Multiparameter Models: Probability Distributions Parameterized by Random Sets

Thomas Fetz

Institut für Grundlagen der Bauingenieurwissenschaften,
Arbeitsbereich Technische Mathematik,
Universität Innsbruck, Austria

5th International Symposium on Imprecise Probability:
Theories and Applications, Prague, Czech Republic, 2007

Propagating Uncertainty Through a Mapping

- Uncertainty modelled by
  - fuzzy sets,
  - random sets,
  - sets of probability measures generated by random sets,
  - sets of probability measures generated by probability measures which are parameterized by random sets.
- Sets of joint probability measures and independence.
- Applications in civil engineering and operations research.

What is Given?

- $g : \mathbb{R}^n \to \mathbb{R} : \ (x_1, \ldots, x_n) \mapsto g(x_1, \ldots, x_n)$.
- Uncertain variables $x_1, \ldots, x_n$.
- Uncertainty modelled by sets of probability measures generated
  - by random sets,
  - by probability distributions parameterized by random sets.

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What Is the Goal?

- The construction of sets of joint probability measures with respect to different types of independence.
- $\overline{P}(g(x_1, \ldots, x_n) \leq 0)$

The Problem

What is Given?

- $g : \mathbb{R}^n \to \mathbb{R} : \ (x_1, \ldots, x_n) \mapsto g(x_1, \ldots, x_n)$. $\leftarrow$ failure function
- Uncertain variables $x_1, \ldots, x_n$.
- Uncertainty modelled by sets of probability measures generated
  - by random sets,
  - by probability distributions parameterized by random sets.

What Is the Goal?

- The construction of sets of joint probability measures with respect to different types of independence.
- $\overline{P}(g(x_1, \ldots, x_n) \leq 0)$ $\leftarrow$ upper probability of failure

Outline of the Talk

1^{st} Part

Sets of marginal probability measures generated

- by random sets,
- by parameterized probability measures.

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1st Part

Sets of marginal probability measures generated

- by random sets,
- by parameterized probability measures.

2nd Part

Construction of sets of joint probability measures.

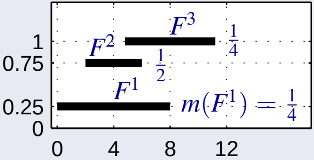
- General formulation using the random set structure.
- Different cases which lead to
  $\longrightarrow$ strong independence (S),
  $\longrightarrow$ random set independence (R),
  $\longrightarrow$ unknown interaction (U).

Random Sets and Upper Probability

A random set $(\mathcal{F}, m)$ consists of a finite class $\mathcal{F}$ of focal sets F^i and of a weight function $m : \mathcal{F} \longrightarrow [0,1] : F \mapsto m(F)$.

The upper probability $\overline{P}$ is defined by

$$\overline{P}(A) = \mathrm{Pl}(A) = \sum_{F^i \cap A \neq \varnothing} m(F^i).$$



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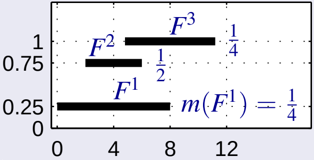
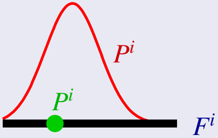


Set of All Probability Measures **on** a Focal Set F^i

$\mathcal{K}(F^i) := \{P^i : \; P^i(F^i) = 1\}$

$m(F^i)\mathcal{K}(F^i)$ is the set of all distributions of the weight $m(F^i)$ on the focal set F^i.



Sets of *Marginal* Probability Measures

Set of Probability Measures Generated by a Random Set $(\mathcal{F}, m)$

$$\mathcal{K}(\mathcal{F}, m) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\, \mathcal{K}(F^i) = \left\{ P : P = \sum_{i=1}^{|\mathcal{F}|} m(F^i) P^i,\ P^i \in \mathcal{K}(F^i) \right\}$$

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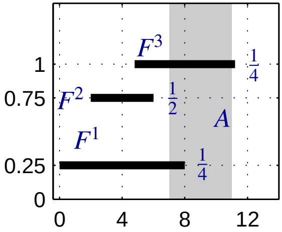

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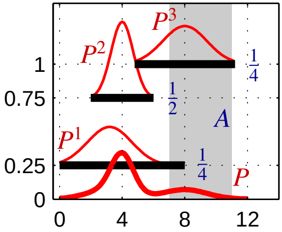

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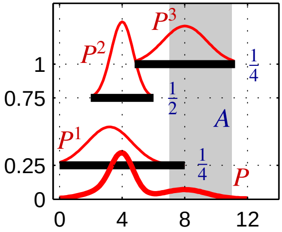

