Multiparameter models: Probability distributions parameterized by random sets

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1 Introduction

Let a mapping

$$g : D \subseteq \mathbb{R}^m \longrightarrow \mathbb{R} : \; (x_1,\dots,x_m) \mapsto g(x_1,\dots,x_m)$$

be given. The variables x_k are assumed to be uncertain where the uncertainty is modelled by sets of probability measures generated by random sets or by probability measures parameterized by random sets. What we want to know is the lower and upper probabilities if the value $g(x_1,\dots,x_m)$, is lower (or greater) than a certain value. Therefore we had to propagate the uncertainty of the variables x_k through this multivariate model g.

Since the uncertainty of the variables is given separately, we have to model the joint uncertainty, that means to construct the set of joint probability measures. To propagate the uncertainty through a multivariate model in a computational efficient way it is essential to make use of the structure of the random sets.

2 Sets of probability measures generated by probability measures parameterized by random sets

2.1 Random sets

A random set $(\mathcal{F}, m)$ on a set Ω consists of a finite class

$$\mathcal{F} = \{F^1, F^2, \dots, F^n\}$$

of focal sets F^i and of a weight function

$$m : \mathcal{F} \longrightarrow [0,1] : F \mapsto m(F).$$

The plausibility measure Pl or upper probability $\overline{P}$ of a set A is defined by

$$\overline{P}(A) = \mathrm{Pl}(A) = \sum_{F^i \cap A \neq \varnothing} m(F^i).$$

2.2 Sets of probability measures generated by random sets

Let $\mathcal{K}(F^i) := \{P : \; P(F^i) = 1\}$ be the set of all probability measures "on" the focal set F^i. Then $m(F^i)\mathcal{K}(F^i)$ is the set of all possible distributions of the weight on the focal set.

A convex set of probability measures is generated by a random set $(\mathcal{F}, m)$ as follows:

$$\mathcal{K}(\mathcal{F}, m) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\mathcal{K}(F^i) = \left\{ P : P = \sum_{i=1}^{|\mathcal{F}|} m(F^i)P^i, P^i \in \mathcal{K}(F^i) \right\}. \tag{1}$$

Upper probability: $\overline{P}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \; \sup\{P^i(A) : \; P^i \in \mathcal{K}(F^i)\}$.

2.3 Sets of parameterized probability measures

- Generation of a set $\mathcal{K}$ of probability measures by a probability measure $p^{\,\theta}$ on Ω which is parameterized by an uncertain parameter $\theta \in \Theta$.
- Modelling the uncertainty of the parameter $\theta \in \Theta$ by a set $\mathfrak{K}$ of probability measures on Θ.
- Generation of $\mathfrak{K}$ by random sets.

The set $\mathcal{K}$ is defined by

$$\mathcal{K} := \mathcal{K}(\mathfrak{K}, p^{\,\theta}) := \left\{ P = \int_\Theta p^{\,\theta}(\cdot)\mu(\mathrm{d}\theta) : \; \mu \in \mathfrak{K} \right\}$$

Upper probability for a set A:

$$\overline{P}(A) = \sup\{P(A) : \; P \in \mathcal{K}\} = \sup_{\mu \in \mathfrak{K}} \int_\Theta p^{\,\theta}(A)\, \mu(\mathrm{d}\theta).$$

$\mathcal{K}$ is always a set of probability measures on Ω and $\mathfrak{K}$ a set of probability measures on the parameter space Θ.

2.4 Generation of $\mathfrak{K}$ by probability measures μ by sets F, $\mathfrak{K} := \mathfrak{K}(F)$

The upper probabilities are given by

$$\overline{P}(A) = \sup_{\mu \in \mathfrak{K}(F)} \int_\Theta p^{\,\theta}(A)\mu(\mathrm{d}\theta) = \sup_{\theta_0 \in F} \int_\Theta p^{\,\theta}(A)\delta_{\theta_0}(\mathrm{d}\theta) = \sup_{\theta_0 \in F} p^{\theta_0}(A).$$

Special case $\Theta := \Omega$ and $p^{\,\omega} := \delta_\omega$: Then $\mathcal{K}(F) = \mathcal{K}(\mathfrak{K}(F), \delta_\omega)$, because

$$\mathcal{K}(\mathfrak{K}(F), \delta_\omega) = \left\{ P = \int_\Omega \delta_\omega(\cdot)\mu(\mathrm{d}\omega) : \mu \in \mathfrak{K}(F) \right\} = \{\mu \in \mathfrak{K}(F)\} = \mathcal{K}(F).$$

