Plastic Structural Analysis under Stochastic Uncertainty

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ABSTRACT

Problems from limit load or shakedown analysis are based on the convex, linear or linearized yield/strength condition and the linear equilibrium equation for the generic stress vector. Having to take into account, in practice, stochastic variations of the model parameters (e.g. yield stresses, plastic capacities) and external loadings, the basic stochastic plastic analysis problem must be replaced by an appropriate deterministic substitute problem. Instead of calculating approximatively the probability of failure based on a certain choice of failure modes, here, a direct approach is presented based on the costs for missing carrying capacity and the failure costs (e.g. costs for damage, repair, compensation for weakness within the structure, etc.). Based on the basic mechanical survival conditions, the failure costs may be represented by the minimum value of a convex and often linear program. Several mathematical properties of this program are shown. Minimizing then the total expected costs subject to the remaining (simple) deterministic constraints, a stochastic optimization problem is obtained which may be represented by a "Stochastic Convex Program (SCP) with recourse". Working with linearized yield/strength conditions, a "Stochastic Linear Program (SLP) with complete fixed recourse" is obtained. In case of a discretely distributed probability distribution or after the discretization of a more general probability distribution of the random structural parameters and loadings as well as certain random cost factors one has a linear program (LP) with a so-called "dual decomposition data" structure. For stochastic programs of this type many theoretical results and efficient numerical solution procedures (LP-solver) are available. The mathematical properties of theses substitute problems are considered. Furthermore approximate analytical formulas for the limit load factor are given.

Key words: Limit load analysis, shakedown analysis, random parameters, stochastic optimization, stochastic linear programs

1 PLASTIC ANALYSIS OF STRUCTURES

According to König [12], many materials, e.g. most of metals, have distinct, plastic properties, i.e., they are ductile. Even after the stress intensity attains the yield point stress, such materials can deform considerably without breaking. This implies that if the stress intensity at a certain point of a hyperstatic

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structure reaches the critical (yield) value, the structure does not necessarily fail or deform excessively. Instead, a certain amount of stress redistribution takes place and some further load increments can be supported. Structural failure does not occur before a kinematic mechanism of unconstrained plastic flow develops. Thus, the actual load-carrying capacity of a structure is higher (in some cases quite considerably) than that derived from classical elastic analysis.

1.1 Plastic limit analysis (analysis under proportional loading)

Limit analysis is concerned [2],[5]-[8],[11],[13],[24]-[29] with establishing the strength of a structure, i.e., its capacity for the supporting of loads. Hence, using the plastic ductility of structural materials in improving the design of structures, limit analysis is not concerned with deformation: it can not therefore provide the load carrying capacity for a structure with elements that have a limited ductility or deformability, nor for a structure which becomes unstable because of the displacements induced by plastic deformation, see [6, 8, 13, 24]

For elastic-perfectly plastic materials, the ultimate load condition corresponding to complete collapse of the structure can be obtained through application of either a pair of dual theorems, i.e., the static (safe or lower bound) theorem and the kinematic (unsafe or upper bound) theorem:

ST) Static Theorem (Lower Bound or Safe Theorem)

If any stress distribution throughout the structure can be found which is everywhere in equilibrium internally and balances the external loads, and at the same time does not violate the yield condition, those external loads will be carried safely by the structure.

KT) Kinematic Theorem (Upper Bound or Unsafe Theorem)

Collapse occurs if a collapse mechanism, i.e., a pair of vectors $(u, \tilde{u})$ exists, fulfilling the compatibility condition, such that the work done by the external loads is larger than the corresponding internal plastic work.

In order to determine the load carrying capacity, i.e., the limit load factor μ^* of structures that are under proportional loading $P = \mu P_0$, where P_0 is the m-vector of external reference loads, and $\mu \geq 0$ denotes the load factor, according to (ST) we have the following maximization problem:

$$\max \mu \tag{1a}$$

s.t.

$$C\sigma - \mu R_0 = 0 \tag{1b}$$

$$R_{id}^{-1}\sigma_i \in K_i, i = 1, \ldots, n_G \tag{1c}$$

or

$$\pi(R_{id}^{-1}\sigma_i | K_i) \leq 1, i = 1, \ldots, n_g \tag{1c'}$$

$$\mu \geq 0. \tag{1d}$$

Here, after a discretization of the structure by Finite Elements (FE), σ_i denotes the n_0-vector of generic stress components (or interior load components) at the i-th reference or "check point" $x_i, i = 1, \ldots, n_G$(number of FE's multiplied by the number of "check points" in each FE), of the structure, $\sigma^T := \left(\sigma_1^T, \sigma_2^T, \ldots, \sigma_{n_G}^T\right)$, R_i is the n_0-vector of positive material strength or resistance parameters (e.g. yield stresses, plastic capacities, etc.) at the i-th reference point, and R_{id} denotes the $n_0 \times n_0$ diagonal matrix with the components of R_i on its main diagonal. Furthermore, C is the $m \times n_0 n_G$ equilibrium matrix of the (discretized) structure, $K_i(\subset \mathbb{R}^{n_0})$ is the bounded, closed convex admissible domain in $\mathbb{R}^{n_0}$ related to the i-th reference point, where the origin is an interior point of K_i, and $\pi = \pi(z|K_i)$ denotes [14, 15] the distance or Minkowski functional of K_i defined by

$$\pi(z|K_i) := \inf\left\{\lambda > 0 : \frac{z}{\lambda} \in K_i\right\}, z \in \mathbb{R}^{n_0}. \tag{1e}$$

Linearizing the convex admissible domain K_i, as is done mostly in practice [3, 4], K_i is approximated $K_i \supset \tilde{K}_i$ by a bounded, closed convex polyhedron

$$\tilde{K}_i := \left\{\sigma_i : N_i \sigma_i \leq h_i\right\} \tag{2a}$$

in $\mathbb{R}^{n_0}$ containing again the origin of $\mathbb{R}^{n_0}$ as an interior point, where $(N_i, h_i), i = 1, \ldots, n_G$, are certain $m_y \times (n_0 + 1)$ matrices; without any restrictions, in the following we may assume that

$$h_i = \mathbf{1} := (1, \ldots, 1)', i = 1, \ldots, n_G. \tag{2b}$$

Replacing in (1c) the yield domain K_i by $\tilde{K}_i$, we obtain the following linearized limit load problem

$$\max \mu \tag{3a}$$

s.t.

$$\begin{aligned} C\sigma - \mu P_0 &= 0 && (3b)\\ N_i R_{id}^{-1} \sigma_i &\leq \mathbf{1}, i = 1, \ldots, n_G && (3c)\\ \mu &\geq 0. && (3d)\end{aligned}$$

Using the Minkowski functional $\pi = \pi(z|\tilde{K}_i)$ of the convex polyhedron $\tilde{K}_i$, then (3c) may also be represented by

$$\pi(R_{id}^{-1}\sigma_i|\tilde{K}_i) \leq \mathbf{1}, i = 1, \ldots, n_G. \tag{3c'}$$

1.2 Shakedown analysis (analysis under variable repeated loading)

If a structure, made of elastic-perfectly plastic materials, is exposed to cyclic loadings, then it may happen [12] that, after some plastic deformation in the initial load cycles, the structural behavior becomes eventually elastic. Such stabilization of plastic deformations is called shakedown or adaptation [3, 4, 8, 12].

Hence, the principal purpose of shakedown analysis is to determine [3]-[5],[7] a limit shakedown domain defined by a limit load factor μ^* such that inside this domain one observes a stabilization of plastic deformations everywhere in the structure.

According to Bleich and Melan's classical theorem [8, 12], a structure will shake down if and only if selfstresses $\rho_i, i = 1, \ldots, n_G$, can be found which lead, when superposed on the elastic response to any load in the load domain, to a stress state that nowhere exceeds the yield limits. Assume that the cycling external loads $P = P(t)$ are contained for each time t in the closed convex load polyhedron in $\mathbb{R}^m$ represented by

$$P(t) = \mu \sum_{j=1}^{n_l} \beta_j(t) P^{(j)} \tag{4a}$$

$$\sum_{j=1}^{n_l} \beta_j(t) = 1, \beta_j(t) \geq 0, j = 1, \ldots, n_l, \tag{4b}$$

where $\mu \geq 0$ is a time-independent load domain multiplier, $P^{(j)}, j = 1, \ldots, n_l$, are fixed reference load m-vectors, and $\beta_j = \beta_j(t), j = 1, \ldots, n_l$, are time-varying coefficients. Let then $\sigma_i^{el(j)}, j = 1, \ldots, n_l$, resp., denotes the elastic structural response with respect to the reference load $P^{(j)}, j = 1, \ldots, n_l$.

