



# Sub- and superradiance in different geometries of interacting two level systems

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## Abstract

In this thesis I study the influence of the interaction of multiple two level systems on the properties of their joint system. I explain the two level system and study their interaction. I motivate the master equation and the Hamiltonian of the system itself. I study the pair of atoms, derive its energy states and its decay rate and show what sub and superadiant states are. I simulate numerically the energy and decay rate of the states with the quantum optics toolbox for the Julia programming language. Introduce the single excitation manifold and derive a effective Hamiltonian for this model. I use the model to describe a ring of atoms and analytically derive the eigenstates and eigenvalues of its effective Hamiltonian. I simulate the dependency of the energy and decay rate of the states on the radius of the ring again using the quantum optics toolbox. I also examine the dependency of the decay rate on the distance of the atoms in the ring. Finally I show the intensity of the electric field of a sub- and a superradiant state of such a ring.

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## 1. Introduction

A big part of the studies in quantum optics is the understanding of the interaction of photons with atoms. Also the interaction between quantum emitters is caused by their dipole moments interacting with the external field and vice versa. This interaction leads to a change in the properties like energy and decay rate of the entire system. The magnitude and effect of the interaction depends on the geometry of the system itself. Especially the effects called sub- and superradiance, states with a decreased and increased decay rate and therefore photon emission are of interest. This may seem like a very theoretical principle and mostly it is but there are also some links to reality. In quantum computing the principle can be used to achieve a suppression of the infidelity of quantum memory [1]. In the field of quantum information it is also important being able to send information over long distances with a preferably small loss of information or energy. It can be seen that interacting rings of dipole atoms show a transfer of subradiant states between the rings with a minimal loss of information [2]. Also in the field of biology does this theory appear. In photosynthesis the light harvesting complexes like LHC-I and LHC-2 are complex ring structures [3]. They use the coherent transfer of states between such rings to transport the energy to the reaction center.

The theory of the dipole interaction between different atoms in different geometries is therefore a very fundamental and present topic in the field of quantum optics. In this Bachelor-Thesis I study the interaction between two level systems and examine the effect on a pair and a ring of atoms to clearly show the appearing subradiant and superradiant states.

## 2. The two level system and spontaneous emission

A quantum-mechanical system can be described by its Hamiltonian  $H$ . The eigenvectors of this hamiltonien give the so called eigenstates of the system, all eigenstates build a basis of the Hilbert space in which the system exists, that means all possible states of the system can be described as a linear combination of these eigenstates. The eigenvalue to a eigenstate gives the energy of the state, a measurement of the energy of the state results always in one of these eigenvalues.

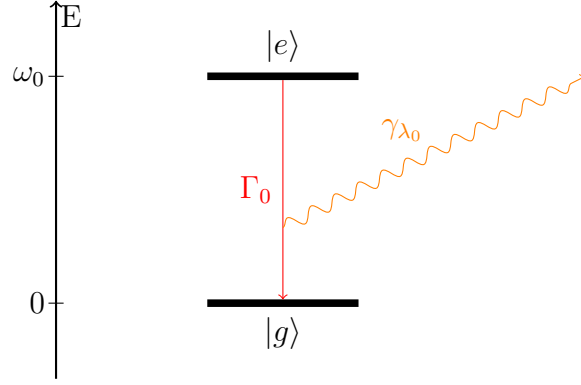
A two level system is a system whose hamiltonian has exactly two eigenstates  $|g\rangle$  an  $|e\rangle$  wit eigenvalues  $E_g$  and  $E_e$  where  $E_g < E_e$ . In the following the zero-point of the energy space is set to  $E_g$  therefore the energy difference between the two states results to  $E_e - E_g = E_e = \omega_0$ . One can also define so called ladder operator. The rising operator  $\sigma^+$  shifts the ground state to the exited state whereas the lowering operator  $\sigma^-$  shifts the exited state to the ground state.

If we describe the time dependency of the system with the time evolution operator with is calculated from the Hamiltonian of solely the two level system it would seem that the ground state as well as the exited are stable. This is certainly not true. The mistake yields from the choice of the system. Because it turns out that the chosen system does not obey energy conservation, to get the correct result for the time evolution one has to include a external electric field with whom the atoms interact. This leads to three phenomena:

- spontaneous emission
- induced emission
- absorption

Spontaneous emission describes the phenomena where the exited state decays into the ground state without any external influences. The possibility of the state decaying in a certain time interval is given by the decay rate  $\Gamma_0$  of the exited state. In case of such a decay the system loses the energy trough emitting a photon into the external field. The photon has to carry away the energy difference of the two states and therefore has a wavelength of  $\lambda_0 = h\omega_0/2\pi$ .

It is also possible that the system makes a transition which is driven by the external field. This is then called a induced emission if the atom goes from exited state to ground state and a absorption if the if it goes from the ground state to the exited state.



### 3. Master equation

In this chapter I want to briefly motivate the master equation. The full derivation can be found in [4].

Now one can look at a system of  $N$  such two level systems. To derive the time evolution of the system we start with the Hamiltonian of the atoms and their environment [4]

$$H_{tot} = \underbrace{\omega_0 \sum_{i=1}^N \sigma_i^+ \sigma_i^-}_{H_0} + \underbrace{\sum_{\vec{k}, \lambda} \omega_{\vec{k}} \hat{a}_{\vec{k}, \lambda}^\dagger \hat{a}_{\vec{k}, \lambda}}_{H_F} - \underbrace{\sum_{i=1}^N \vec{d}_i \cdot E(\vec{r}_i)}_{H_{int}}. \quad (1)$$

$H_0$  describes the sum over all the energies of the atoms themselves.  $H_F$  is the Hamiltonian of the external field calculated by a sum over all modes and the creation and annihilation operators  $\hat{a}_{\vec{k}, \lambda}^\dagger$  and  $\hat{a}_{\vec{k}, \lambda}$ . The last term  $H_{int}$  describes the interaction of the dipole moments of the atoms with the external field. From this Hamiltonian it is possible to calculate the time evolution of the system of atoms without the external field with by using the von Neumann equation

$$\partial_t \rho = i[\rho, H_{tot}] \quad (2)$$

with the density operator  $\rho$ . By performing a trace over the external field, and by using the Makov and rotating waver approximation this leads to the so called Master equation [4]

