Multiparameter Models: Probability Distributions Parameterized by Random Sets

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My Research Area

Propagating Uncertainty Through a Mapping

- Uncertainty modelled by
  - fuzzy sets,
  - random sets,
  - sets of probability measures generated by random sets,
  - sets of probability measures generated by probability measures which are parameterized by random sets.

- Sets of joint probability measures and independence.
- Applications in civil engineering and operations research.

Th. Fetz (Universität Innsbruck)
The Problem

What is Given?

- $g : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, \ldots, x_n) \mapsto g(x_1, \ldots, x_n)$.
- Uncertain variables $x_1, \ldots, x_n$.
- Uncertainty modelled by sets of probability measures generated
  - by random sets,
  - by probability distributions parameterized by random sets.

What is the Goal?

The construction of sets of joint probability measures with respect to different types of independence.
The Problem

What is Given?

- \( g : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto g(x_1, \ldots, x_n) \).
- Uncertain variables \( x_1, \ldots, x_n \).
- Uncertainty modelled by sets of probability measures generated
  - by random sets,
  - by probability distributions parameterized by random sets.

What Is the Goal?

- The construction of sets of joint probability measures with respect to different types of independence.
- \( \overline{P}(g(x_1, \ldots, x_n) \leq 0) \)
The Problem

What is Given?
- \( g : \mathbb{R}^n \rightarrow \mathbb{R} : (x_1, \ldots, x_n) \mapsto g(x_1, \ldots, x_n) \). ← failure function
- Uncertain variables \( x_1, \ldots, x_n \).
- Uncertainty modelled by sets of probability measures generated
  - by random sets,
  - by probability distributions parameterized by random sets.

What Is the Goal?
- The construction of sets of joint probability measures with respect to different types of independence.
- \( \overline{P}(g(x_1, \ldots, x_n) \leq 0) \) ← upper probability of failure
1st Part

Sets of **marginal** probability measures generated
- by random sets,
- by parameterized probability measures.
1\textsuperscript{st} Part

Sets of \textit{marginal} probability measures generated
- by random sets,
- by parameterized probability measures.

2\textsuperscript{nd} Part

Construction of sets of \textit{joint} probability measures.
- General formulation using the random set structure.
- Different cases which lead to
  - strong independence (S),
  - random set independence (R),
  - unknown interaction (U).
A random set \((\mathcal{F}, m)\) consists of a finite class \(\mathcal{F}\) of focal sets \(F^i\) and of a weight function \(m : \mathcal{F} \longrightarrow [0, 1] : F \mapsto m(F)\).

The upper probability \(\overline{P}\) is defined by

\[
\overline{P}(A) = \text{Pl}(A) = \sum_{F^i \cap A \neq \emptyset} m(F^i).
\]
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\[
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\]

The set of all probability measures on a focal set \(F^i\) is

\[
\mathcal{K}(F^i) := \{ P^i : P^i(F^i) = 1 \}
\]

\(m(F^i)\mathcal{K}(F^i)\) is the set of all distributions of the weight \(m(F^i)\) on the focal set \(F^i\).
Set of Probability Measures Generated by a Random Set \((\mathcal{F}, m)\)

\[
\mathcal{K}(\mathcal{F}, m) = \sum_{i=1}^{\vert \mathcal{F} \vert} m(F^i) \mathcal{K}(F^i) = \left\{ P : P = \sum_{i=1}^{\vert \mathcal{F} \vert} m(F^i)P^i, \ P^i \in \mathcal{K}(F^i) \right\}
\]
Sets of *Marginal* Probability Measures

**Set of Probability Measures Generated by a Random Set** $(\mathcal{F}, m)$

$$\mathcal{K}(\mathcal{F}, m) = \bigoplus_{i=1}^{\vert\mathcal{F}\vert} m(F^i) \mathcal{K}(F^i) = \left\{ P : P = \bigoplus_{i=1}^{\vert\mathcal{F}\vert} m(F^i) P^i, \ P^i \in \mathcal{K}(F^i) \right\}$$