Upper Probability

$$\begin{aligned}\overline{P}(A) &= \sum m(F^i) \sup\{P^i(A) : P^i \in \mathcal{K}(F^i)\} \\ &= \sum m(F^i) \sup_{\omega \in F^i} \delta_\omega(A) \\ &= \sum_{F^i \cap A \neq \varnothing} m(F^i) = \mathrm{Pl}(A)\end{aligned}$$

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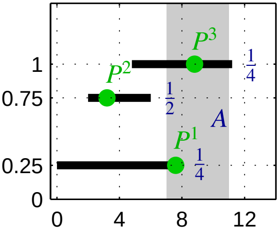

Upper Probability

$$\overline{P}(A) = \sum m(F^i) \sup\{P^i(A) : P^i \in \mathcal{K}(F^i)\}$$

$$= \sum m(F^i) \sup_{\omega \in F^i} \delta_\omega(A)$$

$$= \sum_{F^i \cap A \neq \emptyset} m(F^i) = \mathrm{Pl}(A)$$

Probability Measures Parameterized by Random Sets

- A set $\mathcal{K}$ of probability measures is generated by a probability measure p^{θ} on Ω which is parameterized by an uncertain parameter $\theta \in \Theta$.
- The uncertainty of the parameter $\theta \in \Theta$ is modelled by a set $\mathfrak{K}$ of probability measures on Θ.
- The set $\mathfrak{K}$ is generated by random sets.

Sets of *Marginal* Probability Measures

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Such a Set $\mathcal{K}$ Is Defined by

$$\mathcal{K} := \mathcal{K}(\mathfrak{K}, p^{\theta}) := \left\{ P = \int_{\Theta} p^{\theta}(\cdot)\mu(\mathrm{d}\theta) : \ \mu \in \mathfrak{K} \right\}.$$

Sets of *Marginal* Probability Measures

Upper Probability, General Case

$$\overline{P}(A) = \sup\{P(A) :\ P \in \mathcal{K}\} = \sup_{\mu \in \mathfrak{K}} \int_\Theta p^{\,\theta}(A)\ \mu(\mathrm{d}\theta)$$

Sets of *Marginal* Probability Measures

Upper Probability, General Case

$$\overline{P}(A) = \sup\{P(A) : \; P \in \mathcal{K}\} = \sup_{\mu \in \mathfrak{K}} \int_\Theta p^{\,\theta}(A) \; \mu(\mathrm{d}\theta)$$

Upper Probability for the Case $\mathfrak{K} := \mathfrak{K}(F)$

$$\overline{P}(A) = \sup_{\mu \in \mathfrak{K}(F)} \int_\Theta p^{\,\theta}(A) \; \mu(\mathrm{d}\theta) \quad = \sup_{\theta_0 \in F} \int_\Theta p^{\,\theta}(A) \; \delta_{\theta_0}(\mathrm{d}\theta) = \sup_{\theta_0 \in F} p^{\,\theta_0}(A)$$

Sets of *Marginal* Probability Measures

Set of Probability Measures for $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$

$$\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\theta}) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\, \mathcal{K}(\mathfrak{K}(F^i), p^{\theta})$$

Set of Probability Measures for $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$

$$\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\,\theta}) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\ \mathcal{K}(\mathfrak{K}(F^i), p^{\,\theta})$$

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Sets of *Marginal* Probability Measures

Set of Probability Measures for $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$

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$$\overline{P}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\mu^i \in \mathfrak{K}(F^i)} \int_{\Theta} p^{\theta}(A)\mu^i(\mathrm{d}\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\theta_0 \in F^i} p^{\theta_0}(A)$$

Relation to Pure Random Sets

If $\Theta := \Omega$ and $p^{\omega} := \delta_{\omega}$ then $\mathcal{K}(F) = \mathcal{K}(\mathfrak{K}(F), \delta_{\omega})$.

Notion of Independence

- **Unknown interaction:** The set of joint probability measures according to unknown interaction is generated by

$$\mathcal{K}_{\mathrm{U}} := \{P:\ P(\cdot \times \Omega_2) \in \mathcal{K}_1,\ P(\Omega_1 \times \cdot) \in \mathcal{K}_2\}. \tag{U}$$

- **Strong independence:** The set of joint probability measures according to strong independence is generated by

$$\mathcal{K}_{\mathrm{S}} := \{P_1 \otimes P_2:\ P_1 \in \mathcal{K}_1,\ P_2 \in \mathcal{K}_2\} \subseteq \mathcal{K}_{\mathrm{U}}. \tag{S}$$

- **Random set independence:** Dempster's Rule of Combination.

$$\mathrm{Pl}(A) = \sum_{i,j:\ F_1^i \times F_2^j \cap A \varnothing} m_1(F_1^i)\, m_2(F_2^j) \tag{R}$$

General Formula for a Set of Joint Probaility Measures

$$\mathcal{K}_{?} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_{?}(F_1^i \times F_2^j) \mathcal{K}_{?}(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

General Formula for a Set of Joint Probaility Measures

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The Choice of the Joint Weights $m(F_1^i \times F_2^j)$

Case (U$--$): Unknown interaction, m must satisfy the conditions:

$$m_1(F_1^i) = \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j), \quad i = 1, \ldots, |\mathcal{F}_1|,$$

$$m_2(F_2^j) = \sum_{i=1}^{|\mathcal{F}_1|} m(F_1^i \times F_2^j), \quad j = 1, \ldots, |\mathcal{F}_2|.$$

Case (S$--$): Stochastic independence: $m(F_1^i \times F_2^j) := m_1(F_1^i)m_2(F_2^j)$.