Consequently, working with the linearized yield domain (2a), for the computation of the limit or shakedown load coefficient μ^*, here we obtain the following maximization problem:

$$\max \mu \tag{5a}$$

s.t.

$$C\rho = 0 \tag{5b}$$

$$N_i R_{id}^{-1}\left(\mu \sigma_i^{el(j)} + \rho_i\right) \leq \mathbf{1}, j = 1, \ldots, n_l, i = 1, \ldots, n_G \tag{5c}$$

$$\mu \geq 0, \tag{5d}$$

where $\rho^T := (\rho_1^T, \rho_2^T, \ldots, \rho_{n_G}^T)$.

Corresponding to (3c'), the present condition (5c) can be represented also by

$$\pi\left(R_{id}^{-1}(\mu \sigma_i^{el(j)} + \rho_i) | \tilde{K}_i\right) \leq 1, j = 1, \ldots, n_l, i = 1, \ldots, n_G. \tag{5c'}$$

Defining

$$\eta_i := \max_{1\le j\le n_l} N_i R_{id}^{-1} \sigma_i^{el(j)}, \tag{6}$$

where the maximum in (6) is taken componentwise, the maximum load domain multiplier μ^* can also be found by solving

$$\max \mu \tag{7a}$$

s.t.

$$\begin{aligned} C\rho &= 0 && (7b)\\ \mu\eta_i &+ N_i R_{id}^{-1}\rho_i \le \mathbf{1}, i = 1, \ldots, n_G && (7c)\\ \mu &\ge 0. && (7d) \end{aligned}$$

2 NUMERICAL EVALUATION OF STRUCTURAL SURVIVAL/FAILURE

According to (ST) and (KT), in plastic analysis and design the survival or safety of the structure can be described by means of the equilibrium equation and the yield condition. Hence, working in the following with the linearized yield condition, in limit analysis, in shakedown analysis, resp., we have to deal with the safety conditions (3b,c), (5b,c), respectively.

Consequently, for given load factor $\mu \ge 0$, vector of strength or resistance parameters $R^T := \left(R_1^T, R_2^T, \ldots, R_{n_G}^T\right)$ and reference load vector P_0, in limit analysis we consider the following linear program (LP)

$$\min s \tag{8a}$$

s.t.

$$\begin{aligned} &C\sigma = \mu P_0 && (8b)\\ &N_i R_{id}^{-1}\sigma_i - \mathbf{1} \le s\mathbf{1}, i = 1, \ldots, n_G, && (8c) \end{aligned}$$

Since the equilibrium matrix C has full rank m being smaller than the number $n_0 n_G$ of components of the stress vector σ, LP (8a-c) has a nonempty feasible domain. According to (3c'), the objective function of this linear program is bounded from below on the feasible domain. Hence, the minimum value function

$$\begin{aligned} s_{LA}^* &= s_{LA}^*(\mu, R, P_0) && (9)\\ &= \min\left\{s : C\sigma = \mu P_0, N_i R_{id}^{-1}\sigma_i \le (1+s)\mathbf{1}, i = 1, \ldots, n_G\right\} \end{aligned}$$

exists for arbitrary admissible points (μ, R, P_0).

In case of shakedown analysis we consider the linear program

$$\min s \tag{10a}$$

s.t.

$$\begin{aligned} C\rho &= 0 && (10b)\\ \mu\eta_i &+ N_i R_{id}^{-1}\rho_i - \mathbf{1} \le s\mathbf{1}, i = 1, \ldots, n_G, && (10c) \end{aligned}$$

and the related minimum value function

$$\begin{aligned} s^*_{ShA} &= s^*_{ShA}(\mu, R, \sigma^{el}) && (11)\\ &= \min\left\{s : C\rho = 0, \mu\eta_i + N_i R_{id}^{-1}\rho_i \le (1+s)\mathbf{1}, i = 1, \ldots, n_G\right\}, \end{aligned}$$

where $\sigma^{el} := \left(\sigma_i^{el(j)}\right)$.

For the fulfillment (violation, resp.) of the safety condition (3b,c), (5b,c), resp., we have now the following numerical criterion:

Lemma 2.1 *For given parameter vector $(\mu, R, P_0), (\mu, R, \sigma^{el})$ in limit analysis, shakedown analysis, resp., the safety condition (3b,c), (5b,c) is fulfilled (violated, resp.) if and only if*

$$s^*_{LA}(\mu, R, P_0) \le 0 \left(s^*_{LA}(\mu, R, P_0) > 0\right), \tag{12a}$$

$$s^*_{ShA}(\mu, R, \sigma^{el}) \le 0 \left(s^*_{ShA}(\mu, R, \sigma^{el}) > 0\right), \tag{12b}$$

respectively.

Remark 2.1 *Obviously, because of Lemma 2.1, $s^*_{LA} = s^*_{LA}(\mu, R, P_0), s^*_{ShA} = s^*_{ShA}(\mu, R, \sigma^{el})$, resp., plays the role of the* **limit state function or safety margin** *for the limit, shakedown analysis problem; for simplification, we do not define the limit state function by $t^* := -s^*$, see [21].*

A positive value $s^*_{LA} > 0, s^*_{ShA} > 0$ of the safety margin s^*_{LA}, s^*_{ShA} indicates a violation of the safety condition (3b,c), (5b,c), resp., and therefore a weakness of the structure or a possible structural damage or failure. Hence, the resulting failure costs, the loss or further penalties c^*, e.g. the costs for repairing, strengthening or retrofitting the structure after realization of the random parameters, can be evaluated by means of a certain monotoneous nondecreasing function $\gamma = \gamma(s)$ of the limit state function, hence

$$c^*_{LA} = \gamma(s^*_{LA}), c^*_{ShA} = \gamma(s^*_{ShA}), \tag{13}$$

respectively.

According to Section 4, the limit state function s^* can be interpreted as follows: If $s^* > 0$, then s^* represents the minimum rate of increase of the material resistances (plastic capacities) in the reference points $x_i, i = 1, \ldots, n_G$, needed to maintain again a structure carrying safely the given load; if $s^* < 0$, then $|s^*| = -s^*$ indicates the maximum rate of reserve of the material resistances (plastic capacities) within the structure for carrying safely the given load P. Hence, definition (13) of the cost function c^* is related to the definition of the unit cost function given by Prager et al. in [25].

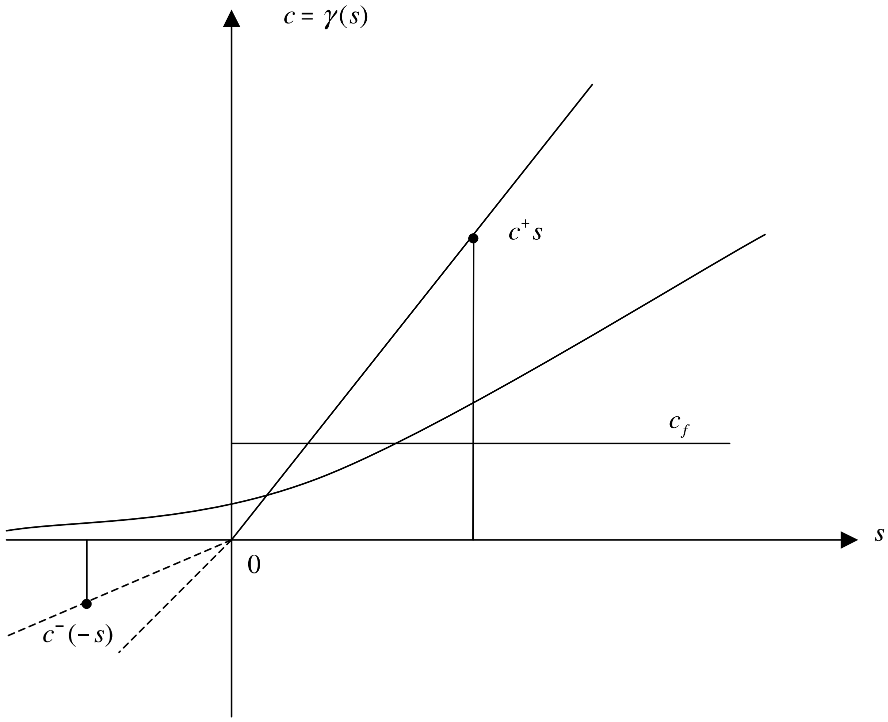

Fig. 2.1 Cost functions γ

Since a minimum point $(s^*, \sigma^*), (s^*, \rho^*)$ exists in the linear programs (8a-c), (10a-c), resp., for the failure costs c^*_{LA}, c^*_{ShA} we have the following representation:

Lemma 2.2 *Let $\gamma = \gamma(s)$ denote any monotoneous nondecreasing function.*

a) For given parameter vector (μ, R, P_0), in limit analysis the failure costs

$c^*_{LA} = \gamma(s^*_{LA}) = \gamma\Big(s^*_{LA}(\mu, R, P_0)\Big)$ *are given by the minimum value of the program*

$$\min \gamma(s) \tag{14a}$$

s.t.

$$C\sigma = \mu P_0 \tag{14b}$$

$$N_i R_{id}^{-1} \sigma_i \le (1+s)\mathbf{1}, i = 1, \ldots, n_G. \tag{14c}$$

b) For given parameter vector (μ, R, σ^{el})*, in shakedown analysis the failure costs* $c^*_{ShA} = \gamma(s^*_{ShA}) = \gamma\Big(s^*_{ShA}(\mu, R, \sigma^{el})\Big)$ *are equal to the minimum value of the program*

$$\min \gamma(s) \tag{15a}$$

s.t.

$$C\rho \; = \; 0 \tag{15b}$$

$$\mu\eta_i \; + \; N_i R_{id}^{-1} \rho_i \le (1+s)\mathbf{1}, i = 1, \ldots, n_G. \tag{15c}$$

Of course, (14a-c), (15a-c) are no linear or convex programs in general. However, if $\gamma = \gamma(s)$ is a convex cost function, then the programs are convex. Moreover, in the important case of sublinear cost functions

$$\gamma(s) = \begin{cases} c^+ s, s \ge 0 \\ c^-(-s), s < 0, \end{cases} \tag{16a}$$

where the cost coefficients c^+, c^- are chosen such that

$$c^+ > 0, \quad 0 \ge c^- \ge -c^+, \tag{16b}$$

cf. Fig. 2.1, for $\gamma(s)$ we have the representation

$$\gamma(s) = \min \left\{ c^+ y^+ + c^- y^- : y^+ - y^- = s, y^+, y^- \ge 0 \right\} \tag{16c}$$

which yields then this representation of the failure costs:

Corollary 2.1 *If* $\gamma = \gamma(s)$ *is given by (16a,b), then* c^*_{LA} *is the minimum value of the linear program*

$$\min c^+ y^+ + c^- y^- \tag{17a}$$

s.t.

$$C\sigma = \mu P_0, \tag{17b}$$

$$N_i R_{id}^{-1} \sigma_i - (y^+ - y^-)\mathbf{1} \le \mathbf{1}, i = 1, \ldots, n_G \tag{17c}$$

$$y^+, y^- \ge 0; \tag{17d}$$

furthermore, c^*_{ShA} is the minimum value of the LP

$$\min c^+ y^+ + c^- y^- \tag{18a}$$

s.t.

$$C\rho = 0 \tag{18b}$$
$$\mu\eta_i + N_i R_{id}^{-1} \rho_i - (y^+ - y^-)\mathbf{1} \leq \mathbf{1}, i = 1, \ldots, n_G \tag{18c}$$
$$y^+, y^- \geq 0. \tag{18d}$$

More general cost functions c^*_{LA}, c^*_{ShA} for the evaluation of the violations of the basic safety conditions of elastic-plastic structures can be obtained as follows:

For this purpose we introduce a vector of cost coefficients q and a so-called *recourse matrix* $M = (M_i)$ such that the LP

$$\min q'v \tag{19a}$$

s.t.

$$M_i v = z_i, i = 1, \ldots, n_G \tag{19b}$$
$$v \geq 0 \tag{19c}$$

has an optimal solution $v^* = v^*(z)$ for any right hand side $z = (z_i)$. Then, the linear program (17a-d) evaluating the violations of the safety condition in limit analysis can be represented by

$$\min q'v \tag{20a}$$

s.t.

$$C\sigma = \mu P_0 \tag{20b}$$
$$N_i R_{id}^{-1} \sigma_i + M_i v = \mathbf{1}, i = 1, \ldots, n_G \tag{20c}$$
$$v \geq 0. \tag{20d}$$

Moreover, in case of shakedown analysis the failure cost representation reads

$$\min q'v \tag{21a}$$

s.t.

$$C\rho = 0 \tag{21b}$$
$$\mu\eta_i + N_i R_{id}^{-1} \rho_i + M_i v = \mathbf{1}, i = 1, \ldots, n_G \tag{21c}$$
$$v \geq 0. \tag{21d}$$

In the above case (17a-d), (18a-d), resp., we have that

$$M_i := (-\mathbf{1}, \mathbf{1}, 0, \ldots, I, \ldots, 0),\ v := \begin{pmatrix} y^+ \\ y^- \\ \delta \end{pmatrix},\ q := \begin{pmatrix} c^+ \\ c^- \\ 0 \end{pmatrix}, \tag{22}$$

where the m_y-vector $\mathbf{1}$ is defined as in (2b), $y^+, y^- \in \mathbb{R}_+$, and the $m_y \times m_y$ identity matrix I in M_i is placed at the ith position, $i = 1, \ldots, n_G$.

Indeed, if M, v and q are selected according to (22), then (20c) reads

$$N_i R_{id}^{-1} \sigma_i - (y^+ - y^-)\mathbf{1} = \mathbf{1} - \delta_i \leq \mathbf{1}, \tag{23a}$$

which coincides with (17c). Dividing (23a) by $1 + y^+ - y^- > 0$, we find the equivalent condition

$$N_i \Big((1 + y^+ - y^-) R_i\Big)_d^{-1} \sigma_i \leq \mathbf{1}, i = 1, \ldots, n_G. \tag{23b}$$

In the same way, (21c) yields

$$\mu \eta_i + N_i R_{id}^{-1} \rho_i - (y^+ - y^-)\mathbf{1} = \mathbf{1} - \delta_i \leq \mathbf{1} \tag{23c}$$

coinciding with (18c); diving still by $1 + y^- - y^- \geq 0$, we get

$$\begin{aligned} \mu \max_{1 \leq j \leq n_l} N_i \Big((1 + y^+ - y^-) R_i\Big)_d^{-1} \sigma_i^{el(j)} & \\ + N_i \Big((1 + y^+ - y^-) R_i\Big)_d^{-1} \rho_i \leq \mathbf{1}, i = 1, \ldots, n_G. & \end{aligned} \tag{23d}$$

For $y^+ - y^- > 0, y^+ - y^- < 0$, resp., $\Delta R_i = (y^+ - y^-) R_i$ can be interpreted as an increase, a reserve of the material resistances of $(y^+ - y^-)100\%$, cf. (49a-c).

A more general situation arises if M, v, q are chosen such that

$$M_i := (-\mathbf{1}, \mathbf{1}, \ldots, -N_i N_i', I, \ldots) \tag{24a}$$

and

$$v := (y^+, y^-, \lambda_1, \delta_1, \lambda_2, \delta_2, \ldots, \lambda_{n_G}, \delta_{n_G}) \tag{24b}$$

$$q := (c^+, c^-, q_1, 0, q_2, 0, \ldots, q_{n_G}, 0). \tag{24c}$$

In this case (20c) reads

$$N_i R_{id}^{-1} \sigma_i - (y^+ - y^-)\mathbf{1} - N_i N_i' \lambda_i = \mathbf{1} - \delta_i \leq \mathbf{1}$$

and therefore

$$N_i R_{id}^{-1} (\sigma_i - R_{id} N_i' \lambda_i) \leq (1 + y^+ - y^-)\mathbf{1}.$$

Dividing again by $1+y^+-y^->0$, we finally have that

$$N_i\Big((1+y^+-y^-)R_i\Big)_d^{-1}(\sigma_i - R_{id}N_i'\lambda_i) \le \mathbf{1}. \tag{25}$$

Of course, in case of (24a-c), a quite similar representation is obtained for (21c).

Obviously, (25) can be interpreted as the result of the change of the yield surface ∂K_i, the admissible domain K_i, resp., due to mixed isotropic and kinematic hardening [6, 8].