2.5 Generation of $\mathfrak{K}$ by random sets, $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$

Here $\mathfrak{K} := \mathfrak{K}(\mathcal{F}, m)$ and $\mathcal{K} := \mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\,\theta})$. A probability measure $P \in \mathcal{K}$ is written as follows:

$$P = \int_\Theta p^{\,\theta}(\cdot)\; \mu(\mathrm{d}\theta) = \int_\Theta p^{\,\theta}(\cdot) \left(\sum_{i=1}^{|\mathcal{F}|} m(F^i)\; \mu^i(\mathrm{d}\theta) \right) =$$
$$= \sum_{i=1}^{|\mathcal{F}|} m(F^i) \int_\Theta p^{\,\theta}(\cdot)\; \mu^i(\mathrm{d}\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) P^i$$

where $\mu \in \mathfrak{K}(\mathcal{F}, m)$. $\mu = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\mu^i$ is a decomposition of μ according to the focal sets and $P^i = \int_\Theta p^{\,\theta}(\cdot)\mu^i(\mathrm{d}\theta)$ is a probability measure in $\mathcal{K}(\mathfrak{K}(F^i), p^{\,\theta})$. So for the set $\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\,\theta})$ we also can write

$$\mathcal{K}(\mathfrak{K}(\mathcal{F}, m), p^{\,\theta}) = \sum_{i=1}^{|\mathcal{F}|} m(F^i)\; \mathcal{K}(\mathfrak{K}(F^i), p^{\,\theta}) \tag{2}$$

which is Eq. (1) but with $\mathcal{K}(F^i)$ replaced by $\mathcal{K}(\mathfrak{K}(F^i), p^{\,\theta})$.

Similar to the Section above we have for the upper probability:

$$\overline{P}(A) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\mu^i \in \mathfrak{K}(F^i)} \int_\Theta p^{\,\theta}(A)\mu^i(\mathrm{d}\theta) = \sum_{i=1}^{|\mathcal{F}|} m(F^i) \sup_{\theta_0 \in F^i} p^{\,\theta_0}(A).$$

3 General formulation of the generation of sets of joint probability measures by random sets

Let random sets $(\mathcal{F}_k, m_k)$, $k = 1, 2$, be given for modelling the uncertainty of the variables x_1 and x_2. As a consequence of Dempster's rule of combination the joint random set $(\mathcal{F}, m)$ is defined by

$$\mathcal{F} = \{F^{ij} : \; i = 1,\dots,n_1; \; j = 1,\dots,n_2\}$$

where $F^{ij} := F_1^i \times F_2^j$ and $m(F_1^i \times F_2^j) := m_1(F_1^i)m_2(F_2^j)$ which is the case of random set independence (RS-independence).

For our more general approach we start with the multivariate analogon of Eq. (1) or Eq. (2):

$$\mathcal{K}_? = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_?(F_1^i \times F_2^j)\mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

where the question marks indicates the possibility of different choices:

1. Choice of the joint weights.
2. Choice in combining the sets of probability measures $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ associated with the marginal focal sets F_1^i and F_2^j.
3. Possible interactions between probability measures in the different sets $\mathcal{K}_?(\mathcal{K}_1^i, \mathcal{K}_2^j)$

The consequences of these different choices are different sets of joint probability measures $\mathcal{K}_?$ and the goal is to generate sets according to strong independence (S), unknown interaction (U) and RS-independence (R).

3.1 The choice of the joint weights $m(F_1^i \times F_2^j)$

Case (S−−): Stochastic independence: $m(F_1^i \times F_2^j) := m_1(F_1^i)m_2(F_2^j)$.

Case (U−−): Unknown interaction, m must satisfy the following conditions:

$$m_1(F_1^i) = \sum_{j=1}^{|\mathcal{F}_2|} m(F_1^i \times F_2^j), \quad i = 1, \dots, |\mathcal{F}_1|,$$

$$m_2(F_2^j) = \sum_{i=1}^{|\mathcal{F}_1|} m(F_1^i \times F_2^j), \quad j = 1, \dots, |\mathcal{F}_2|.$$

3.2 The choice of P^{ij}, $\mathcal{K}^{ij}$, respectively

$P^{ij} \in \mathcal{K}^{ij}$ is a probability measure associated to the joint focal set $F_1^i \times F_2^j$. How a P^{ij} looks like depends on how $\mathcal{K}^{ij}$ is constructed from $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$.

Case (−U−): $\mathcal{K}_{\mathrm{U}}^{ij} := \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j))$ which is the set of all joint probability measures generated by the sets $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ according to condition (U).

Case (−S−): $\mathcal{K}_{\mathrm{S}}^{ij} := \mathcal{K}_{\mathrm{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$ which is the set generated according to strong independence (S).