General Formula for a Set of Joint Probaility Measures

$$\mathcal{K}_{?} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_{?}(F_1^i \times F_2^j) \mathcal{K}_{?}(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

The Choice of P^{ij}, $\mathcal{K}^{ij}$, respectively

Case (–U–): $\mathcal{K}_{\mathrm{U}}^{ij} := \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$

which is the set of all joint probability measures generated by the sets $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ according to condition (U).

Case (–S–): $\mathcal{K}_{\mathrm{S}}^{ij} := \mathcal{K}_{\mathrm{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$

which is the set generated according to strong independence (S).

General Formula for a Set of Joint Probaility Measures

$$\mathcal{K}_? = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_?(F_1^i \times F_2^j) \mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

The Choice of Interactions between the P^{ij}

Case ($--$0): No interactions.
We can choose a $P^{ij} \in \mathcal{K}^{ij}$ on $F_1^i \times F_2^j$ irrespective of the probability measures chosen on other joint focal sets.
Case ($--$1): Row- and columnwise equality conditions on the marginals of the probability measures on the joint focal sets:

$$P_1^i := P_1^{i,i1} = \cdots = P_i^{i,in_2},\ i = 1, \ldots, n_1,$$
$$P_2^j := P_2^{j,1j} = \cdots = P_i^{j,n_1 j},\ j = 1, \ldots, n_2$$

where $P_1^{i,ik} = P_1^{ik}(\cdot \times \Omega_2)$ and $P_2^{j,kj} = P_2^{kj}(\Omega_1 \times \cdot)$.

Notation

The cases are indicated by a tripel where for example (SU0) means

- m according (S$--$)
- P^{ij} according to ($-$U$-$)
- No interaction between the P^{ij}, ($--$0).

Goal

Cases which lead to

- Strong independence (S)
- Random set independence (R)
- Unknown interaction (U)

General Formulation

The sets $\mathcal{K}_{\text{SU0}}$ and $\mathcal{K}_{\text{SS0}}$ of joint probability measures are generated by

$$\begin{matrix}\mathcal{K}_{\text{SU0}} \\ \mathcal{K}_{\text{SS0}}\end{matrix} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \begin{matrix}\mathcal{K}_{\text{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j) \\ \mathcal{K}_{\text{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j)\end{matrix}$$

The Cases (SU0) + (SS0) and Random Set Independence

General Formulation

The sets $\mathcal{K}_{\text{SU0}}$ and $\mathcal{K}_{\text{SS0}}$ of joint probability measures are generated by

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For $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

$\mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i,\mathcal{K}_2^j) = \mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1^i),\mathcal{K}(F_2^j)) = \mathcal{K}(F_1^i \times F_2^j)$

$\Longrightarrow$ Random set independence

Results: $\mathcal{K}_{\mathrm{R}} = \mathcal{K}_{\mathrm{SU0}} \supseteq \mathcal{K}_{\mathrm{SS0}}$ and $\overline{P}_{\mathrm{R}} = \overline{P}_{\mathrm{SU0}} = \overline{P}_{\mathrm{SS0}}$

General Formulation

The sets $\mathcal{K}_{\text{SU0}}$ and $\mathcal{K}_{\text{SS0}}$ of joint probability measures are generated by

$$\begin{array}{c} \mathcal{K}_{\text{SU0}} \\ \mathcal{K}_{\text{SS0}} \end{array} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \begin{array}{r} \mathcal{K}_{\text{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j) \\ \mathcal{K}_{\text{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j) \end{array}$$

For $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

Problem: What is $\mathcal{K}_{\text{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$ now?

$\mathcal{K}_{\text{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$ cannot be described by a set $\mathcal{K}(\mathfrak{K}, p^{\theta})$ because there is not only one joint probability measure p^{θ} determined by $p_1^{\theta_1}$ and $p_2^{\theta_2}$.

We define $\mathcal{K}_{\text{R}} := \mathcal{K}_{\text{S(US)0}}$ where (US) means unknown interaction (U) for the parameters and a product measure $p_1^{\theta_1} \otimes p_2^{\theta_2}$ (S).