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}[\rho]. \quad (3)$$

with the new Hamiltonian of the system [4]

$$H = \omega_0 \sum_{i=0}^N \sigma_i^+ \sigma_i^- + \sum_{i \neq j} \Omega_{ij} \sigma_i^- \sigma_j^+ \quad (4)$$

where  $\Omega_{ij}$  describes the energy shift caused by only the interaction between atom  $i$  and atom  $j$ . This matrix is calculated with the help of the Greens tensor  $G_{ij}$  which describes the interaction of dipoles through [2]

$$\Omega_{ij} = -\frac{3\pi\Gamma_0}{k_0} \text{Re}(\vec{\mu}_i^* \cdot \mathbf{G}(\vec{r}_i - \vec{r}_j, \omega_0) \cdot \vec{\mu}_j). \quad (5)$$

While  $H_{tot}$  describes a system of  $N$  atoms where the atoms interact only with the external field, is  $H$  now the Hamiltonian of  $N$  atoms which interact with each other. The first sum of the Hamiltonian describes the energy of all atoms while the second part describes the Interaction. Because this new Hamiltonian does not obey energy conservation the time evolution is described by the Master equation and can not be calculated with the help of the evolution operator or the von Neumann equation. The second part of the Master equation is the so called Lindblad term [4]

$$\mathcal{L}[\rho] = \frac{1}{2} \sum_{i,j} \Gamma_{ij} \left( \underbrace{2\sigma_i^- \rho \sigma_j^+}_{\text{recycling Term}} - \sigma_i^+ \sigma_j^- \rho - \rho \sigma_i^+ \sigma_j^- \right) \quad (6)$$

where  $\Gamma_{ij}$  describes the shift of the decay rate caused by only the interaction between atom  $i$  and atom  $j$ . It is also calculated with the Greens tensor through [2]

$$\Gamma_{ij} = \frac{6\pi\Gamma_0}{k_0} \text{Im}(\vec{\mu}_i^* \cdot \mathbf{G}(\vec{r}_i - \vec{r}_j, \omega_0) \cdot \vec{\mu}_j). \quad (7)$$

The recycling term in the Lindblad equation makes sure that the total occupation number of the system is constant, this means that if a higher state decays, its occupation number must be distributed over the states which have the next lower energy.

## 4. Pair of atoms

### 4.1. Sub- and superradiant eigenstates

The simplest geometry to look at, is the pair of atoms. It only depends on the distance of the two atoms and the orientation of their dipole-moments relative to the vector that connects the two atoms. Both two level Systems have a basis  $b$  which spans a Hilbert space  $\tilde{\mathcal{H}}$  with dimension 2. The space of the total system  $\mathcal{H}$  of both atoms is formed with the tensor-product of both Hilbert spaces

$$\mathcal{H} = \tilde{\mathcal{H}} \otimes \tilde{\mathcal{H}} \quad \text{mit} \quad \dim(\mathcal{H}) = 4 \quad (8)$$

and is spanned by the basis  $(|gg\rangle, |eg\rangle, |ge\rangle, |ee\rangle)$ . The rising and lowering-operators of the isolated two level system  $\sigma^+$  and  $\sigma^-$  have to be expanded onto the common Hilbert space.

$$\begin{aligned} \sigma_1^{+,-} &= \sigma^{+,-} \otimes \mathbb{1} \\ \sigma_2^{+,-} &= \mathbb{1} \otimes \sigma^{+,-} \end{aligned}$$

Now it is possible to form the Hamiltonian for the pair of atoms with the formula for the Hamiltonian of the N-atom system 4. Because of the Symmetry under exchange of the atoms, the equality  $\Omega_{12} = \Omega_{21}$  holds. The Hamiltonian can now be described as a

matrix of the form

$$H = \begin{pmatrix} 2\omega_0 & 0 & 0 & 0 \\ 0 & \omega_0 & \Omega_{12} & 0 \\ 0 & \Omega_{12} & \omega_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This leads to the eigenvalues  $2\omega_0, \omega_0 + \Omega_{12}, \omega_0 - \Omega_{12}$  and 0 and the associated eigenstates

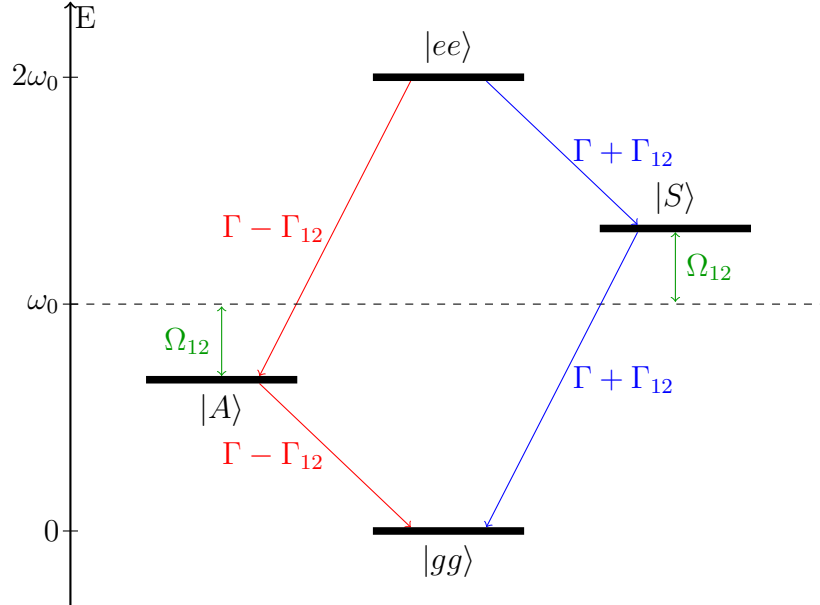
$$EV(2\omega_0) = |ee\rangle$$

$$EV(\omega_0 + \Omega_{12}) = |S\rangle = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)$$

$$EV(\omega_0 - \Omega_{12}) = |A\rangle = \frac{1}{\sqrt{2}}(|eg\rangle - |ge\rangle)$$

$$EV(0) = |gg\rangle$$

It can be noticed that the interaction doesn't influence the energy of the  $|gg\rangle$  and the  $|ee\rangle$  state. The energy of the symmetric triplet-state  $|S\rangle$  gets shifted upwards by  $\Omega_{12}$  whereas the anti symmetric singlet-state  $|A\rangle$  gets shifted downwards by  $\Omega_{12}$ . The decay rate can be determined with the time development [4]. This leads to two decay channels one through  $|S\rangle$  whose decay rate is increased by  $\Gamma_{12}$  and one through  $|A\rangle$  which is decreased by  $\Gamma_{12}$ .