3.3 The choice of interactions between the P^{ij}

Case (−−1): Row- and columnwise equality conditions on the marginals of the probability measures on the joint focal sets:

$$P_1^i := P_1^{i,i1} = \dots = P_i^{i,in_2}, \; i = 1, \dots, n_1,$$
$$P_2^j := P_2^{j,1j} = \dots = P_i^{j,n_1 j}, \; j = 1, \dots, n_2$$

where

$$P_1^{i,ik} = P_1^{ik}(\cdot \times \Omega_2) \text{ and } P_2^{j,kj} = P_2^{kj}(\Omega_1 \times \cdot).$$

Case (−−0): No interactions, this means that we can choose a $P^{ij} \in \mathcal{K}^{ij}$ on $F_1^i \times F_2^j$ irrespective of the probability measures chosen on other joint focal sets.

4 The different cases

Now we will discuss combinations of the above cases which lead to random set independence (R), unknown interaction (U) and strong independence (S). The cases are indicated by indices of the form (ABC) where for example (SU0) means m according (S−−), P^{ij} according to (−U−) and no interaction between the P^{ij}, (−−0).

4.1 (SU0), (SS0) and RS-independence

4.1.1 General formulation

The sets $\mathcal{K}_{\mathrm{SU0}}$ and $\mathcal{K}_{\mathrm{SS0}}$ of joint probability measures are generated by

$$\mathcal{K}_{\mathrm{SU0}} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$$

$$\mathcal{K}_{\mathrm{SS0}} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j) m_2(F_2^j) \mathcal{K}_{\mathrm{S}}(\mathcal{K}_1^i, \mathcal{K}_2^j).$$

4.1.2 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

We obtain the upper probability $\overline{P}_{\mathrm{SU0}}(A)$ for a set A by

$$\overline{P}_{\mathrm{SU0}}(A) = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \overline{P}_{\mathrm{U}}^{ij}(A)$$

where

$$\overline{P}_{\mathrm{U}}^{ij}(A) = \sup\left\{P_{\mathrm{U}}^{ij}(A) : \; P_{\mathrm{U}}^{ij} \in \mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1^i), \mathcal{K}(F_2^j))\right\}$$

and

$$\mathcal{K}_{\mathrm{U}}(\mathcal{K}(F_1^i), \mathcal{K}(F_2^j)) = \mathcal{K}(F_1^i \times F_2^j).$$

So $\overline{P}_{\mathrm{U}}^{ij}(A)$ is computed very easily by

$$\overline{P}_{\mathrm{U}}^{ij}(A) = \sup\{\delta_\omega(A) : \; \omega \in F_1^i \times F_2^j\} = \begin{cases} 1 & \exists\, \omega \in A \cap F_1^i \times F_2^j, \\ 0 & \text{else,} \end{cases}$$

which leads to the formula for the joint plausibility measure

$$\overline{P}_{\mathrm{R}}(A) := \overline{P}_{\mathrm{SU0}}(A) = \mathrm{Pl}(A) = \sum_{i,j:\; F_1^i \times F_2^j \cap A \neq \varnothing} m_1(F_1^i) m_2(F_2^j)$$

which is the joint upper probability in the case of RS-independence. Further we have $\overline{P}_{\mathrm{SU0}} = \overline{P}_{\mathrm{SS0}}$ because δ_ω is a product measure (case (−S−)).

For the corresponding sets of joint probability measures we have

$$\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{SU0}} \supseteq \mathcal{K}_{\mathrm{SS0}}.$$

4.1.3 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

An idea would be to define $\mathcal{K}_{\mathrm{R}}$ by $\mathcal{K}_{\mathrm{SU0}}$ as before.

Problem: The set $\mathcal{K}_{\mathrm{U}}(\mathcal{K}_1^i, \mathcal{K}_2^j)$ cannot be described by a set $\mathcal{K}(\mathfrak{K}, p^{\,\theta})$ because there is not only one joint probability measure $p^{\,\theta}$, but a set of all possible joint probability measures determined by $p_1^{\theta_1}$ and $p_2^{\theta_2}$. So another possibility would be to define $\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{S(US)0}}$ where (US) means that we have unknown interaction (U) for the parameters only and a product measure $p_1^{\theta_1} \otimes p_2^{\theta_2}$ (S).

We start with the case of (SS0) and get

$$\mathcal{K}_{\mathrm{SS0}} = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j)\, \mathcal{K}_{\mathrm{S}}^{ij} \subseteq \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \mathcal{K}_{\mathrm{(US)}}^{ij} =$$
$$= \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \mathcal{K}\big(\mathfrak{K}(F_1^i \times F_2^j), p_1^{\theta_1} \otimes p_2^{\theta_2}\big) =$$
$$= \mathcal{K}\big(\mathfrak{K}(\mathcal{F}, m), p_1^{\theta_1} \otimes p_2^{\theta_2}\big) =: \mathcal{K}_{\mathrm{S(US)0}} =: \mathcal{K}_{\mathrm{R}},$$

with $\mathcal{K}_{\mathrm{S}}^{ij} := \mathcal{K}_{\mathrm{S}}\big(\mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1}), \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})\big)$ and $\mathcal{K}_{\mathrm{(US)}}^{ij} := \mathcal{K}\big(\mathfrak{K}(F_1^i \times F_2^j), p_1^{\theta_1} \otimes p_2^{\theta_2}\big)$ and Eq. (2).