**Upper Probability**

$$P(A) = \sum_{i=1}^{\vert\mathcal{F}\vert} m(F^i) \sup_{P^i \in \mathcal{K}(F^i)} \left\{ P^i(A) \right\} = \sum_{i=1}^{\vert\mathcal{F}\vert} m(F^i) \sup_{\omega \in F^i} \delta_\omega(A)$$

**Example Diagram**

- $F^1$: Interval from 0 to 4
- $F^2$: Interval from 4 to 8
- $F^3$: Interval from 8 to 12

- $A$: Area between $F^1$ and $F^2$
Sets of *Marginal* Probability Measures

**Set of Probability Measures Generated by a Random Set \((\mathcal{F}, m)\)**

\[
\mathcal{K}(\mathcal{F}, m) = \sum_{i=1}^{\left|\mathcal{F}\right|} m(F^i) \mathcal{K}(F^i) = \left\{ P : P = \sum_{i=1}^{\left|\mathcal{F}\right|} m(F^i)P^i, \ P^i \in \mathcal{K}(F^i) \right\}
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Sets of *Marginal* Probability Measures

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\]

**Upper Probability**

\[
\overline{P}(A) = \sum m(F^i) \sup \left\{ P^i(A) : P^i \in \mathcal{K}(F^i) \right\}
= \sum m(F^i) \sup_{\omega \in F^i} \delta_\omega(A)
= \sum_{F^i \cap A \neq \emptyset} m(F^i) = \text{Pl}(A)
\]
Sets of *Marginal* Probability Measures

**Set of Probability Measures Generated by a Random Set** \((\mathcal{F}, m)\)

\[
\mathcal{K}(\mathcal{F}, m) = \sum_{i=1}^{\lvert \mathcal{F} \rvert} m(F^i) \mathcal{K}(F^i) = \left\{ P : P = \sum_{i=1}^{\lvert \mathcal{F} \rvert} m(F^i)P^i, \ P^i \in \mathcal{K}(F^i) \right\}
\]

**Upper Probability**

\[
\overline{P}(A) = \sum m(F^i) \sup \{P^i(A) : P^i \in \mathcal{K}(F^i)\} = \sum m(F^i) \sup_{\omega \in F^i} \delta_{\omega}(A) = \sum_{F^i \cap A \neq \emptyset} m(F^i) = Pl(A)
\]
A set $\mathcal{K}$ of probability measures is generated by a probability measure $p^\theta$ on $\Omega$ which is parameterized by an uncertain parameter $\theta \in \Theta$.

The uncertainty of the parameter $\theta \in \Theta$ is modelled by a set $\mathcal{K}$ of probability measures on $\Theta$.

The set $\mathcal{K}$ is generated by random sets.
Sets of *Marginal* Probability Measures

**Probability Measures Parameterized by Random Sets**

- A set \( \mathcal{K} \) of probability measures is generated by a probability measure \( p^\theta \) on \( \Omega \) which is parameterized by an uncertain parameter \( \theta \in \Theta \).
- The uncertainty of the parameter \( \theta \in \Theta \) is modelled by a set \( \mathcal{K} \) of probability measures on \( \Theta \).
- The set \( \mathcal{K} \) is generated by random sets.

**Such a Set \( \mathcal{K} \) Is Defined by**

\[
\mathcal{K} := \mathcal{K}(\mathcal{K}, p^\theta) := \left\{ P = \int_{\Theta} p^\theta(\cdot) \mu(d\theta) : \mu \in \mathcal{K} \right\}.
\]
Upper Probability, General Case

\[ \overline{P}(A) = \sup\{P(A) : P \in \mathcal{K}\} = \sup_{\mu \in \mathcal{K}} \int_{\Theta} p^\theta(A) \mu(d\theta) \]
Sets of *Marginal* Probability Measures

**Upper Probability, General Case**

\[
\overline{P}(A) = \sup\{P(A) : P \in \mathcal{K}\} = \sup_{\mu \in \mathcal{K}} \int \Theta p^\theta(A) \mu(d\theta)
\]