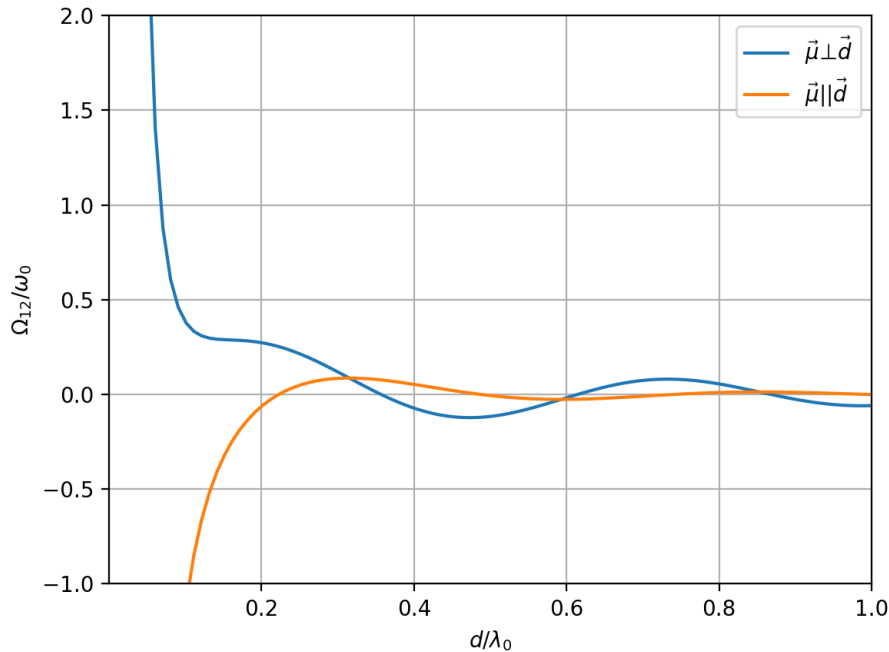
A decreased decay rate leads to a smaller phonon emission rate, therefore the state with a suppressed decay rate is called subradiant and the state with a elevated decay rate is called superradiant.

#### 4.2. Energy shift

In figure 1 is the energy-shift  $\Omega_{12}$ , scaled with the decay rate of the isolated two level system  $\Gamma_0$  as a function of the distance of the atoms shown. The graph  $\vec{\mu} \perp \vec{d}$  describes



the dependency for the the dipole moments directed perpendicular to the connecting vector of the Atoms whereas  $\vec{\mu} \parallel \vec{d}$  describes it for the parallel alignment.

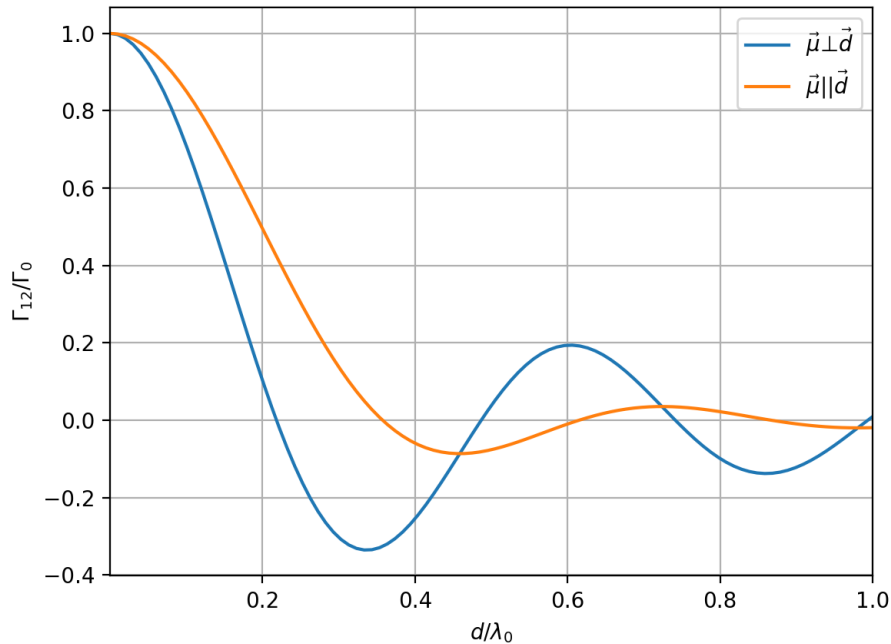


**Figure 1.** In the figure is the dependency of the energy shift on the distance of the atoms, for a orthogonal (blue) and a parallel (orange) alignment of the dipoles , shown.

It can be noticed that the absolute value of the energy shift increases for small distances whereas it becomes zero for large distances. For large distances the interaction between the atoms decreases and therefore they should behave like two isolated systems at a infinite distance whereby the energy of the sub and the superradiant state becomes  $\omega_0$ .

#### 4.3. Shift of the decay rate

In figure 2 is the shift of the decay rate, again scaled with the  $\Gamma_0$  for both alignments of the dipole moments shown. Also this shift approaches zero, hence the isolated case, for large distances. For small distances it approaches  $\Gamma_0$  for both orientations. This means the decay rate of the superradiant state nearly becomes  $2\Gamma_0$ , which is the maximum possible value for the system, whereas the subradiant decay rate approaches 0, the state becomes nearly stable. In this case the subradiant state also gets called dark because it shows a very suppressed photon emission.



**Figure 2.** In the figure is the dependency of the shift of the decay rate on the distance of the atoms, for a orthogonal (blue) and a parallel (orange) alignment of the dipoles, shown.

The sign of the energy shift depends on the orientation of the dipole moments whereas the shift of the decay rate for small distances is in both cases positive. This means that the state with higher energy does not always have a increased decay rate, for some orientation of the dipole moments and distances of the atoms the subradiant state is energetically higher then the superradiant state.