4.2 (UU0), (US0) and unknown interaction

4.2.1 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

Similar to RS-independence, but (U−−), the joint weights have to be determined by solving a linear optimization problem subject to condition (U−−).

Results: $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{\mathrm{UU0}} \supseteq \mathcal{K}_{\mathrm{US0}}$ and $\overline{P}_{\mathrm{UU0}} = \overline{P}_{\mathrm{US0}}$.

4.2.2 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

Unfortunately we do not have $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{\mathrm{UU0}}$ in general.

4.3 The case (SS1), strong independence

4.3.1 $\mathcal{K}_1^i := \mathcal{K}(F_1^i)$, $\mathcal{K}_2^j := \mathcal{K}(F_2^j)$

We write a probability measure $P_{\mathrm{SS1}} \in \mathcal{K}_{\mathrm{SS1}}$ in the following way:

$$P_{\mathrm{SS1}}(A) = \sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j)\, (P_1^i \otimes P_2^j)(A) =$$
$$= \left(\sum_{i=1}^{|\mathcal{F}_1|} m_1(F_1^i)\, P_1^i\right) \otimes \left(\sum_{j=1}^{|\mathcal{F}_2|} m_2(F_2^j)\, P_2^j\right)(A) = (P_1 \otimes P_2)(A) = P_{\mathrm{S}}(A)$$

with $P_1 \in \mathcal{K}(\mathcal{F}_1, m_1)$ and $P_2 \in \mathcal{K}(\mathcal{F}_2, m_2)$. This leads to $\mathcal{K}_{\mathrm{SS1}} = \mathcal{K}_{\mathrm{S}}$.

Computational method: The upper probability $\overline{P}_{\mathrm{S}}(A)$ is the solution of the following optimization problem:

$$\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j)\; \chi_A(\omega_1^i, \omega_2^j) \; = \max!$$

subject to $\omega_1^i \in F_1^i$ and $\omega_2^j \in F_2^j$. χ_A is the indicator function of the set A.

4.3.2 $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$, $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$

Computational method: We get the following optimization problem for the computation of $\overline{P}_{\mathrm{S}}(A)$:

$$\sum_{i=1}^{|\mathcal{F}_1|} \sum_{j=1}^{|\mathcal{F}_2|} m_1(F_1^i) m_2(F_2^j) \; \left(p_1^{\theta_1^i} \otimes p_2^{\theta_2^j}\right)(A) \; = \max!$$

subject to $\theta_1^i \in F_1^i$ and $\theta_2^j \in F_2^j$. This leads again to $\mathcal{K}_{\mathrm{SS1}} = \mathcal{K}_{\mathrm{S}}$.

5 Summary

Relations between the sets of probability measures and between the upper probabilities for $\mathcal{K}_1^i := \mathcal{K}(\mathfrak{K}(F_1^i), p_1^{\theta_1})$ and $\mathcal{K}_2^j := \mathcal{K}(\mathfrak{K}(F_2^j), p_2^{\theta_2})$.

$\mathcal{K}_{\mathrm{U}} \supseteq \mathcal{K}_{\mathrm{UU0}}$	$\supseteq$	$\mathcal{K}_{\mathrm{U(US)0}}$	$\supseteq$	$\mathcal{K}_{\mathrm{US0}}$
$\cup\!\shortmid$		$\cup\!\shortmid$		$\cup\!\shortmid$
$\mathcal{K}_{\mathrm{SU0}}$	$\supseteq$	$\mathcal{K}_{\mathrm{R}} := \mathcal{K}_{\mathrm{S(US)0}}$	$\supseteq$	$\mathcal{K}_{\mathrm{SS0}}$
				$\cup\!\shortmid$
				$\mathcal{K}_{\mathrm{S}} = \mathcal{K}_{\mathrm{SS1}}$

$\overline{P}_{\mathrm{U}} \geq \overline{P}_{\mathrm{UU0}}$	$\geq$	$\overline{P}_{\mathrm{U(US)0}}$	$=$	$\overline{P}_{\mathrm{US0}}$
$\vee\!\shortmid$		$\vee\!\shortmid$		$\vee\!\shortmid$
$\overline{P}_{\mathrm{SU0}}$	$\geq$	$\overline{P}_{\mathrm{R}} := \overline{P}_{\mathrm{S(US)0}}$	$=$	$\overline{P}_{\mathrm{SS0}}$
				$\vee\!\shortmid$
				$\overline{P}_{\mathrm{S}} = \overline{P}_{\mathrm{SS1}}$