Now we have to define the cost coefficients c^+, c^- in (22) and $c^+, c^-, q_i, i = 1, \dots, n_G$, in (24c). The cost/return contribution of the terms

$$\Delta R_i^+ := y^+ R_i, \Delta R_i^- := y^- R_i, a_i := R_{id}N_i'\lambda_i \tag{26}$$

with the plastic multipliers [6, 8] $y^+ \ge 0, y^- \ge 0$ and $\lambda_i \ge 0, i = 1, \dots, n_G$, can be obtained as follows:

The vector $-\Delta R_i^- = -y^- R_i$ indicates strength reserves of the amount ΔR_i^- corresponding to elasticity reserves represented by the elastic strains [6, 8]

$$\Delta\epsilon_i^e := E_i^{-1}(-\Delta R_i^-) = -y^- E_i^{-1} R_i, \tag{27a}$$

where E_i denotes the elastic moduli matrix. Furthermore, isotropic/kinematic hardening [6, 8] is represented by the plastic strains

$$\Delta\epsilon_i^{p,\text{iso}} := E_{Ti}^{-1}\Delta R_i^+ = y^+ E_{Ti}^{-1} R_i, \tag{27b}$$

$$\Delta\epsilon_i^{p,\text{kin}} := E_{Ti}^{-1} a_i = E_{Ti}^{-1} R_{id}^{-1} R_{id} N_i'\lambda_i, \tag{27c}$$

where E_{Ti} is a symmetric, positive definite matrix depending on certain hardening moduli [6, 8]. Having the possible elastic/plastic strains (27a-c), the corresponding return/costs can be evaluated now as follows:

$$\Delta c_i^- := (E_i^{-1}R_i)'\Delta\epsilon_i^e = -y^-\|E_i^{-1}R_i\|^2, \tag{28a}$$

$$\Delta c_i^+ := (E_{Ti}^{-1}R_i)'\Delta\epsilon_i^{p,\text{iso}} = y^+\|E_{Ti}^{-1}R_i\|^2, \tag{28b}$$

$$\Delta c_i^{\text{kin}} := \lambda_i'\, tr\,(N_i R_{id} E_{Ti}^{-1})(N_i R_{id} E_{Ti}^{-1})', i = 1, \dots, n_G. \tag{28c}$$

Hence, with certain weight factors $\gamma_i^-, \gamma_i^+ > 0, i = 1, \dots, n_G$, the cost coefficients c^-, c^+ can be defined now as follows:

$$c^- := -\sum_{i=1}^{n_G} \gamma_i^- \|E_{Ti}^{-1}R_i\|^2, \tag{29a}$$

$$c^+ := \sum_{i=1}^{n_G} \gamma_i^+ \|E_{Ti}^{-1}R_i\|^2; \tag{29b}$$

moreover, the vector q_i of cost coefficients reads

$$q_i := \gamma_i \, tr \, (N_i R_{id} E_{Ti}^{-1})(N_i R_{id} E_{Ti}^{-1})', i = 1, \ldots, n_G, \tag{29c}$$

where $\gamma_i > 0, i = 1, \ldots, n_G$, are further weight factors.

Remark 2.2 *The linear program (19a-c), called "2nd stage LP" in stochastic linear programming, may be used also to describe the minimum costs for strengthening or retrofitting [22] the structure after knowing the material and loading parameters in the following two-stage decision situation: After an initial design of the structure the random parameters are realized, and the initial design (or predesign) may be improved or repaired by strengthening the layout, e.g. by inserting better materials or additional elements.*

An important property of the failure costs c^*_{LA}, c^*_{ShA}, with respect to the load factor $\mu \geq 0$ is shown next:

Lemma 2.3 *Let the failure costs c^*_{LA}, c^*_{ShA}, resp., be defined either by program (14a-c), (15a-c), using a convex function $\gamma = \gamma(s)$, or by the LP (20a-d), (21a-d), resp., based on the 2nd stage LP (19a-c). For given parameter vector $(R, P_0), (R, \sigma^{el})$, resp., the failure costs $c^*_{LA} = c^*_{LA}(\mu, R, P_0), c^*_{ShA} = c^*_{Sh}(\mu, R, \sigma^{el})$, resp., are convex with respect to the load factor μ.*

Proof. The assertion is shown for $c^*_{LA} = c^*_{LA}(\mu, R, P_0)$ defined by (20a-d), the other cases follow in the same way. For given $\mu_1, \mu_2 \geq 0$ and $\alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$, let $(v, \sigma), (w, \eta)$ be feasible points in (20a-d) for $\mu = \mu_1, \mu = \mu_2$, respectively. Then $(\alpha_1 v + \alpha_2 w, \alpha_1 \sigma + \alpha_2 \eta)$ is feasible in (20a-d) for $\mu = \alpha_1 \mu_1 + \alpha_2 \mu_2$. Hence,

$$c^*_{LA}(\alpha \mu_1 + \alpha_2 \mu_2, R, P_0) \leq \alpha_1 q' v + \alpha_2 q' w,$$

which yields the assertion.

3 LIMIT AND SHAKEDOWN ANALYSIS UNDER STOCHASTIC UNCERTAINTY

Due to random variations of the material and the loading, due to manufacturing and modelling errors and further random disturbances, the (reference) load vector(s) $P_0, P^{(j)}, j = 1, 2, \ldots, n_l$, resp., and the vector R of material strength or resistance parameters are not given, fixed quantities, but must be modelled $P_0 = P_0(\omega), P^{(j)} = P^{(j)}(\omega), j = 1, \ldots, n_l$, resp., $R = R(\omega)$ through random variables on a certain probability space $(\Omega, \mathcal{A}, P)$. Hence, the limit analysis as well as the shakedown analysis problem (3a-d), (5a-d), resp., under stochastic uncertainty must be replaced by appropriate deterministic substitute problems which are provided by using stochastic optimization techniques [1],[14]-[22]. Based on the numerical evaluation of a structural failure/survival described in Section 2, in the following we consider now the expected costs of failure defined by

$$\Gamma_{LA} = \Gamma_{LA}(\mu) := Ec^*_{LA}\Big(\mu, R(\omega), P_0(\omega)\Big), \ \mu \geq 0, \tag{30a}$$

in limit analysis and

$$\Gamma_{ShA} = \Gamma_{ShA}(\mu) := Ec^*_{ShA}\Big(\mu, R(\omega), \sigma^{el}(\omega)\Big), \ \mu \geq 0, \tag{30b}$$

for shakedown analysis. The random limit, shakedown analysis problem (3a-d), (5a-d), resp., is then replaced by the deterministic substitute problem

$$\max \mu \tag{31a}$$

s.t.

$$\Gamma_{LA}(\mu) \ \leq \ \Gamma^{\max}_{LA} \tag{31b}$$

$$\mu \ \geq \ 0 \tag{31c}$$

for limit analysis and

$$\max \mu \tag{32a}$$

s.t.

$$\Gamma_{ShA}(\mu) \ \leq \ \Gamma^{\max}_{ShA} \tag{32b}$$

$$\mu \ \geq \ 0 \tag{32c}$$

for shakedown analysis, where $\Gamma^{\max}_{LA}, \Gamma^{\max}_{ShA}$ denote the maximum (expected) loss.

A first important property of the above substitute problems is obtained from Lemma 2.3:

Corollary 3.1 *If the assumptions of Lemma 2.3 hold, then (31a-c), (32a-c), resp., is a convex deterministic substitute problem for the limit, shakedown analysis problem.*

As can be seen from the following examples, cf. Fig. 2.1, important types of substitute problems for (3a-d), (5a-d), resp., are covered by the programs (31a-c) and (32a-c), where we assume here that the failure costs are represented by program (14a-c), (15a-c), respectively.

Example 3.1 *If $\gamma(s) = 0$ for $s \le 0$ and $\gamma(s) = c_f$ for $s > 0$ with a given cost factor c_f, then*

$$\Gamma_{LA}(\mu) = c_f P\left(s^*_{LA}\Big(\mu, R(\omega), P_0(\omega)\Big) > 0\right) = c_f p_{f,LA}(\mu), \tag{33a}$$

where

$$p_{f,LA}(\mu) := P(\text{there is no stress vector (internal load vector) } \sigma \text{ fulfilling the safety condition (3b,c)}),\ \mu \ge 0, \tag{33b}$$

is the probability of failure function of the limit analysis problem (3a-d). Furthermore, we have that

$$\Gamma_{ShA}(\mu) = c_f P\left(s^*_{ShA}\Big(\mu, R(\omega), \sigma^{el}(\omega)\Big) > 0\right) = c_f p_{f,ShA}(\mu), \tag{33c}$$

where $p_{f,ShA} = p_{f,ShA}(\mu), \mu \ge 0$, is the probability of failure function related to the shakedown analysis problem (7a-d):

$$p_{f,ShA}(\mu) := P(\text{there is no } \rho \text{ fulfilling the safety condition (5b,c)}),\ \mu \ge 0. \tag{33d}$$

Consequently, if $\Gamma^{\max}_{LA} := c_f \alpha^{\max}_{LA}, \Gamma^{\max}_{ShA} := c_f \alpha^{\max}_{ShA}$, where $\alpha^{\max}_{LA}, \alpha^{\max}_{ShA}$ are maximum failure probabilities, then the following chance-constrained programming problems are obtained:

$$\max \mu \tag{34a}$$

s.t.

$$p_{f,LA}(\mu) \le \alpha^{\max}_{LA} \tag{34b}$$
$$\mu \ge 0, \tag{34c}$$

and

$$\max \mu \tag{35a}$$

s.t.

$$p_{f,ShA}(\mu) \le \alpha^{\max}_{ShA} \tag{35b}$$
$$\mu \ge 0. \tag{35c}$$

In general, the probability functions $p_{f,LA}(\mu), p_{f,ShA}(\mu), \mu \ge 0$, are not convex. However, using probability inequalities, the probability of failure functions can be approximated from above by certain convex expected cost functions $\Gamma_{LA}(\mu), \Gamma_{ShA}(\mu), \mu \ge 0$.