**Upper Probability for the Case \( \mathcal{K} := \mathcal{K}(F) \)**

\[
\overline{P}(A) = \sup_{\mu \in \mathcal{K}(F)} \int \Theta p^\theta(A) \mu(d\theta) = \sup_{\theta_0 \in F} \int \Theta p^\theta(A) \delta_{\theta_0}(d\theta) = \sup_{\theta_0 \in F} p^{\theta_0}(A)
\]
Sets of *Marginal* Probability Measures

Set of Probability Measures for $\mathcal{K} := \mathcal{K}(\mathcal{F}, m)$

$$
\mathcal{K}(\mathcal{K}(\mathcal{F}, m), p^\theta) = \sum_{i=1}^{\left|\mathcal{F}\right|} m(F^i) \mathcal{K}(\mathcal{K}(F^i), p^\theta)
$$
**Sets of *Marginal* Probability Measures**

### Set of Probability Measures for $\mathcal{K} := \mathcal{K}(\mathcal{F}, m)$

$$
\mathcal{K}(\mathcal{K}(\mathcal{F}, m), p^\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F_i) \mathcal{K}(\mathcal{K}(F_i), p^\theta)
$$

### Upper Probability for $\mathcal{K} := \mathcal{K}(\mathcal{F}, m)$

$$
\overline{P}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F_i) \sup_{\mu^i \in \mathcal{K}(F_i)} \int_{\Theta} p^\theta(A) \mu^i(d\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F_i) \sup_{\theta_0 \in F_i} p^{\theta_0}(A)
$$
Sets of *Marginal* Probability Measures

**Set of Probability Measures for** $\mathcal{K} := \mathcal{K}(\mathcal{F}, m)$

$$\mathcal{K}(\mathcal{K}(\mathcal{F}, m), p^\theta) = \sum_{i=1}^{\lvert \mathcal{F} \rvert} m(F^i) \mathcal{K}(\mathcal{K}(F^i), p^\theta)$$

**Upper Probability for** $\mathcal{K} := \mathcal{K}(\mathcal{F}, m)$

$$\overline{P}(A) = \sum_{i=1}^{\lvert \mathcal{F} \rvert} m(F^i) \sup_{\mu^i \in \mathcal{K}(F^i)} \int \Theta p^\theta(A) \mu^i(d\theta) = \sum_{i=1}^{\lvert \mathcal{F} \rvert} m(F^i) \sup_{\theta_0 \in F^i} p^{\theta_0}(A)$$

**Relation to Pure Random Sets**

If $\Theta := \Omega$ and $p^\omega := \delta_\omega$ then $\mathcal{K}(F) = \mathcal{K}(\mathcal{K}(F), \delta_\omega)$. 
Sets of *Joint* Probability Measures

**Notion of Independence**

- **Unknown interaction:** The set of joint probability measures according to unknown interaction is generated by
  \[ \mathcal{K}_U := \{ P : P(\cdot \times \Omega_2) \in \mathcal{K}_1, P(\Omega_1 \times \cdot) \in \mathcal{K}_2 \}. \] (U)

- **Strong independence:** The set of joint probability measures according to strong independence is generated by
  \[ \mathcal{K}_S := \{ P_1 \otimes P_2 : P_1 \in \mathcal{K}_1, P_2 \in \mathcal{K}_2 \} \subseteq \mathcal{K}_U. \] (S)

- **Random set independence:** Dempster’s Rule of Combination.
  \[ \text{Pl}(A) = \sum_{i,j: F^i_1 \times F^j_2 \cap A \varnothing} m_1(F^i_1)m_2(F^j_2) \] (R)
General Formula for a Set of Joint Probability Measures

\[ \mathcal{K}_? = \sum_{i=1}^{\mathcal{F}_1} \sum_{j=1}^{\mathcal{F}_2} m_? (F^i_1 \times F^j_2) \mathcal{K}_? (\mathcal{K}^i_1, \mathcal{K}^j_2) \]
General Formula for a Set of Joint Probability Measures

\[ K_? = \sum_{i=1}^{\mid F_1 \mid} \sum_{j=1}^{\mid F_2 \mid} m_?(F_i^1 \times F_j^2) K_?(K_i^1, K_j^2) \]