#### 4.4. Julia code

The following script calculates the eigenvalues and eigenstates of a pair of atoms. The energy shift and the shift of the decay rate can be calculated with the functions `Omega2_ij` and `Gamma2_ij`.

```

1 using QuantumOptics
2 using PyPlot;pygui(true)
3 using LinearAlgebra # for scalar product and normalize
4 using CollectiveSpins # fo Omega and Gamma Funktion
5
6 #Define gemoetry of system
7 k0 = 2pi
8 d = 2
9
10 #define positions both atoms
11 pos = []
12 push!(pos, [d, 0, 0])
13 push!(pos, [-d, 0, 0])
14

```

```

15 #Define dipolemoment of atoms (orthogonal case)
16 dips = []
17 push!(dips,[0,0,1])
18 push!(dips,[0,0,1])
19
20 #defines basis ,Hilbert space, and operators
21 b = NLevelBasis(2) # single two-level atom
22  $\sigma$  = transition(b, 1,2) # lowering operator
23  $\sigma_p$  = dagger( $\sigma$ ) # raising operator
24
25 ba = b $\otimes$ b # total hilbert space
26
27 # vector with the operators means translate operator
28 #from basis of single atom to basis of hilbert space
29 sm = [];sp=[]
30 for i = 1:N
31     push!(sm,embed(ba,i, $\sigma$ ))
32     push!(sp,embed(ba,i, $\sigma_p$ ))
33 end
34
35 # Omega and Gamma functions using GreenTensor() from CollectiveSpins
36 function Omega2_ij(r1, r2,  $\mu_1$ ,  $\mu_2$ ; k0 = 2 $\pi$ )
37     if r1 == r2
38         return 0.0
39     else
40         G_re = real(GreenTensor(r1 - r2, k0))
41         return -0.75 * (dot( $\mu_1$ , G_re,  $\mu_2$ ))
42     end
43 end
44 function Gamma2_ij(r1, r2,  $\mu_1$ ,  $\mu_2$ ; k0 = 2 $\pi$ )
45     if (r1 == r2) && ( $\mu_1$  ==  $\mu_2$ )
46         return 1.0
47     elseif (r1 == r2) && ( $\mu_1$  !=  $\mu_2$ )
48         return 0.0
49     else
50         G_im = imag(GreenTensor(r1 - r2, k0))
51         return 1.5 * real(dot( $\mu_1$ , G_im,  $\mu_2$ ))
52     end
53 end
54
55  $\Omega$ (i, j) = Omega2_ij(pos[i], pos[j], dips[i], dips[j]; k0 = 2 $\pi$ )
56  $\Gamma$ (i, j) = Gamma2_ij(pos[i], pos[j], dips[i], dips[j]; k0 = 2 $\pi$ )
57
58 #calculate hamiltonian
59 H = sum(sp[i]*sm[i] for i=1:N) + sum( $\Omega$ (i, j)*sp[i]*sm[j] for i=1:N, j=1:N)
60
61 #calculate eigenstates and eigenvalues
62  $\lambda\_val, \lambda\_vec$  = eigenstates(dense(H))

```

## 5. First-Excitation

In this chapter I am going to look at a model which is used to simplify systems with a larger number of atoms by looking only at the states in the so called first excitation. For large distances the system has to behave like  $N$  isolated two level systems. For the non-interacting case the states show a binomial distribution in the energy space. This means that  $\binom{N}{n}$  states have an energy of  $n\omega_0$ . This leads to a number of  $N$  states in the first excitation with energy  $\omega_0$ . Now the goal is to build a system which only includes these  $N$  states. This system now has a dimension of  $N$  and the states in single excitation form a basis for the new space of the system. All states in first excitation decay into the ground state, which is not part of the new system. This means every decay in our

new system leads to a loss of occupation in the new system. In the original system the recycling term ensured the conservation of the occupation number, in the new system it is not conserved anymore and the recycling term is set to zero. This leads to [2]

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{1}{2} \sum_{i,j} \Gamma_{ij} \{\sigma_i^+ \sigma_j^-, \rho\} \quad (9)$$

$$= -i[H_{eff}, \rho] \quad (10)$$

with the commutator  $[A, B] = A^*B - AB^*$ . This effective Hamiltonian  $H_{eff}$  describes our new system. Because the systems occupation number is not conserved its Hamiltonian is not hermitian and its eigenvalues are not real. The real part of the  $i$ th eigenvalue gives the energy shift of the  $i$ th eigenstate around the single excitation  $\omega_0$  whereas the imaginary part gives the total decay rate of the  $i$ th eigenstate [2]

$$H_{eff} = \sum_{ij} \left( \Omega_{ij} - \frac{i}{2} \Gamma_{ij} \right) \sigma_i^+ \sigma_j^- \quad (11)$$

$$\Omega_i = \text{Re}(\lambda_i) \quad \Gamma_i = -2\text{Im}(\lambda_i) \quad i = 1 \dots N. \quad (12)$$

The energies of the atoms themselves are not included in the Hamiltonian because we now describe the system in a frame which is rotating with the the atomic transition frequency  $\omega_0$  [2].

## 6. Ring of atoms

In this part I am going to use the single excitation model to examine a ring of interacting atoms. The atoms are distributed symmetrically across the ring, this means that the angle  $\theta$  and the distance  $d$  between two neighboring atoms is equal for all atoms.

### 6.1. Analytical eigenstates and -values

From a numerical point of view the work would be done. With the effective Hamiltonian 11 we are now able to describe every system with a finite number of particles and calculate numerically the eigenvalues to determine the shift of the energy and the decay rate with equations 12. But the ring has a special property, because of its strong symmetry it is even possible to calculate its eigenstates and -values analytically. First I rewrite the effective Hamiltonian by inserting the definition for  $\Omega_{ij}$  5 and  $\Gamma_{ij}$  7. This leads to

$$H_{eff} = \frac{3\pi\Gamma_0}{k_0} \sum_{i,j} G_{ij} \sigma_i^+ \sigma_j^- \quad (13)$$

with

$$G_{ij} = \vec{\mu}_i^* \cdot G(\vec{r}_i - \vec{r}_j, \omega_0) \vec{\mu}_j. \quad (14)$$