Example 3.2 *If $\gamma = \gamma(s)$ is a nonnegative, monotoneous nondecreasing loss function such that $\gamma(0) > 0$, then*

$$p_{f,LA}(\mu) \;\le\; \frac{1}{\gamma(0)}\Gamma_{LA}(\mu), \mu \ge 0, \tag{36a}$$

$$p_{f,ShA}(\mu) \;\le\; \frac{1}{\gamma(0)}\Gamma_{ShA}(\mu), \mu \ge 0, \tag{36b}$$

and, according to Corollary 3.1, the right hand side in (36a), (36b), resp., is convex in μ, provided that γ is convex.

If $\gamma(0) = 0$ as in case (16a-c), then we may proceed as follows:

Example 3.3 *Let $\gamma = \gamma(s)$ be any nonnegative, monotoneous nondecreasing loss function such that $\gamma(s_0) > 0$ for a certain positive threshold $s_0 > 0$. Defining the approximate failure probability functions*

$$p^{(s_0)}_{f,LA}(\mu) \;:=\; P\left(s^*_{LA}\Big(\mu, R(\omega), P_0(\omega)\Big) > s_0\right), \;\mu \ge 0, \tag{37a}$$

$$p^{(s_0)}_{f,ShA}(\mu) \;:=\; P\left(s^*_{ShA}\Big(\mu, R(\omega), \sigma^{el}(\omega)\Big) > s_0\right), \;\mu \ge 0, \tag{37b}$$

we have that $p^{(s_0)}_{f,LA} \le p_{f,LA}(\mu)$, $p^{(s_0)}_{f,ShA}(\mu) \le p_{f,ShA}(\mu), \mu \ge 0$, and

$$p^{(s_0)}_{f,LA}(\mu) \;\le\; \frac{1}{\gamma(s_0)}\Gamma_{LA}(\mu), \;\mu \ge 0, \tag{37c}$$

$$p^{(s_0)}_{f,ShA}(\mu) \;\le\; \frac{1}{\gamma(s_0)}\Gamma_{ShA}(\mu), \;\mu \ge 0. \tag{37d}$$

4 NUMERICAL SOLUTION OF PLASTIC STRUCTURAL ANALYSIS PROBLEMS

An important representation of the limit state functions s^*_{LA}, s^*_{ShA} and the (minimum) failure cost functions c^*_{LA}, c^*_{ShA}, see (9),(11), resp., and (13), is shown first.

Considering [14, 15] the Minkowski functional $\pi = \pi(z|\tilde{K}_i)$ of the convex polyhedron $\tilde{K}_i \subset K_i, i = 1, \ldots, n_G$, we observe first, cf. (2a,b), that

$$\begin{aligned}\pi(z|\tilde{K}_i) \;&=\; \inf\left\{\lambda > 0 : \frac{z}{\lambda} \in \tilde{K}_i\right\} \qquad (38a)\\ &=\; \inf\left\{\lambda > 0 : N_{ij}\frac{z}{\lambda} \le 1, j = 1, \ldots, m_y\right\} = \max_{1\le j\le m_y} N_{ij}z, z \in \mathbb{R}^{n_0},\end{aligned}$$

where $N_{ij}, j = 1, \ldots, m_y$, are the rows of the $m_y \times n_0$ matrix N_i. Since $\tilde{K}_i \subset K_i, \tilde{K}_i$ contains a ball with radius $\eta_0 > 0$, and K_i is contained in a ball with radius $\eta_1 > \eta_0$, we have that

$$\frac{1}{\eta_1}\|z\| \leq \pi(z|K_i) \leq \pi(z|\tilde{K}_i) \leq \frac{1}{\eta_0}\|z\|, z \in \mathbb{R}^{n_0}, \tag{38b}$$

where $\pi(z|K_i)$ denotes the Minkowski functional of K_i, see (1e), and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{n_0}$.

The LP (8a-c) defining the limit state function $s^*_{LA} = s^*_{LA}(\mu, R, P_0|\tilde{K}_i)$ can be represented now by

$$\min s \tag{39a}$$

s.t.

$$C\sigma = \mu P_0 \tag{39b}$$

$$\pi(R_{id}^{-1}\sigma_i|\tilde{K}_i) \leq 1 + s, i = 1, \ldots, n_G. \tag{39c}$$

Working directly with the exact yield domain K_i, the corresponding convex program for the safety margin $s^*_{LA} = s^*_{LA}(\mu, R, P_0|K_i)$ reads, cf. (1b,c,c'),

$$\min s \tag{39a'}$$

s.t.

$$C\sigma = \mu P_0 \tag{39b'}$$

$$\pi(R_{id}^{-1}\sigma_i|K_i) \leq 1 + s, i = 1, \ldots, n_G. \tag{39c'}$$

In the same way, the limit state function $s^*_{LA} = s^*_{LA}(\mu, R, \sigma^{el})$ in shakedown analysis can be represented, cf. (5a-d), (1c'), by

$$\min s \tag{40a}$$

s.t.

$$C\rho = 0 \tag{40b}$$

$$\pi\Big(R_{id}^{-1}\,\Big(\mu\sigma_i^{el(j)} + \rho_i\Big)\Big|\,\tilde{K}_i\Big) \leq 1 + s, j = 1, \ldots, n_l, i = 1, \ldots, n_G, \tag{40c}$$

working with the linearized yield domain $\tilde{K}_i$, and

$$\pi\Big(R_{id}^{-1}\,\Big(\mu\sigma_i^{el(j)} + \rho_i\Big)\Big|\,K_i\Big) \leq 1 + s, j = 1, \ldots, n_l, i = 1, \ldots, n_G, \tag{40c'}$$

using the exact yield domains $K_i, i = 1, \ldots, n_G$.

Remark 4.1 *For simplification, in the following $\pi = \pi(z_i|K_i^y)$ denotes the Minkowski functional of the exact as well as the linearized yield domain $K_i, \tilde{K}_i$, respectively.*

Introducing now the positively homogeneous, subadditive functional

$$\pi(z|K) := \max_{1 \le i \le n_G} \pi(z_i|K_i^y),\ z = (z_i) \in (\mathbb{R}^{n_0})^{n_G}, \tag{41a}$$

we first observe that $\pi(z) = \pi(z|K)$ is the Minkowski functional of the product yield domain

$$K = \bigotimes_{i=1}^{n_G} K_i^y \tag{41b}$$

in $\mathbb{R}^{n_0 n_G} = (\mathbb{R}^{n_0})^{n_G}$. Consequently, conditions (39c') and (40c') can be represented by

$$\pi(R_d^{-1}\sigma|K) \le 1 + s \tag{41c}$$

$$\pi\Big(R_d^{-1}\Big(\mu\sigma^{el(j)} + \rho\Big)\Big|\,K\Big) \le 1 + s,\ j = 1, \ldots, n_l, \tag{41d}$$

where $R^T := (R_1^T, R_2^T, \ldots, R_{n_G}^T)$, and R_d denotes again the diagonal matrix containing the elements of R on its main diagonal.

Since $\pi(z) \ge 0$ and $\pi(z) = 0$ if and only if $z = 0$, cf. (38b), for each feasible point $(s, \sigma), (s, \rho)$ in (39a'-c'), (40a,40b,40c'), resp., we have that $s \ge -1$ and $s = -1$ if and only if $\sigma = 0, \mu\sigma^{el(j)} + \rho = 0, j = 1, \ldots, n_l$, respectively.

