}}\right)$$

- $\rho(t)$ $(\mathcal{C},\mathcal{D})$-entangled: $\rho(t) = \sum_{k,\ell=0}^{N}\rho_{k\ell}(t)\,|k,N-k\rangle_{(\mathcal{C},\mathcal{D})}\langle k,N-k|$

  (G. Argentieri, F.B., R. Floreanini, U. Marzolino: IJQI **9** (2011))

QFI and dephasing

Can a $(\mathcal{C}, \mathcal{D})$-entangling environment increase QFI? NO

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$$F[\mathbb{G}[\rho], J_{\vec{n}}] \leq F[\rho, J_{\vec{n}}]$$

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- From monotonicity and $\mathbb{G}_t = \mathbb{G}_{t-s} \circ \mathbb{G}_s$: if $0 \leq s \leq t$

$$F[\rho(t), J_{\vec{n}}] = F[\mathbb{G}_{t-s+s}[\rho], J_{\vec{n}}] \leq F[\mathbb{G}_s[\rho], J_{\vec{n}}] = F[\rho(s), J_{\vec{n}}]$$

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- The noise-generated entanglement is not metrologically useful