Sets of joint probability measures generated by random sets

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Given: Two uncertain parameters a_1 and a_2 with values $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

The uncertainty is modelled by random sets $(\mathcal{F}_k, m_k)$ which are interpreted as sets of probability measures

$$\mathcal{M}_k = \{P = \sum_i m(F_k^i) P_k^i\}.$$

Goal: Computation of $\overline{P}\{y \geq y_f\}$ for a function $y = g(a_1, a_2)$.

$$\overline{P}(A) = \sup\{P(A) : \ P \in \mathcal{M}\}$$

with $A = g^{-1}([y_f, \infty))$.

$\mathcal{M}$ is the set of joint probability measures generated by $\mathcal{M}_1$ and $\mathcal{M}_2$. $\mathcal{M}$ is not uniquely determined.

Notion of independence for sets of probability measures

What does independence mean?

If we learn the value ω_1 of the uncertain parameter a_1, our knowledge about the uncertainty of parameter a_2 does not change.

Types of independence for sets of probability measures:

- Unknown interaction
- Epistemic independence
- Strong independence
  (Couso, Moral, Walley 1999)
- Random set independence
- Fuzzy set independence

Single probability measures P_1 and P_2 given for **stochastically independent** parameters a_1 and a_2

Joint probability measure $P = P_1 \otimes P_2$.

The conditional probabilities satisfy

$$P(\Omega_1 \times A_2 \mid \{\omega_1\} \times \Omega_2) = P_2(A_2)$$
$$P(A_1 \times \Omega_2 \mid \Omega_1 \times \{\omega_2\}) = P_1(A_1)$$

for all $(\omega_1, \omega_2) \in \Omega$.
The conditional probabilities are equal to the marginal probabilities.

Unknown interaction

$\mathcal{M}_{\mathrm{U}}$ is the set of all probability measures P which satisfy

$$P(\cdot \times \Omega_2) \in \mathcal{M}_1 \quad \text{and} \quad P(\Omega_1 \times \cdot) \in \mathcal{M}_2. \tag{U}$$

- Every $P_1 \in \mathcal{M}_1$ can be combined with every $P_2 \in \mathcal{M}_2$.
- All possible joint probability measures P are allowed.

$\mathcal{M}_{\mathrm{U}}$ is an appropriate choice if it is not known

- which probability measures can be combined
- if the uncertain parameters are independent or if they are correlated in some way.

Epistemic independence

$\mathcal{M}_{\mathrm{E}}$ is the set of all probability measures P which satisfy

$$P(\cdot \times \Omega_2) \in \mathcal{M}_1 \quad \text{and} \quad P(\Omega_1 \times \cdot) \in \mathcal{M}_2. \tag{U}$$

and for which the conditional probabilities satisfy

$$\begin{aligned} P(\cdot \times \Omega_2 \mid \Omega_1 \times \{\omega_2\}) &\in \mathcal{M}_1, \\ P(\Omega_1 \times \cdot \mid \{\omega_1\} \times \Omega_2) &\in \mathcal{M}_2 \end{aligned} \tag{E'}$$

for all $(\omega_1, \omega_2) \in \Omega$.

Epistemic independence

Conditions (E') can be rewritten in the following way:

P with marginal probability measures P_1 and P_2 must satisfy for all $(\omega_1, \omega_2) \in \Omega$

$$P(A_1 \times \{\omega_2\}) = P_1^{|\omega_2}(A_1) P_2(\{\omega_2\}) \qquad \text{(E'')}$$
$$P(\{\omega_1\} \times A_2) = P_1(\{\omega_1\}) P_2^{|\omega_1}(A_2)$$

with conditional probability measures $P_1^{|\omega_2} \in \mathcal{M}_1$ and $P_2^{|\omega_1} \in \mathcal{M}_2$.

For $\mathcal{M}_i = \{P_i\}$, $i = 1, 2$, the condition (E'') leads to stochastic independence, because $P_1^{|\omega_2} = P_1$ and $P_2^{|\omega_1} = P_2$ for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$.

$\Longrightarrow$ Epistemic independence is a sort of generalization of the notion of independence to sets of probability measures.

Epistemic independence

If we have learned the value ω_1 of a_1, then the probability measure for the value ω_2 of parameter a_2 is again one of the probability measures in $\mathcal{M}_2$, but in general not always the same for different ω_1.

And vice versa.