However, (39b') yields then $\mu P_0 = 0$ and therefore $\mu = 0$, since $P_0 \ne 0$. In the other case, from $C\sigma^{el(j)} = P^{(j)}, j = 1, \ldots, n_l$, and (40b) we obtain $\mu P^{(j)} = 0, j = 1, \ldots, n_l$, hence also $\mu = 0$, since $P^{(j)} \ne 0$ for at least one $j, 1 \le j \le n_l$. Conversely, if $\mu = 0$, then $(s, \sigma) = (-1, 0), (s, \rho) = (-1, 0)$, is feasible in (39a'-c'), (40a,40b,40c'), resp., which shows that $s_{LA}^* = -1, s_{ShA}^* = -1$ for $\mu = 0$. Summarizing the above considerations, we find the next lemma, where we still introduce the following notation:

$$s^*(\mu, R, S) := \begin{cases} s_{LA}^*(\mu, P, P_0) & \text{for } S = P_0 \\ s_{ShA}^*(\mu, P, \sigma^{el}) & \text{for } S = \sigma^{el}. \end{cases} \tag{41e}$$

Lemma 4.1 *Let $P_0 \ne 0, P^{(j)} \ne 0$ for at least $1 \le j \le n_l$, respectively. Then $s^*(\mu, R, S) > -1$ for $\mu > 0$ and $s^*(\mu, R, S) = -1$ for $\mu = 0$.*

For given $\mu > 0$ we consider now the transformations

$$\hat{\sigma} := \frac{\sigma}{\mu}, \hat{s} := \frac{1+s}{\mu} \tag{42a}$$

for limit analysis, and

$$\hat{\rho} := \frac{\rho}{\mu}, \hat{s} := \frac{1+s}{\mu} \tag{42b}$$

for shakedown analysis.

Applying (42a), problem (39a'-c') in limit analysis can be represented, cf. (41c), by

$$\min \mu\hat{s} - 1 \tag{43a}$$

s.t.

$$C\hat{\sigma} = P_0 \tag{43b}$$

$$\pi(R_d^{-1}\hat{\sigma}|K) \leq \hat{s}. \tag{43c}$$

Moreover, using (42b) and (41b), problem (40a,40b,40c') in shakedown analysis reads

$$\min \mu\hat{s} - 1 \tag{44a}$$

s.t.

$$C\hat{\rho} = 0 \tag{44b}$$

$$\pi\left(R_d^{-1}\left(\sigma^{el(j)} + \hat{\rho}\right)\Big| K\right) \leq \hat{s}, j = 1, \ldots, n_l. \tag{44c}$$

Consequently, the following representation of $s^* = s^*_{LA}, s^* = s^*_{ShA}$, resp. holds:

Theorem 4.1 *Let $P_0 \neq 0, P^{(j)} \neq 0$ for at least one $1 \leq j \leq n_l$. Then*

$$s^*(\mu, R, S) = \mu\hat{s}^*(R, S) - 1, \mu \geq 0, \tag{45a}$$

where for $\hat{s}^* = \hat{s}^*_{LA}, \hat{s}^* = \hat{s}^*_{ShA}$, resp., we have that

$$\hat{s}^*_{LA}(R, P_0) := \min\left\{\hat{s} : C\hat{\sigma} = P_0, \pi(R_d^{-1}\hat{\sigma}|K) \leq \hat{s}\right\}, \tag{45b}$$

$$\hat{s}^*_{ShA}(R, \sigma^d) := \min\left\{\hat{s} : C\hat{\rho} = 0, \pi\left(R_d^{-1}\left(\sigma^{el(j)} + \hat{\rho}\right)\Big| K\right) \leq \hat{s}, 1 \leq j \leq n_l\right\}. \tag{45c}$$

and $\hat{s}^*_{LA}(R, P_0) > 0, \hat{s}^*_{Sh}(R, \sigma^{el}) > 0$.

Some immediate consequences of the above theorem are shown next:

Corollary 4.1 *For nonzero loadings, the limit state functions s^*_{LA} and s^*_{ShA} are affine-linear, strictly monotoneous increasing functions of the load factor μ.*

Having the above Theorem 4.1 and Corollary 4.1, according to safety condition (12a,b) in Lemma 2.1 the maximum load factor μ^* can be represented - for given parameters R and S - as follows:

Theorem 4.2 *For given vectors R, S, resp. of material strength parameters and external loadings, the maximum load factor $\mu^* = \mu^*(R,S)$ is given by*

$$\mu^*(R,S) := \frac{1}{\hat{s}^*(R,S)}, \tag{46}$$

where $\hat{s}^* = \hat{s}^*(R,S)$ is defined by (45b,c).

Since $\pi = \pi(z|K)$ is a positively homogeneous, subadditive function, hence, a semi-norm on $\mathbb{R}^{n_0 n_G}$, the minimization problem

$$\min \pi(R_d^{-1}\hat{\sigma}|K) \tag{47a}$$

s.t.

$$C\hat{\sigma} = P_0, \tag{47b}$$

defining the failure index $\hat{s}^*_{LA}(R, P_0)$, can be interpreted as the **"weighted projection"** of the origin of $\mathbb{R}^{n_0 n_G}$ onto the linear manifold defined by (47b) in $\mathbb{R}^{n_0 n_G}$. A quite similar interpretation can be given also in the second case.

Approximating the semi-norm $\pi = \pi(z|K)$ by

$$\pi(z|K) \approx \sqrt{z'Qz}, \tag{48a}$$

where Q is a positive definite symmetric matrix, we find ($S = P_0$) that

$$\hat{s}^*_{LA}(R, P_0) \approx \sqrt{P_0'(CR_dQ^{-1}R_dC')^{-1}P_0}. \tag{48b}$$

Finally, the following consideration shows the meaning of the limit state functions s^*_{LA}, s^*_{ShA} for the cost-evaluation of the violation/fulfillment of the safety condition: Dividing for a given $\mu > 0$ the yield condition (41c), (41d), resp., by $1 + s > 0$, we get the following inequalities

$$\pi\left(\left(R + sR\right)_d^{-1}\sigma \Big| K\right) \leq 1, \tag{49a}$$

$$\pi\left(\left(R + sR\right)_d^{-1}\left(\mu\sigma^{el(j)} + \rho\right)\Big| K\right) \leq 1, j = 1, \ldots, n_l. \tag{49b}$$

However, considering the vector

$$\Delta R := s^*R,\ s^* = s^*(\mu, R, S), \tag{49c}$$

we find that

i) **Case $\mathbf{s^* > 0}$**

s^* is the minimum rate of increase of the present material resistance (plastic capacity) R maintaining again a structure carrying safely the load μP_0, shaking down under the variable load given by (4a,b), respectively;

ii) **Case $\mathbf{s^* < 0}$**

$|s^*|$ is the maximum rate of reserve within the present material resistance (plastic capacity) of the structure for carrying safely the given load μP_0, shaking down under the variable loading (4a,b), respectively.

Hence, this is a direct justification of the cost approach (13).

4.1 Cost model $c^* = \gamma(s^*)$

Suppose now that the failure costs $c^* = c^*_{LA}, c^* = c^*_{ShA}$, resp., are given by (13). Hence, according to (45a,b) and by Theorem 4.1 the expected costs of failure function $\Gamma = \Gamma_{LA}, \Gamma = \Gamma_{ShA}$ can be represented in both cases by

$$\begin{aligned} \Gamma(\mu) &= E\gamma\left(s^*\left(\mu, R(\omega), S(\omega)\right)\right) \\ &= E\gamma\left(\mu\hat{s}^*\left(R(\omega), S(\omega)\right) - 1\right), \end{aligned} \tag{50}$$

cf. (41e). Corollary 4.1 yields then, cf. Fig. 4.1, the following important property of the expected failure cost function Γ:

Corollary 4.2 *Suppose that $\gamma = \gamma(s)$ is a convex, monotoneous nondecreasing loss function. Then $\Gamma = \Gamma(\mu)$ is a convex, monotoneous nondecreasing function on $[0, +\infty)$. If $\gamma = \gamma(s)$ is strictly monotoneous increasing, then $\Gamma = \Gamma(\mu)$ has the same property, provided that the reference loads are nonzero with positive probability.*