Then i perform a Fourier-Transform over the creation and annihilation operators like

$$\sigma_m^+ = \frac{1}{\sqrt{N}} \sum_i e^{-im\theta_i} \sigma_i^+ \quad (15)$$

where  $\theta_i$  is the angle of atom  $i$  in the ring. Now it is also possible to constitute  $\sigma_i^+$  as a Fourier-transform of  $\sigma_m^+$

$$\sigma_i^+ = \frac{1}{\sqrt{N}} \sum_m e^{im\theta_i} \sigma_m^+, \quad (16)$$

and insert it into the effective Hamiltonian. Doing the same for the operator  $\sigma_j^-$  i get a effective Hamiltonian of the form

$$H_{eff} = \frac{3\pi\Gamma_0}{k_0N} \sum_{m,\tilde{m}} \sum_{i,j} G_{ij} e^{im\theta_i} e^{i\tilde{m}\theta_j} \sigma_m^+ \sigma_{\tilde{m}}^- \quad (17)$$

$$= \frac{3\pi\Gamma_0}{k_0N} \sum_{m,\tilde{m}} \sum_{i,j} G_{ij} e^{im\theta_i} e^{i\tilde{m}\theta_j} e^{im\theta_j} e^{-i\tilde{m}\theta_j} \sigma_m^+ \sigma_{\tilde{m}}^- \quad (18)$$

$$= \frac{3\pi\Gamma_0}{k_0N} \sum_{m,\tilde{m}} \sum_{i,j} G_{ij} e^{im(\theta_i-\theta_j)} e^{i\tilde{m}(m-\tilde{m})} \sigma_m^+ \sigma_{\tilde{m}}^-. \quad (19)$$

Now i use the symmetry of the ring to rewrite the sum over  $i$  and  $j$

$$\sum_j \sum_i G_{ij} e^{im(\theta_i-\theta_j)} = \sum_j \sum_k G_{1k} e^{imk\frac{2\pi}{N}} \quad (20)$$

where the atom with index 1 is at the angle  $\theta_1 = 0$ . Using this it is possible to diagonalize the effective Hamiltonian

$$H_{eff} = \frac{3\pi\Gamma_0}{k_0N} \sum_{m,\tilde{m}} \sum_{j,k} G_{1k} e^{imk\frac{2\pi}{N}} e^{i\tilde{m}(m-\tilde{m})} \sigma_m^+ \sigma_{\tilde{m}}^- \quad (21)$$

$$= \frac{3\pi\Gamma_0}{k_0N} \sum_{m,\tilde{m}} \sum_{j,k} G_{1k} e^{imk\frac{2\pi}{N}} \delta_{m,\tilde{m}} \sigma_m^+ \sigma_{\tilde{m}}^- \quad (22)$$

$$= \frac{3\pi\Gamma_0}{k_0N} \sum_m \sum_{j,k} G_{1k} e^{imk\frac{2\pi}{N}} \sigma_m^+ \sigma_m^- \quad (23)$$

Therefore the eigenvalues of the effective hamiltonian are

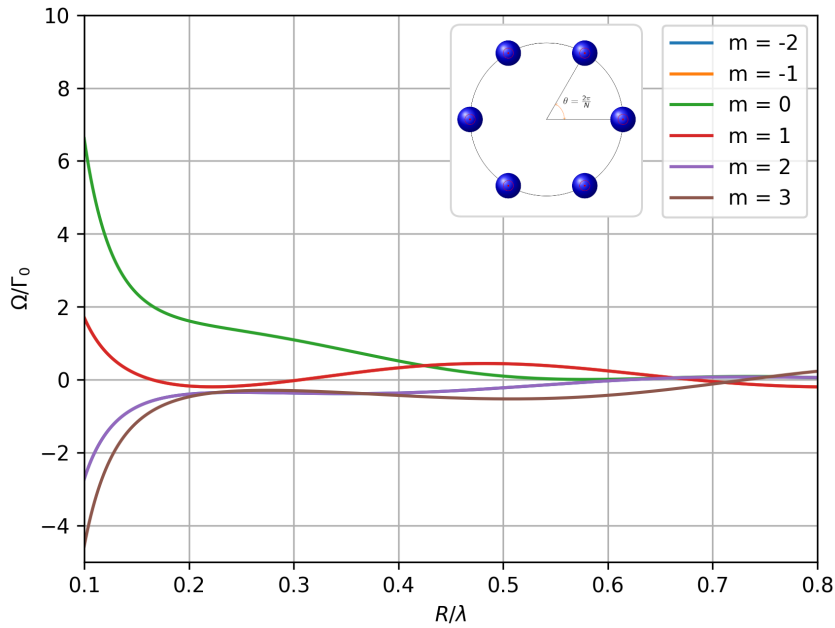
$$\lambda_m = -\frac{3\pi\Gamma_0}{k_0N} \sum_{i,j} e^{im*(\theta_i-\theta_j)} G_{ij} \quad (24)$$

where  $m = \left\{ \left\lceil \frac{-(N-1)}{2} \right\rceil, \dots, \left\lfloor \frac{N}{2} \right\rfloor \right\}$ . The corresponding eigenstates can be calculated by using the new creation operator on the ground state

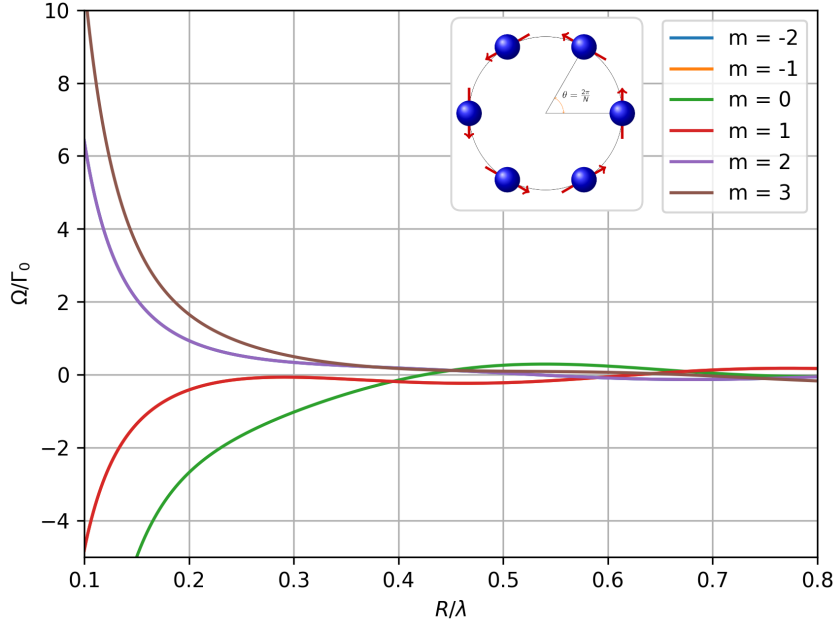
$$|m\rangle = \sigma_m^+ |g\rangle. \quad (25)$$

### 6.2. Energy shift

In figure 3 is the energy shift  $\Omega$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the perpendicular orientation of the dipole moments, shown. In figure 4 is the energy shift  $\Omega$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the tangential orientation of the dipole moments shown. It can be noticed that some states are degenerate which means they have the same eigenvalues. This is the case for the eigenstates with  $m = i$  and  $m = -i$  for  $-\frac{N}{2} + 1 \leq i \leq \frac{N}{2} - 1$  for a even number of atoms  $N$ . Also the state with  $m = -\frac{N}{2}$  has the same eigenvalue as the state with  $m = \frac{N}{2}$  but they only differ by a global phase and describe the same eigenstate. More generally said, every state with  $m = i$  describes the same eigenstate as the state with  $m = i + N$ , because of the rotation symmetry of the ring. Also for the ring geometry the interaction and therefore the energy shift gets smaller for big radii and consequently larger distances of the atoms. For small radii the absolute value of the increases. The sign of the of the shift for each state depends on the orientation of the dipoleoments but for the perpendicular as well as the parallel orientation the energy of half of the states get shifted upwards while the energy of the other half gets shifted downwards.