Strong independence

$\mathcal{M}_{\mathrm{S}} = \{P = P_1 \otimes P_2 : \; P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2\}$

is the set of all product measures which can be generated by probability measures in $\mathcal{M}_1$ and $\mathcal{M}_2$.

This is a stronger generalization of stochastic independence.

We also get the set $\mathcal{M}_{\mathrm{S}}$, if we add the following conditions

$$\begin{aligned} \forall \omega_2 \in \Omega_2 : \quad P_1^{|\omega_2} &= P_1, \\ \forall \omega_1 \in \Omega_1 : \quad P_2^{|\omega_1} &= P_2 \end{aligned} \tag{S}$$

to the conditions for epistemic independence.

Strong independence

If we have learned the value ω_1 of a_1, then the probability measure for the value ω_2 of parameter a_2 is again one of the probability measures in $\mathcal{M}_2$.
But it is always the same probability measure for different ω_1.

And vice versa.

Strong independence

Strong independence is an appropriate choice, if the following assumptions are satisfied:

1. The values of parameter a_1 and a_2 are random, each described by a unique but unknown probability distribution.

2. These probability distributions belong to the sets $\mathcal{M}_1$, $\mathcal{M}_2$, respectively.

3. The parameters a_1 and a_2 are stochstically independent.

4. We do not know which $P_1 \in \mathcal{M}_1$ is allowed to be combined with $P_2 \in \mathcal{M}_2$. So we take all combinations.

Relations between the types of indpendence

- $\mathcal{M}_{\mathrm{U}}$ is the set of P subject to conditions (U).
- $\mathcal{M}_{\mathrm{E}}$ is the set of P subject to conditions (U)+(E").
- $\mathcal{M}_{\mathrm{S}}$ is the set of P subject to conditions (U)+(E")+(S).

The conditions are successively added.

$$\Longrightarrow \qquad \mathcal{M}_{\mathrm{U}} \supseteq \mathcal{M}_{\mathrm{E}} \supseteq \mathcal{M}_{\mathrm{S}},$$

$$\Longrightarrow \qquad \underline{P}_{\mathrm{U}} \leq \underline{P}_{\mathrm{E}} \leq \underline{P}_{\mathrm{S}} \leq \overline{P}_{\mathrm{S}} \leq \overline{P}_{\mathrm{E}} \leq \overline{P}_{\mathrm{U}}.$$

Set $\mathcal{M}_k$ of probability measures generated by random sets $(\mathcal{F}_k, m_k)$ for parameter a_k, $k = 1, 2$

Focal sets: $\mathcal{F}_k = \{F_k^1, F_k^2, \ldots, F_k^{n_k}\}$.

Set of probability measures on F_k^i: $\mathcal{M}_k^i = \{P : \ P(F_k^i) = 1\}$.

Set $\mathcal{M}_k$: $\mathcal{M}_k = \{P : \ P = \sum_i m_k(F_k^i) P_k^i, \ P_k^i \in \mathcal{M}_k^i\}$.

Upper probability: $\overline{P}_k(A_k) = \sup\{P(A_k) : \ P \in \mathcal{M}_k\}$.

Here plausibility is equal to upper probability:

$$\mathrm{Pl}_k(A_k) = \sum_{i:\ A_k \cap F_k^i \neq \varnothing} m_k(F_k^i) = \sum_i m_k(F_k^i) \delta_{\omega_k^i}(A_k) = \overline{P}_k(A_k).$$

$\delta_{\omega_k^i} \in \mathcal{M}_k^i$ Dirac measures in $\omega_k^i \in A_k \cap F_k^i$, if $A_k \cap F_k^i \neq \varnothing$.

Set $\mathcal{M}$ of joint probability measures generated by $\mathcal{M}_1$ and $\mathcal{M}_2$:

We write a joint probability measure P as

$$P = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) P^{ij}$$

where $F_1^i \times F_2^j$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$, are the joint focal sets and where P^{ij} are probability measures on $F_1^i \times F_2^j$.

We have to choice

- the joint weights $m(F_1^i \times F_2^j)$
- the probability measures P^{ij} and
- how the P^{ij} interact.

The choice of $m(F_1^i \times F_2^j)$

Case (U$--$):

Unknown interaction. m must satisfy condition (U).
The marginals of m must be m_1 and m_2.

Case (S$--$):

Stochastically independent choice of the focals:

$$m(F_1^i \times F_2^j) = m_1(F_1^i)m_2(F_2^j).$$

Case (F$--$):

m_1 and m_2 are correlated in a way which leads to the joint possibility measure.