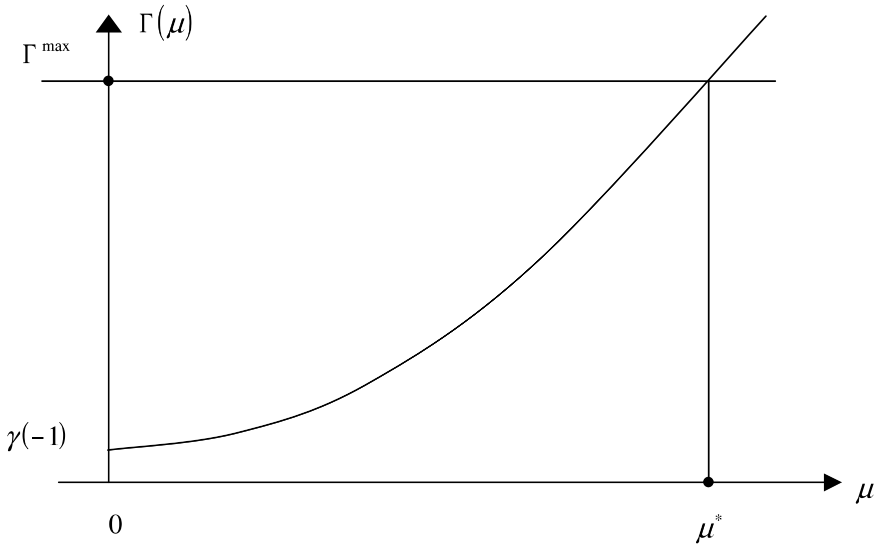

Fig. 4.1. Expected costs of failure function

Based on the above considerations, the deterministic substitute problems (31a-c), (32a-c) for the limit load, shakedown analysis problem (3a-d), (5a-d), resp., under stochastic uncertainty can be represented jointly by

$$\max \mu \tag{51a}$$

s.t.

$$\Gamma(\mu) \leq \Gamma^{\max} \tag{51b}$$

$$\mu \geq 0, \tag{51c}$$

where $\Gamma = \Gamma(\mu)$ is given by (50) and $\Gamma^{\max} = \Gamma^{\max}_{LA}, \Gamma^{\max} = \Gamma^{\max}_{ShA}$, respectively.

Because of Corollary 4.2, a maximal load factor $\mu^* > 0$ is determined, cf. Fig. 4.1, also by the equation

$$\Gamma(\mu) = \Gamma^{\max}. \tag{52}$$

Example 4.1 *If $\gamma(s) := (s+1)^2 - 1$, then (50) yields*

$$\Gamma(\mu) = \mu^2 E\hat{s}^*\Big(R(\omega), S(\omega)\Big)^2 - 1. \tag{53a}$$

Consequently, the solution μ^* of (52) reads

$$\mu^* = \left(\frac{1+\Gamma^{\max}}{E\hat{s}^*\Big(R(\omega), S(\omega)\Big)^2}\right)^{1/2}. \tag{53b}$$

From (53b) and $E\hat{s}^{*2} = (E\hat{s}^*)^2 + \sigma^2_{\hat{s}^*}$ *it can be seen that increasing uncertainty of the random variable* $\hat{s}^* = \hat{s}^*\Big(R(\omega), S(\omega)\Big)$ *decreases the robust maximum load factor* μ^**, cf. (46). Using now the approximation (48b), for* $\hat{s}^* = \hat{s}^*_{LA}(R, P_0)$ *we approximatively find*

$$E\hat{s}^*_{LA}\Big(R(\omega), P_0(\omega)\Big) \approx EP_0(\omega)'\Big(CR(\omega)_d Q^{-1} R(\omega)_d C'\Big)^{-1} P_0(\omega). \tag{53c}$$

Assuming that the vector $P_0(\omega)$ *of reference loadings and the vector* $R(\omega)$ *of material strength parameters are stochastically independent, then an upper bound of the right hand side of (53c) is given by*

$$E\|P_0(\omega)\|^2 \;\; E\left\|\Big(CR(\omega)_d Q^{-1} R(\omega)_d C'\Big)^{-1}\right\| \le \frac{1}{\rho} E\|P_0(\omega)\|^2 E\frac{1}{\|R(\omega)\|^2}$$

with a certain positive constant ρ*. Consequently, an approximative lower bound for* μ^*_{LA} *is given by*

$$\mu^*_{LA} \ge_{(\text{appr})} \sqrt{\rho(1+\Gamma^{\max})}\frac{1}{\sqrt{E\Big\|P_0(\omega)\Big\|^2}}\frac{1}{\sqrt{E\frac{1}{\Big\|R(\omega)\Big\|^2}}}. \tag{53d}$$

4.2 Computation of robust maximum load factors

Suppose now that $\gamma = \gamma(s)$ is a convex, (strictly) monotoneous increasing function.

Risk constraints

In case of the risk constrained problems (31a-c), (32a-c), a robust or reliable maximum load factor μ^* can be obtained by one of the following search routines $\mathcal{A}_1, \mathcal{A}_2$, see Fig. 4.2:

$\mathcal{A}1$) Interval bisection starting with an interval $[\mu_a, \mu_b]$ containing μ^*;

$\mathcal{A}2$) Descent/ascent method of the type

$$\mu_{n+1} := \mu_n - \delta_n sgn(\Gamma^*_n - \Gamma^{\max}), n = 1, 2, \ldots, \tag{54a}$$

where

$$\Gamma^*_n := \Gamma(\mu_n) = \min\left\{\Gamma(\mu) : \mu \ge \mu_n\right\}, \tag{54b}$$

and $\delta_n > 0$ is a step size.

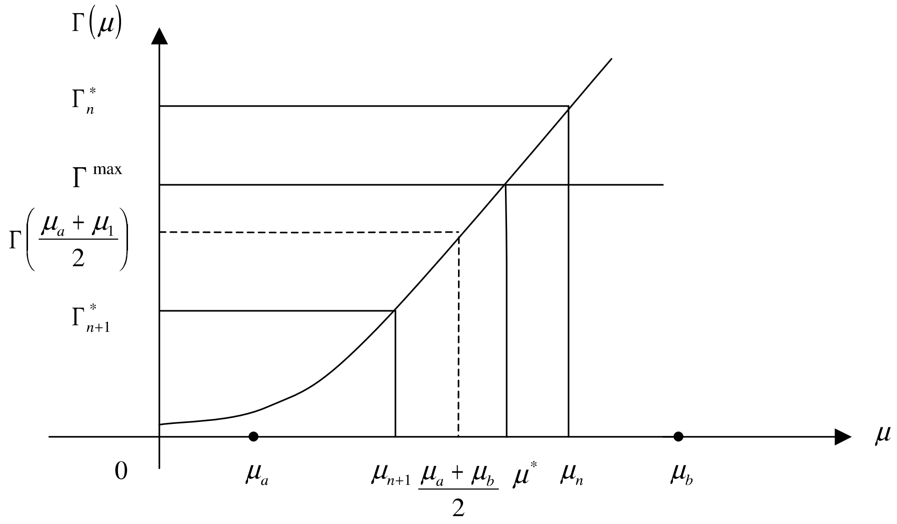

Fig. 4.2. Computation of robust maximum load factors μ

The main problem in the search methods described above shortly is the computation of the expected (minimum) failure costs

$$\Gamma_n^* = \Gamma(\mu_n) = E\gamma\left(\mu_n \hat{s}^*\Big(R(\omega), S(\omega)\Big) - 1\right). \tag{54c}$$

Consider now a sublinear cost function of the type (16a-c). According to (17a-d), in limit load analysis the function value $\Gamma_n^* = \Gamma(\mu_n)$ can be obtained by solving the following stochastic optimization problem

$$\min E\Big(c^+ y^+(\omega) + c^- y^-(\omega)\Big) \tag{55a}$$

s.t.

$$C\sigma(\omega) = \mu_n P_0(\omega) \text{ a.s.} \tag{55b}$$

$$N_i R_{id}^{-1}(\omega)\sigma_i(\omega) - \Big(y^+(\omega) - y^-(\omega)\Big)\mathbf{1} \le \mathbf{1} \text{ a.s.}, i = 1, \ldots, n_G \tag{55c}$$

$$y^+(\omega), y^-(\omega) \ge 0 \text{ a.s.}, \tag{55d}$$

where "a.s." means "almost sure". However, (55a-d) is a stochastic optimization problem having a well known structure:

Theorem 4.3 *The stochastic optimization problem (55a-d) representing the expected failure costs $\Gamma(\mu_n)$ for limit load analysis problems can be represented by a* **stochastic linear program (SLP) with complete fixed recourse**.

In case of a discrete probability parameter distribution (55a-d) may be described by a (large scale) linear program (LP) having a **dual decomposition data structure**. *A corresponding representation holds also for shakedown analysis problems.*

Proof. Using representation (20a-d), (21a-d), resp., of the failure costs $c^* = \gamma(s^*)$, the assertion follows e.g. from [9, 10, 23].