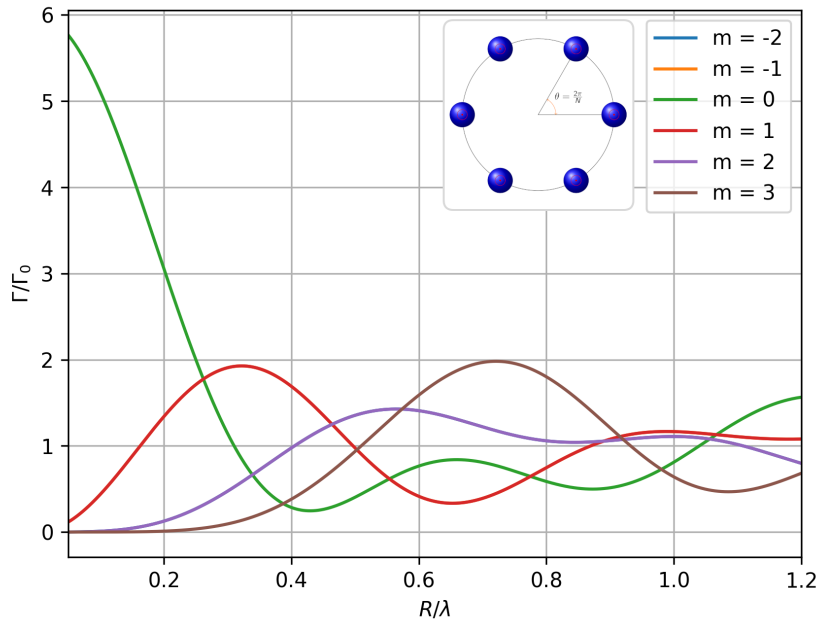
**Figure 3.** In the figure is the energy shift  $\Omega$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the perpendicular orientation of the dipole moments shown.



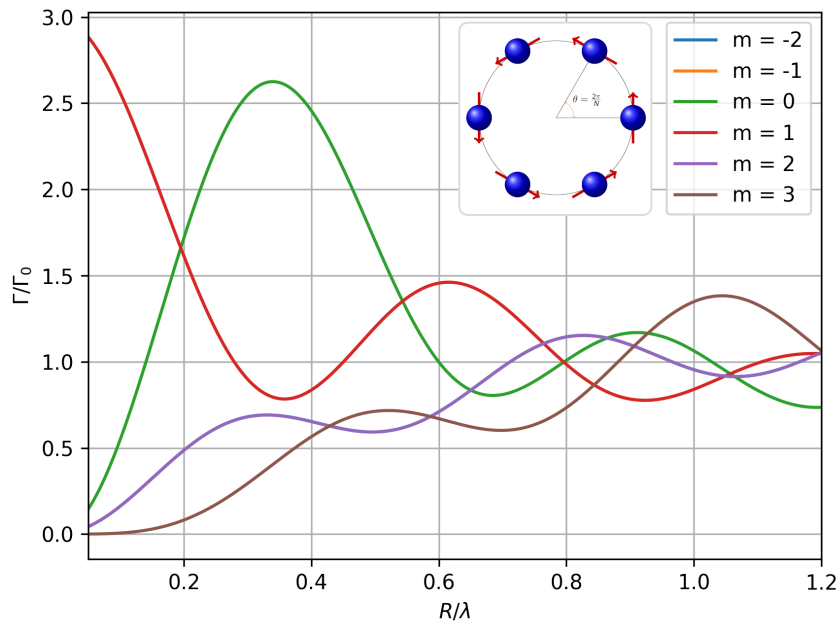
**Figure 4.** In the figure is the energy shift  $\Omega$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the tangential orientation of the dipole moments shown.

### 6.3. Decay rate

In figure 5 is the decay rate  $\Gamma$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the perpendicular orientation of the dipole moments shown. In figure 6 is the decay rate  $\Gamma$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the tangential orientation of the dipole moments shown. For large radii the decay rate approaches  $\Gamma_0$ , hence the non interacting case. In the perpendicular case the the decay rate of the non degenerate state  $m = 0$  approaches  $6\Gamma_0$  and the decay rate of all other states goes to zero. For the tangential orientation the decay rate of the degenerate states  $m = 1$  and  $m = -1$  approaches  $3\Gamma_0$  while the decay rate of the other states goes to zero. But the sum over all decay rates gives always the maximum decay rate of  $N\Gamma_0 = 6\Gamma_0$ . It can also be noticed that for different radii, different states become super and subradiant.



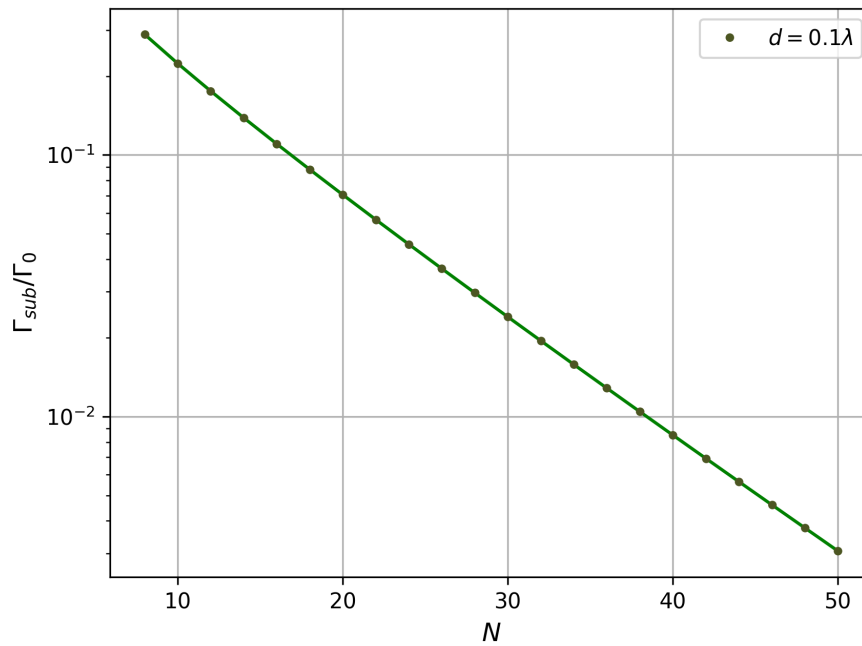
**Figure 5.** In the figure is the decay rate  $\Gamma$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the perpendicular orientation of the dipole moments shown.