The choice of $\mathcal{M}^{ij}$

Case $(-$U$-)$:

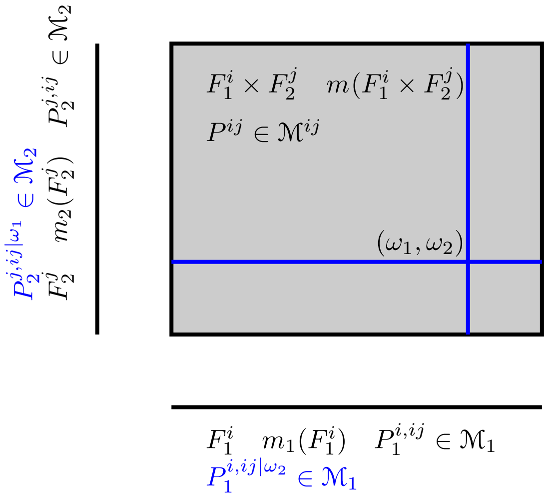
$\mathcal{M}^{ij} := \mathcal{M}_{\mathrm{U}}(\mathcal{M}_1^i, \mathcal{M}_2^j)$ which the set of all joint probability measures generated by the sets $\mathcal{M}(F_1^i)$ and $\mathcal{M}(F_2^j)$ according to conditions U.

Case $(-$E$-)$:

$\mathcal{M}^{ij} := \mathcal{M}_{\mathrm{E}}(\mathcal{M}_1^i, \mathcal{M}_2^j)$ which is the set generated according to conditions U+E".

Case $(-$S$-)$:

$\mathcal{M}^{ij} := \mathcal{M}_{\mathrm{S}}(\mathcal{M}_1^i, \mathcal{M}_2^j)$ which is the set generated according to conditions U+E"+S.



$F_1^i \times F_2^j \quad m(F_1^i \times F_2^j)$
$P^{ij} \in \mathcal{M}^{ij}$
(ω_1, ω_2)
$F_2^j \quad m_2(F_2^j) \quad P_2^{j,ij} \in \mathcal{M}_2$
$P_2^{j,ij|\omega_1} \in \mathcal{M}_2$
$F_1^i \quad m_1(F_1^i) \quad P_1^{i,ij} \in \mathcal{M}_1$
$P_1^{i,ij|\omega_2} \in \mathcal{M}_1$

Interactions between the P^{ij}

Case ($--$1):

$$\begin{aligned} P_1^i &:= P_1^{i,1j} &= \cdots = P_i^{i,n_1 j}, \\ P_2^j &:= P_2^{j,i1} &= \cdots = P_i^{j,in_2}, \\ P_1^{i|\omega_2} &:= P_1^{i,1j|\omega_2} &= \cdots = P_i^{i,n_1 j|\omega_2}, \\ P_2^{j|\omega_1} &:= P_2^{j,i1|\omega_1} &= \cdots = P_i^{j,in_2|\omega_1} \end{aligned}$$

for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

Case ($--$0):

No interactions.

The case (SU0), random set independence

$$
\begin{aligned}
\overline{P}_{\mathrm{SU0}}(A) &= \max\{P(A) : P \in \mathcal{M}_{\mathrm{SU0}}\} = \\
&= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) P^{ij*}(A) = \\
&= \sum_{F_1^i \times F_2^j \cap A \neq \varnothing} m_1(F_1^i) m_2(F_2^j) = \mathrm{Pl}(A)
\end{aligned}
$$

where P^{ij*} are appropriate Dirac measures.

The case (UU0), unknown interaction

The weights m^* are obtained by solving

$$\sum_{F_1^i \times F_2^j \cap A \neq \varnothing} m(F_1^i \times F_2^j) = \max$$

subject to

$$m_1(F_1^i) = \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j), \qquad m_2(F_2^j) = \sum_{i=1}^{|\mathcal{F}_1|} m(F_1^i \times F_2^j).$$

$\mathcal{M}_{\mathrm{UU0}} = \mathcal{M}_{\mathrm{U}} := \{P : P(\cdot \times \Omega_2) \in \mathcal{M}_1,\ P(\Omega_1 \times \cdot) \in \mathcal{M}_2\}$