The Choice of the Joint Weights \( m(F_i^1 \times F_j^2) \)

Case (U→→): Unknown interaction, \( m \) must satisfy the conditions:

\[ m_1(F_1^i) = \sum_{j=1}^{\mid F_2 \mid} m(F_i^1 \times F_j^2), \quad i = 1, \ldots, \mid F_1 \mid, \]

\[ m_2(F_2^j) = \sum_{i=1}^{\mid F_1 \mid} m(F_i^1 \times F_j^2), \quad j = 1, \ldots, \mid F_2 \mid. \]

Case (S→→): Stochastic independence: \( m(F_i^1 \times F_j^2) := m_1(F_i^1)m_2(F_j^2). \)
General Formula for a Set of Joint Probability Measures

\[ \mathcal{K}_? = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m? (F^i_1 \times F^j_2) \mathcal{K}_? (\mathcal{K}^i_1, \mathcal{K}^j_2) \]

The Choice of \( P^{ij} \), \( \mathcal{K}^{ij} \), respectively

**Case (−U−):** \( \mathcal{K}^{ij}_U := \mathcal{K}_U (\mathcal{K}^i_1, \mathcal{K}^j_2) \)

which is the set of all joint probability measures generated by the sets \( \mathcal{K}^i_1 \) and \( \mathcal{K}^j_2 \) according to condition (U).

**Case (−S−):** \( \mathcal{K}^{ij}_S := \mathcal{K}_S (\mathcal{K}^i_1, \mathcal{K}^j_2) \)

which is the set generated according to strong independence (S).
General Formula for a Set of Joint Probability Measures

\[ \mathcal{K}_? = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m? (F^i_1 \times F^j_2) \mathcal{K}_? (\mathcal{K}^i_1, \mathcal{K}^j_2) \]

The Choice of Interactions between the \( P^{ij} \)

**Case (−−0): No interactions.**

We can choose a \( P^{ij} \in \mathcal{K}^{ij} \) on \( F^i_1 \times F^j_2 \) irrespective of the probability measures chosen on other joint focal sets.

**Case (−−1): Row- and columnwise equality conditions** on the marginals of the probability measures on the joint focal sets:

\[
\begin{align*}
P^i_1 &:= P^{i,i1}_1 = \cdots = P^{i,in_2}_i, \quad i = 1, \ldots, n_1, \\
P^j_2 &:= P^{j,1j}_2 = \cdots = P^{j,m_1j}_i, \quad j = 1, \ldots, n_2
\end{align*}
\]

where \( P^{i,ik}_1 = P^{ik}_1 (\cdot \times \Omega_2) \) and \( P^{j,kj}_2 = P^{kj}_2 (\Omega_1 \times \cdot) \).
The Different Cases

Notation

The cases are indicated by a triple where for example (SU0) means

- \( m \) according \((S\--\)\)
- \( P_{ij} \) according to \((-U-)\)
- No interaction between the \( P_{ij} \), \((-0)\).

Goal

Cases which lead to

- Strong independence \((S)\)
- Random set independence \((R)\)
- Unknown interaction \((U)\)
The Cases (SU0) + (SS0) and Random Set Independence

General Formulation

The sets $\mathcal{K}_{SU0}$ and $\mathcal{K}_{SS0}$ of joint probability measures are generated by

\[
\mathcal{K}_{SU0} = \sum_{i=1}^{\left| \mathcal{F}_1 \right|} \sum_{j=1}^{\left| \mathcal{F}_2 \right|} m_1(F^i_1)m_2(F^j_2)\mathcal{K}_U(\mathcal{K}^i_1, \mathcal{K}^j_2)
\]

\[
\mathcal{K}_{SS0} = \mathcal{K}_S(\mathcal{K}^i_1, \mathcal{K}^j_2)
\]
The Cases (SU0) + (SS0) and Random Set Independence