Remark 4.2 *For the class of SLP with complete fixed recourse and for LP with a dual decomposition data structure a large amount of theoretical results, approximation techniques and numerical solution procedures is available, see e.g. [10, 23].*

Minimization of the expected total loss

Measuring the expected costs for the lack of carrying capacity by $-c\mu, \mu \geq 0$, where $c > 0$ is a positive coefficient, for the minimization of the expected total costs we obtain then the following convex deterministic substitute problem

$$\min -c\mu + \Gamma(\mu) \tag{56a}$$

s.t.

$$\mu \geq 0. \tag{56b}$$

Note that (56a,b) results by applying a Lagrange function approach to (31a-c), (32a-c), respectively.

If $\Gamma = \Gamma(\mu)$ is a differentiable, convex expected cost function, then a necessary and sufficient condition for a postitive optimal solution μ_c^* of (56a,b) is the expected cost rate condition $\Gamma'(\mu) = c$, see Fig. 4.3.

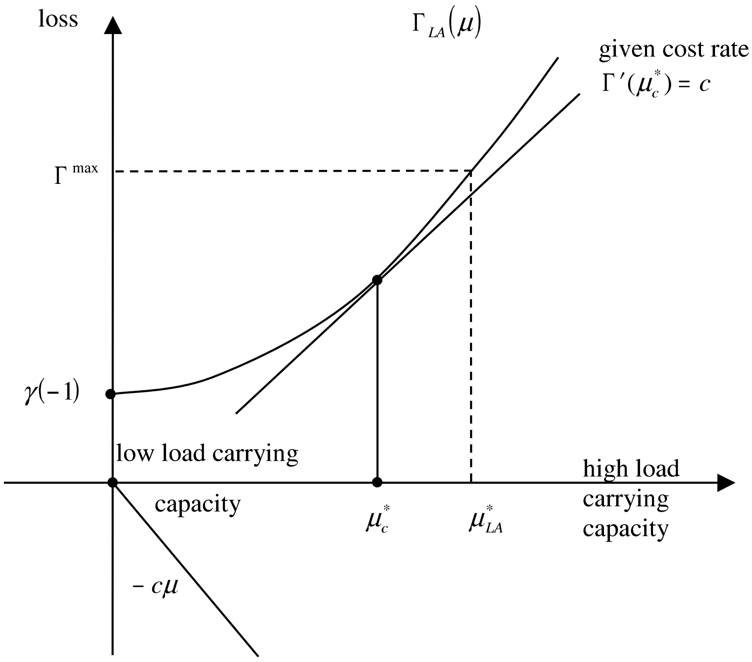

Fig. 4.3. Optimal solution of (54a,b)

Corresponding to Theorem 4.3, for (56a,b) we have, cf. (20a-d), (21a-d), resp., the following SLP-representation:

Theorem 4.4 *In case of limit load analysis problem (56a,b) can be represented by the following* **SLP with complete fixed recourse***:*

$$\min E\Big(-c\mu + c^+y^+(\omega) + c^-y^-(\omega)\Big) \tag{57a}$$

s.t.

$$C\sigma(\omega) - \mu P_0(\omega) = 0 \ a.s. \tag{57b}$$

$$N_i R_{id}^{-1}(\omega)\sigma_i(\omega) - \Big(y^+(\omega) - y^-(\omega)\Big)\mathbf{1} \le \mathbf{1} \ a.s.,\ i = 1, \dots, n_G \tag{57c}$$

$$\mu \ge 0, y^+(\omega), y^-(\omega) \ge 0 \ a.s., \tag{57d}$$

and a corresponding representation holds for shakedown analysis problems.

REFERENCES

1. Augusti, G., Baratta, A. and Casciati, F. (1984). Probabilistic Methods in Structural Engineering. London/New York: Chapman and Hall.
2. Charnes, A., Lemke, C.E. and Zienkiewicz, O.C. (1959). Virtual work, linear programming and plastic limit analysis. *Proceedings of the Royal Soc. A* 25: 110-116.
3. Cocchetti, G. and Maier, G. (1988). Static shakedown theorems in piecewise linearized poroplasticity. *Archive of Applied Mechanics* 68: 651-661.
4. Corradi, L. and Zavelani, A. (1974). A Linear Programming Approach to Shakedown Analysis of Structures. *Computer Methods in Applied Mechanics and Engineering* 3: 37-53.
5. Damkilde, L. and Kreuk, S. (1997). Limits - A System for Limit State Analysis and Optimal Material Layout. *Computers and Structures* 64 (1-4): 709-718.
6. Han, H. and Reddy, B.D. (1999). Plasticity - Mathematical Theory and Numerical Analysis. New York [etc.]: Springer-Verlag.
7. Heitzer, M. (1999). Traglast- und Einspielanalyse zur Bewertung der Sicherheit passiver Komponenten. Dissertation RWTH Aachen.
8. Kaliszky, S. (1984). Plastizitätslehre. Düsseldorf: VDI-Verlag.
9. Kall, P. (1976). Stochastic Linear Programming. Berlin/Heidelberg/New York: Springer-Verlag.
10. Kall, P. and Wallace, S.W. (1994). Stochastic Programming. Chichester: J. Wiley.
11. Kirsch, U. (1993). Structural Optimization. Berlin/Heidelberg/New York: Springer-Verlag.
12. König, J.A. (1987). Shakedown of Elastic-Plastic Structures. Amsterdam, etc.: Elsevier.
13. Lawo, M. (1987). Optimierung im kunstruktiven Ingenieurbau. Braunschweig/Wiesbaden: Vieweg.
14. Marti, K. (1972). Entscheidungsprobleme mit linearem Aktionen- und Ergebnisraum. *Z. Wahrscheinlichkeitstheorie verw. Geb.* 23: 133-147.
15. Marti, K. (1975). Approximationen der Entscheidungsprobleme mit linearer Ergebnisfunktion und positiv homogener, subadditiver Verlustfunktion. *Z. Wahrscheinlichkeitstheorie verw. Geb.* 31: 203-233.
16. Marti, K. (1979). Diskretisierung stochastischer Programme unter Berücksichtigung der Problemstruktur. *Zeitschrift für Angewandte Mathematik und Mechanik* 59: T105-T108.
17. Marti, K. (1979). Approximationen stochastischer Optimierungsprobleme. Königstein/Ts.: Verlag Anton Hain Meisenheim GmbH.
18. Marti, K. (1985). Computation of Descent Directions in Stochastic Optimization Problems with Invariant Distributions. *Zeitschrift für Angewandte Mathematik und Mechanik* 65: 355-378.
19. Marti, K. (1988). Descent Directions and Efficient Solutions in Disretely Distributed Stochastic Programs. *Lecture Notes in Economics and Mathematical Systems* Vol. 299.Berlin/Heidelberg/New York: Springer-Verlag.
20. Marti, K. (1992). Computation of Efficient Solutions of Discretely Distributed Stochastic Optimization Problems. *Mathematical Methods of Operations Research* 36: 259-294.
21. Marti, K. (1997). Approximation and derivatives of probabilities of survival in structural analysis and design. *Structural Optimization* 13: 230-243.
22. Marti, K. (1998). Optimal Design of Trusses as a Stochastic Linear Programming Problem. In Nowak, A.S., ed., *Reliability and Optimization of Structural*

Systems. Ann Arbor, Michigan: University of Michigan Press. 231–239.
23. Mayer, J. (1998). Stochastic Linear Programming Algorithms. Gordon and Breach.
24. Nielsen, M.P. (1984). Limit Anlaysis and Concrete Plasticity. New Jersey: Prentice Hall, Englewood Cliffs.
25. Prager, W. and Shield, R.T. (1967). A General Theory of Optimal Plastic Design. *Journal Applied Mechanics* 34 (1): 184-186.
26. Smith, D.L. (ed.) (1990). Mathematical Programming Methods in Structural Plasticity. *CISM Courses and Lectures* 299. Wien/New York: Springer-Verlag.
27. Smith, D.L. and Munro, J. (1976). Plastic Analysis and Synthesis of Frames subject to Multiple Loadings. *Engineering Optimization* 2: 145-157.
28. Tin–Loi, F. (1990). On the optimal plastic synthesis of frames. *Engineering Optimization* 16: 91–108.
29. Tin–Loi, F. (1995). Plastic limit analysis of plane frames and grids using GAMS. *Computers & Structures* 54: 15-25.