The case (SE1), epistemic independence

$$\mathcal{M}_{\mathrm{SE1}} \ni P_{\mathrm{SE1}}(\cdot \times \{\omega_2\}) = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) P^{ij}(\cdot \times \{\omega_2\}) =$$

$$= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) P_1^{i|\omega_2} P_2^j(\{\omega_2\}) =$$

$$= \left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i) P_1^{i|\omega_2} \right) \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j) P_2^j(\{\omega_2\}) \right) =$$

$$= P_1^{|\omega_2} P_2(\{\omega_2\}).$$

$$\Longrightarrow P_{\mathrm{SE1}} = P_{\mathrm{E}} \in \mathcal{M}_{\mathrm{E}} \qquad \ldots \qquad \mathcal{M}_{\mathrm{SE1}} \subseteq \mathcal{M}_{\mathrm{E}}.$$

The case (SS1), strong independence

$$\mathcal{M}_{\mathrm{SS1}} \ni P_{\mathrm{SS1}}(A) = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) P^{ij}(A) =$$

$$= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) (P_1^i \otimes P_2^j)(A) =$$

$$= \left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i) P_1^i \right) \otimes \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j) P_2^j \right)(A) =$$

$$= (P_1 \otimes P_2)(A) = P_{\mathrm{S}} \in \mathcal{M}_{\mathrm{S}}.$$

$$\Longrightarrow \mathcal{M}_{\mathrm{SS1}} = \mathcal{M}_{\mathrm{S}}$$

The case (SS1), strong independence

The upper probability $\overline{P}_{\mathrm{S}}(A) = \sup\{P(A) : \; P \in \mathcal{M}_{\mathrm{S}}\}$ is the solution of the optimization problem:

$$\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \chi_A(\omega_1^i, \omega_2^j) = \max!$$

subject to

$$\begin{aligned} \omega_1^i &\in F_1^i, \; i = 1, \ldots, |\mathcal{F}_1|, \\ \omega_2^j &\in F_2^j, \; j = 1, \ldots, |\mathcal{F}_2|, \end{aligned}$$

where χ_A is the indicator function of the set A.

Summary

$$\begin{array}{ccccc}
\boxed{\mathcal{M}_{\mathrm{FU0}} = \mathcal{M}_{\mathrm{F}}} & \supseteq & \mathcal{M}_{\mathrm{FE0}} & \supseteq & \mathcal{M}_{\mathrm{FS0}} \\
\cap| & & \cap| & & \cap| \\
\boxed{\mathcal{M}_{\mathrm{UU0}} = \mathcal{M}_{\mathrm{U}}} & \supseteq & \mathcal{M}_{\mathrm{UE0}} & \supseteq & \mathcal{M}_{\mathrm{US0}} \\
\cup| & & \cup| & & \cup| \\
\boxed{\mathcal{M}_{\mathrm{SU0}} = \mathcal{M}_{\mathrm{R}}} & \supseteq & \mathcal{M}_{\mathrm{SE0}} & \supseteq & \mathcal{M}_{\mathrm{SS0}} \\
 & & \cup| & & \cup| \\
 & & \boxed{\mathcal{M}_{\mathrm{SE1}} \;\; (\subseteq \mathcal{M}_{\mathrm{E}})} & \supseteq & \boxed{\mathcal{M}_{\mathrm{SS1}} = \mathcal{M}_{\mathrm{S}}}
\end{array}$$

Summary

$$
\begin{array}{ccccc}
\boxed{\overline{P}_{\mathrm{FU0}} = \overline{P}_{\mathrm{F}}} & = & \overline{P}_{\mathrm{FE0}} & = & \overline{P}_{\mathrm{FS0}} \\
\wedge\vert & & \wedge\vert & & \wedge\vert \\
\boxed{\overline{P}_{\mathrm{UU0}} = \overline{P}_{\mathrm{U}}} & = & \overline{P}_{\mathrm{UE0}} & = & \overline{P}_{\mathrm{US0}} \\
\vee\vert & & \vee\vert & & \vee\vert \\
\boxed{\overline{P}_{\mathrm{SU0}} = \overline{P}_{\mathrm{R}}} & = & \overline{P}_{\mathrm{SE0}} & = & \overline{P}_{\mathrm{SS0}} \\
 & & \vee\vert & & \vee\vert \\
 & & \boxed{\overline{P}_{\mathrm{SE1}} \quad (\le \overline{P}_{\mathrm{E}})} & \ge & \boxed{\overline{P}_{\mathrm{SS1}} = \overline{P}_{\mathrm{S}}}
\end{array}
$$