General Formulation

The sets $\mathcal{K}_{SU0}$ and $\mathcal{K}_{SS0}$ of joint probability measures are generated by

$$
\mathcal{K}_{SU0} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F^i_1)m_2(F^j_2) \mathcal{K}_U(\mathcal{K}_1^i, \mathcal{K}_2^j)
$$

$$
\mathcal{K}_{SS0} = \mathcal{K}_S(\mathcal{K}_1^i, \mathcal{K}_2^j)
$$

For $\mathcal{K}_1^i := \mathcal{K}(F^i_1)$, $\mathcal{K}_2^j := \mathcal{K}(F^j_2)$

$$
\mathcal{K}_U(\mathcal{K}_1^i, \mathcal{K}_2^j) = \mathcal{K}_U(\mathcal{K}(F^i_1), \mathcal{K}(F^j_2)) = \mathcal{K}(F^i_1 \times F^j_2)
$$

$\implies$ Random set independence

Results: $\mathcal{K}_R = \mathcal{K}_{SU0} \supseteq \mathcal{K}_{SS0}$ and $\overline{P}_R = \overline{P}_{SU0} = \overline{P}_{SS0}$
The Cases $(SU0) + (SS0)$ and Random Set Independence

### General Formulation

The sets $\mathcal{K}_{SU0}$ and $\mathcal{K}_{SS0}$ of joint probability measures are generated by

$$
\mathcal{K}_{SU0} = \sum_{i=1}^{\left|\mathcal{F}_1\right|} \sum_{j=1}^{\left|\mathcal{F}_2\right|} m_1(F^i_1)m_2(F^j_2)\mathcal{K}_U(\mathcal{K}^i_1, \mathcal{K}^j_2)
$$

$$
\mathcal{K}_{SS0} = \mathcal{K}_S(\mathcal{K}^i_1, \mathcal{K}^j_2)
$$

For $\mathcal{K}^i_1 := \mathcal{K}(\mathcal{K}(F^i_1), p^\theta_1)$, $\mathcal{K}^j_2 := \mathcal{K}(\mathcal{K}(F^j_2), p^\theta_2)$

### Problem:

What is $\mathcal{K}_U(\mathcal{K}^i_1, \mathcal{K}^j_2)$ now?

$\mathcal{K}_U(\mathcal{K}^i_1, \mathcal{K}^j_2)$ cannot be described by a set $\mathcal{K}(\mathcal{K}, p^\theta)$ because there is not only one joint probability measure $p^\theta$ determined by $p^\theta_1$ and $p^\theta_2$.

We define $\mathcal{K}_R := \mathcal{K}_{S(US)0}$ where $(US)$ means unknown interaction (U) for the parameters and a product measure $p^\theta_1 \otimes p^\theta_2$ (S).

### Then:

$\mathcal{K}_{S(US)0} = \mathcal{K}(\mathcal{K}(\mathcal{F}, m), p^\theta_1 \otimes p^\theta_2)$
The Cases (UU0) + (US0) and Unknown Interaction (U)

For \( \mathcal{K}_1^i := \mathcal{K}(F_1^i) \), \( \mathcal{K}_2^j := \mathcal{K}(F_2^j) \)

Similar to RS-independence, but with condition (U---).

The joint weights have to be determined by solving a linear optimization problem subject to condition (U---).

Results: \( \mathcal{K}_U = \mathcal{K}_{UU0} \supseteq \mathcal{K}_{US0} \) and \( P_{UU0} = P_{US0} \).
The Cases (UU0) + (US0) and Unknown Interaction (U)

For \( \mathcal{K}_1^i := \mathcal{K}(F_1^i) \), \( \mathcal{K}_2^j := \mathcal{K}(F_2^j) \)

Similar to RS-independence, but with condition (U−−).