Then: $\mathcal{K}_{\text{S(US)0}} = \mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p_1^{\theta_1} \otimes p_2^{\theta_2})$

For $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

Similar to RS-independence, but with condition (U$--$).

The joint weights have to be determined by solving a linear optimization problem subject to condition (U$--$).

Results: $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{\mathrm{UU0}} \supseteq \mathcal{K}_{\mathrm{US0}}$ and $\overline{P}_{\mathrm{UU0}} = \overline{P}_{\mathrm{US0}}$.

The Cases (UU0) + (US0) and Unknown Interaction (U)

For $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

Similar to RS-independence, but with condition (U$--$).

The joint weights have to be determined by solving a linear optimization problem subject to condition (U$--$).

Results: $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{\mathrm{UU0}} \supseteq \mathcal{K}_{\mathrm{US0}}$ and $\overline{P}_{\mathrm{UU0}} = \overline{P}_{\mathrm{US0}}$.

For $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

Unfortunately we do not have $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{\mathrm{UU0}}$ in general.

$\mathcal{K}_{\text{SS1}} = \mathcal{K}_{\text{S}}$

Let $P_1 \in \mathcal{K}(\mathcal{F}_1, m_1)$ and $P_2 \in \mathcal{K}(\mathcal{F}_2, m_2)$.

$$\mathcal{K}_{\text{S}} \ni P_{\text{S}}(A) = (P_1 \otimes P_2)(A) = \left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i)\, P_1^i \right) \otimes \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j)\, P_2^j \right)(A) =$$

$$= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j)\, (P_1^i \otimes P_2^j)(A) = P_{\text{SS1}}(A) \in \mathcal{K}_{\text{SS1}}$$

Relations between the Sets of Probability Measures

$$
\begin{array}{ccccc}
\boxed{\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{K}_{\mathrm{UU0}}} & \supseteq & \mathcal{K}_{\mathrm{U(US)0}} & \supseteq & \mathcal{K}_{\mathrm{US0}} \\
\cup\!| & & \cup\!| & & \cup\!| \\
\mathcal{K}_{\mathrm{SU0}} & \supseteq & \boxed{\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{S(US)0}}} & \supseteq & \mathcal{K}_{\mathrm{SS0}} \\
 & & & & \cup\!| \\
 & & & & \boxed{\mathcal{K}_{\mathrm{S}} = \mathcal{K}_{\mathrm{SS1}}}
\end{array}
$$

Summary for Sets of Parameterized Probability Measures

Relations between the Sets of Probability Measures

$$
\begin{array}{ccccc}
\boxed{\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{K}_{\mathrm{UU0}}} & \supseteq & \mathcal{K}_{\mathrm{U(US)0}} & \supseteq & \mathcal{K}_{\mathrm{US0}} \\
\cup\!| & & \cup\!| & & \cup\!| \\
\mathcal{K}_{\mathrm{SU0}} & \supseteq & \boxed{\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{S(US)0}}} & \supseteq & \mathcal{K}_{\mathrm{SS0}} \\
 & & & & \cup\!| \\
 & & & & \boxed{\mathcal{K}_{\mathrm{S}} = \mathcal{K}_{\mathrm{SS1}}}
\end{array}
$$

Relations between the Upper Probabilities

$$
\begin{array}{ccccc}
\boxed{\overline{P}_{\mathrm{U}} \geq \overline{P}_{\mathrm{UU0}}} & \geq & \overline{P}_{\mathrm{U(US)0}} & = & \overline{P}_{\mathrm{US0}} \\
\vee\!| & & \vee\!| & & \vee\!| \\
\overline{P}_{\mathrm{SU0}} & \geq & \boxed{\overline{P}_{\mathrm{R}} := \overline{P}_{\mathrm{S(US)0}}} & = & \overline{P}_{\mathrm{SS0}} \\
 & & & & \vee\!| \\
 & & & & \boxed{\overline{P}_{\mathrm{S}} = \overline{P}_{\mathrm{SS1}}}
\end{array}
$$

For $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

$\overline{P}_{\mathrm{S}}(A)$ is the solution of the following optimization problem:

$$\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j)\, \chi_A(\omega_1^i, \omega_2^j) = \max!$$

subject to $\omega_1^i \in F_1^i$ and $\omega_2^j \in F_2^j$. χ_A is the indicator function of the set A.

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subject to $\omega_1^i \in F_1^i$ and $\omega_2^j \in F_2^j$. χ_A is the indicator function of the set A.

For $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

$\overline{P}_{\text{S}}(A)$ is the solution of the following optimization problem:

$$\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \left(p_1^{\theta_1^i} \otimes p_2^{\theta_2^j}\right)(A) = \max!$$

subject to $\theta_1^i \in F_1^i$ and $\theta_2^j \in F_2^j$.