**Figure 6.** In the figure is the decay rate  $\Gamma$  in dependence of the radius of the ring for all eigenstates in the single excitation manifold, for the tangential orientation of the dipole moments shown.



In figure 7 is the minimal decay of all states  $\Gamma_{sub}$  in dependence of the number of atoms on the ring on a logarithmic scale shown. The distance  $d$  between the atom is held at a constant value. It can be noticed that the subradiant decay rate drops exponentially with the number of atoms on the ring.



**Figure 7.** In the figure is the minimal decay of all states in dependence of the number of atoms on the ring on a logarithmic scale shown. The distance  $d$  between the atom is held at a constant value.

#### 6.4. Julia code

```

1  using QuantumOptics
2  using PyPlot;pygui(true)
3  using LinearAlgebra # for scalar product and normalize
4  using CollectiveSpins # fo Omega and Gamma Funktion
5
6  #define number of atoms and size of Ring
7  N=6
8  R = 1
9
10 #define angles of atoms on ring
11 angles = []
12 for i = 1:N
13     φ = i*2*pi/N
14     push!(angles,φ)
15 end
16
17
18 #define positions of atoms on ring
19 pos = []
20 for φ in angles
21     push!(pos, [R*cos(φ),R*sin(φ),0])

```

```

22 end
23
24 #define dipoles of atoms (tangential case)
25 dips = []
26 for  $\varphi$  in angles
27     push!(dips, [-sin( $\varphi$ ), cos( $\varphi$ ), 0])
28 end
29
30 #define new system mit raising/lowering operator from/to ground state
31 #ground state not in new system
32 bs = NLevelBasis(N+1)
33 sm = []
34 sp = []
35 for i = 1:N
36     sm_val = transition(bs, 1, N+2-i)
37     push!(sm, sm_val)
38     push!(sp, dagger(sm_val))
39 end
40
41 #define analytically calculated eigenstates
42 function psi(m)
43     return 1/sqrt(N) * sum(exp(im*m*angles[j])*sp[j] for j = 1:N) * nlevelstate(bs, 1)
44 end
45
46
47 #calculate effective hamiltonien
48  $\Omega(i, j) = \Omega_{2\_ij}(\text{pos}[i], \text{pos}[j], \text{dips}[i], \text{dips}[j]; k_0 = 2\pi)$ 
49  $\Gamma(i, j) = \Gamma_{2\_ij}(\text{pos}[i], \text{pos}[j], \text{dips}[i], \text{dips}[j]; k_0 = 2\pi)$ 
50  $H_{\text{eff}} = \text{sum}((\Omega(i, j) - im/2 * \Gamma(i, j)) * sp[i] * sm[j] \text{ for } i = 1:N, j = 1:N)$ 
51
52
53 if N%2 != 0
54     m_min = -Int((N-1)/2)
55     m_max = -m_min
56 else
57     m_min = -Int(N/2)+1
58     m_max = Int(N/2)
59 end
60
61 #calculate all eigenvalues
62  $\lambda_{\text{array}} = []$ 
63 for m = m_min:m_max
64      $\psi_m = \text{psi}(m)$ 
65      $\lambda_m = \text{dagger}(\psi_m) * H_{\text{eff}} * \psi_m$ 
66     push!( $\lambda_{\text{array}}$ ,  $\lambda_m$ )
67 end

```

The  $\Omega_{2\_ij}$  and  $\Gamma_{2\_ij}$  function are defined the same way as in the code of the pair, therefore the definitions were omitted in this code example. Although I described the first excitation model in the previous chapter as a space with dimension  $N$  I defined in the code in line 32 a  $N + 1$  dimensional space. Because the states with single excitation decay into the ground state, which is not in the system I would have had to use creation and annihilation operators and a Fock space. Therefore I added an extra dimension for the ground state and was able to use simple lowering and raising operators. I corrected this extra dimension by defining the ladder operators only for the excited states.

### 6.5. Electric field

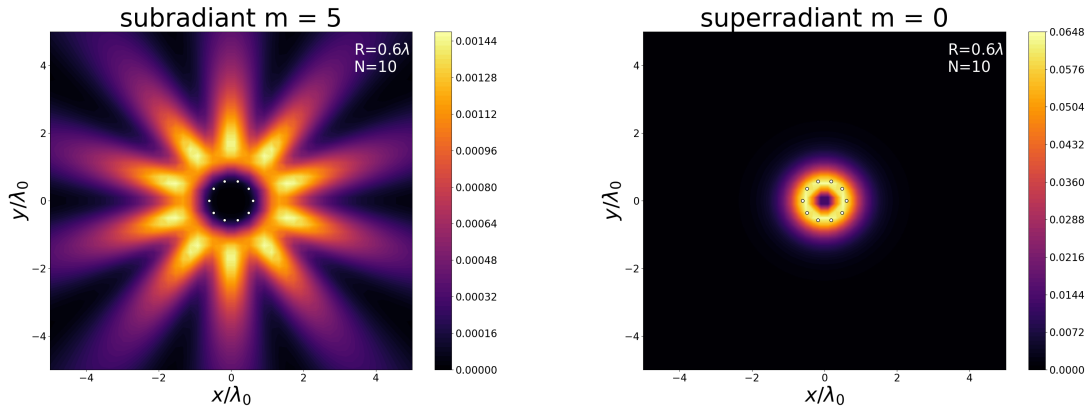
The electric field of an ensemble of  $N$  two level systems can be calculated with the general input-output equation in the time domain without external field [1]

$$\hat{\mathbf{E}}^+(\mathbf{r}) = \mu_0 \omega_0^2 \sum_{i=1}^N \mathbf{G}(\mathbf{r}, \mathbf{r}_i, \omega_0) \mu_i \sigma_i^+. \quad (26)$$