The joint weights have to be determined by solving a linear optimization problem subject to condition (U−−).

Results: \( \mathcal{K}_U = \mathcal{K}_{UU0} \supseteq \mathcal{K}_{US0} \) and \( \overline{P}_{UU0} = \overline{P}_{US0} \).

For \( \mathcal{K}_1^i := \mathcal{K}(\mathcal{R}(F_1^i), p_1^{\theta_1}) \), \( \mathcal{K}_2^j := \mathcal{K}(\mathcal{R}(F_2^j), p_2^{\theta_2}) \)

Unfortunately we do not have \( \mathcal{K}_U = \mathcal{K}_{UU0} \) in general.
\( \mathcal{K}_{SS1} = \mathcal{K}_S \)

Let \( P_1 \in \mathcal{K}(\mathcal{F}_1, m_1) \) and \( P_2 \in \mathcal{K}(\mathcal{F}_2, m_2) \).

\[
\mathcal{K}_S \ni P_S(A) = (P_1 \otimes P_2)(A) = \left( \sum_{i=1}^{\left| \mathcal{F}_1 \right|} m_1(F_1^i) P_1^i \right) \otimes \left( \sum_{j=1}^{\left| \mathcal{F}_2 \right|} m_2(F_2^j) P_2^j \right)(A) = \\
\sum_{i=1}^{\left| \mathcal{F}_1 \right|} \sum_{j=1}^{\left| \mathcal{F}_2 \right|} m_1(F_1^i) m_2(F_2^j) (P_1^i \otimes P_2^j)(A) = P_{SS1}(A) \in \mathcal{K}_{SS1}
\]
### Relations between the Sets of Probability Measures

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<tr>
<th>( \mathcal{K}<em>U \supseteq \mathcal{K}</em>{UU0} )</th>
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<th>( \subseteq )</th>
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<th>( \mathcal{K}_{SS0} \subseteq \mathcal{K}<em>S = \mathcal{K}</em>{SS1} )</th>
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<td>$\mathcal{K}<em>S = \mathcal{K}</em>{SS1}$</td>
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### Relations between the Upper Probabilities

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<td>$\overline{P}<em>S = \overline{P}</em>{SS1}$</td>
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The Case (SS1), Strong Independence, Computational Method

For $\mathcal{K}_1 := \mathcal{K}(F_1^i)$, $\mathcal{K}_2 := \mathcal{K}(F_2^j)$

$P_S(A)$ is the solution of the following optimization problem:

$$
\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j) \chi_A(\omega_1^i, \omega_2^j) = \max!
$$

subject to $\omega_1^i \in F_1^i$ and $\omega_2^j \in F_2^j$. $\chi_A$ is the indicator function of the set $A$. 
The Case (SS1), Strong Independence, Computational Method

For $\mathcal{K}_i^1 := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

$\overline{P}_S(A)$ is the solution of the following optimization problem:

$$\sum_{i=1}^{\vert F_1 \vert} \sum_{j=1}^{\vert F_2 \vert} m(F_1^i \times F_2^j) \chi_A(\omega_1^i, \omega_2^j) = \text{max}!$$

subject to $\omega_1^i \in F_1^i$ and $\omega_2^j \in F_2^j$. $\chi_A$ is the indicator function of the set $A$.

For $\mathcal{K}_i^1 := \mathcal{K}(\mathcal{K}(F_1^i), p_{\theta_1}^1)$, $\mathcal{K}_2^j := \mathcal{K}(\mathcal{K}(F_2^j), p_{\theta_2}^2)$

$\overline{P}_S(A)$ is the solution of the following optimization problem:

$$\sum_{i=1}^{\vert F_1 \vert} \sum_{j=1}^{\vert F_2 \vert} m_1(F_1^i)m_2(F_2^j) \left(p_{\theta_1}^i \otimes p_{\theta_2}^j\right)(A) = \text{max}!$$

subject to $\theta_1^i \in F_1^i$ and $\theta_2^j \in F_2^j$.