Then the intensity can be calculated through

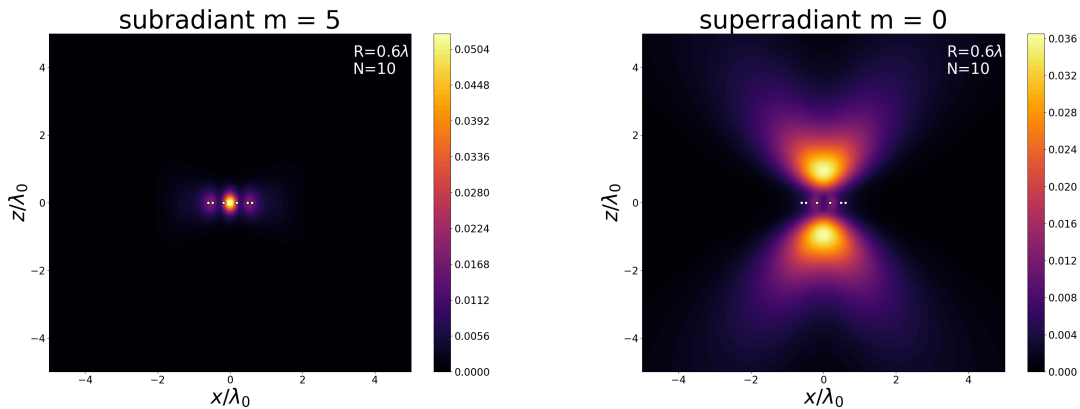
$$\hat{I} = \left( \hat{E}_x^+ \right)^\dagger \hat{E}_x^+ + \left( \hat{E}_y^+ \right)^\dagger \hat{E}_y^+ + \left( \hat{E}_z^+ \right)^\dagger \hat{E}_z^+. \quad (27)$$

In figure 8 the expectation value of the intensity operator on the xy-plane for a subradiant (left) and a superradiant (right) state of a ring with 10 atoms.



**Figure 8.** In the figure is the expectation value of the intensity operator in the xy-plane for a subradiant (left) and a superradiant (right) state of a ring with 10 atoms plotted.

In figure 9 the expectation value of the intensity operator on the xz-plane for a subradiant (left) and a superradiant (right) state of a ring with 10 atoms.



**Figure 9.** In the figure is the expectation value of the intensity operator in the  $xz$ -plane for a subradiant (left) and a superradiant (right) state of a ring with 10 atoms plotted.

## 7. Conclusion

First I looked at the single two level system and introduced some concepts like the spontaneous emission. Then I expanded the system onto a larger number of atoms and discussed the master equation as the time evolution of the system. Then I looked at the Pair of atoms and calculated the eigenstates and eigenvalues of the system. I showed that the interaction results in a subradiant and a superradiant state with a decreased and a increased decay rate. The interaction also results in a shift of the energy of the  $|S\rangle$  and the  $|A\rangle$  state. I numerically calculated the shift of the decay rate and the shift of the energy with the help of the quantum optics toolbox for the Julia programming language. It turned out that the energy shift increases for small distances of the atoms and approaches zero for large distances, furthermore I saw that the sign of the shift depends on the orientation of the dipole moments of the atoms. The shift of the decayrate also approaches zero for large distances but for smaller distances it goes to  $\Gamma_0$ . Witch leads to a decay rate  $\Gamma = 2\Gamma_0$  for the symmetric eigenstate  $|S\rangle$  and  $\Gamma = 0$  for the anti symmetric eigenstate  $|A\rangle$ . The I introduced the single excitation manifold, defined a new effective Hamiltonian of this model and discussed shortly what advantages it brings. I introduced the geometry of the ring and showed the analytical calculation of eigenstates and eigenvectors of the system. I again examined the proprieties of energy and decay rate numerically. The results show that also for the ring the absolute value of the energy shift increases for small radii. The shift is positive for half of the states in the first excitation manifold while it is negative for the others. Which states shift upwards depends on the orientation of the dipole moments. For large radii the energy shift goes again to zero. The decay rate of all states with single excitation approaches  $\Gamma_0$  for large radii whereas for small radii all state except for one sate or two degenerate states become dark. The decay rate of the other state becomes  $N\Gamma_0$  or  $\frac{N}{2}\Gamma_0$  in the degenerate case. I showed that the decay rate also depends on the number of atoms on

the ring. If the distance between the atoms on the ring is kept constant the smallest decay rate of all states decreases exponentially with the number of atoms on the Ring. In the end I looked at the intensity of the electric field of the ring in a subradiant and a superradiant state. It shows, as expected, that the superradiant state produces an electric field with much higher intensity than the subradiant one.

In summary I was able to understand the underlying theory of the interaction of multiple two level system and the idea and the theory of the master equation. I successfully simulated the different geometry, analyzed their propensities and was able to clearly show the effect of sub- and superradiance.

## 8. Appendix

The scripts used for the numeric calculation where all executed on Julia version 1.4.1. For a big part of the numerical calculations I used QuantumOptics.jl, a Julia framework developed in Helmut Ritsch's CQED group at the Institute for Theoretical Physics of the University of Innsbruck. It was created to simplify the simulation of different open quantum systems. More information can be found in [5] and on the official website [6].

## 9. References

- [1] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, D. Chang, and H.J. Kimble. Exponential improvement in photon storage fidelities using subradiance and “selective radiance” in atomic arrays. August 2017.
- [2] M. Moreno-Cardoner, D. Plankensteiner, L. Ostermann, D. Chang, and H. Ritsch. Extraordinary subradiance with lossless excitation transfer in dipole-coupled nano-rings of quantum emitters. January 2019.
- [3] R.J. Cogdell, A. Gall, and J. Köhler. The architecture and function of the light-harvesting apparatus of purple bacteria: from single molecules to in vivo membranes. 2006.
- [4] L. Ostermann. Collective radiation of coupled atomic dipoles and the precise measurement of time. June 2016.
- [5] Sebastian Krämer, David Plankensteiner, Laurin Ostermann, and Helmut Ritsch. Quantumoptics.jl: A julia framework for simulating open quantum systems. *Computer Physics Communications*, 227:109–116, 2018.
- [6] QuantumOptics.jl, A Julia Framework for Open Quantum Dynamics. <https://qojulia.org/>.