Sets of Joint Probability Measures Generated by Weighted Marginal Focal Sets

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Abstract

This paper is devoted to the construction of sets of joint probability measures for the case that the marginal sets of probability measures are generated by weighted focal sets. Different conditions on the choice of the weights of the joint focal sets and on the probability measures on these sets lead to different types of independence such as strong independence, random set independence, fuzzy set independence and unknown interaction. As an application the upper probabilities of failure of a beam are computed.

Keywords. Weighted focal sets, possibility measures, plausibility measures, lower and upper probabilities, sets of probability measures.

1 Introduction

Precise probability theory alone often does not suffice for modeling the uncertainties arising in civil engineering problems such as the reliability analysis of structures and much more in soil mechanics. One of the most difficult problems here is to analyze the behavior of the soil or rock during the construction of a tunnel where the soil properties are only very imprecisely known.

The goal is to have practicable measures for the risk of failure in the case where the material properties are not or only partly given by precise values or probability measures. One should also have the possibility to assess subjective knowledge and expert estimates. It is usually easy for the planning engineer to provide such information by using weighted focal sets to model the fluctuations of the parameters involved. In most cases intervals are used for the focal sets which has the advantage that the computations can be performed by interval analysis [12]. This leads, if the intervals are nested, to fuzzy numbers and possibility measures or, if not nested, to plausibility measures and evidence theory. Fuzzy numbers or possibility measures [17, 18] are used in [3, 4, 5, 9, 10, 11]. Plausibility measures [17] are used in [6, 14, 15]. Using the more general plausibility measure has the advantage that we can mix e.g. fuzzy numbers with histograms or probability measures directly without transforming the probability measures into fuzzy numbers and neglecting information.

It is often more practicable to interpret these measures as upper probabilities as done in [6, 14, 15]. It is easy to do that if one can start the computations with given weighted joint focal sets, c.f. [6].

But in many cases the weighted focal sets are given only for each uncertain parameter separately. If the marginal focal sets are nested, we have a joint possibility measure and a joint plausibility measure. Unfortunately these two measures are not the same in general, which leads to ambiguities in interpreting both measures as upper probabilities.

The plan of this paper is as follows:

Section 2 is devoted to weighted focals sets and to the notation we will use. In Section 3 we present a civil engineering example with two uncertain parameter separately given by weighted focal sets.

In Section 4 we construct sets of joint probability measures by means of weighted joint focal sets. We list different conditions on choosing the weights of the joint focal sets and the probability measures on these sets. Depending on these conditions we get different sets of joint probability measures and different types of independence, respectively. We show that some of these cases lead to types of independence as described in [1] such as strong independence, random set independence and unknown interaction. Further we investigate how the joint possibility measure fit into this scheme, if the marginal focal sets are nested. For each discussed case the upper probability of failure of the beam given in Section 3 will be computed.

In the last section we show in a summary how the sets of joint probability measures and the upper probabilities are related to each other.

2 Sets of marginal probability measures generated by weighted focal sets

We consider two uncertain values or parameters λ_1 and λ_2. The possible realizations ω_k of an uncertain parameter λ_k belong to a measurable space $(\Omega_k, \mathcal{C}_k)$ with σ-algebra $\mathcal{C}_k$.

Here the uncertainty of a parameter λ_k is always modeled by a finite class $\mathcal{A}_k = \{A_k^1, \dots, A_k^{n_k}\} \subseteq \mathcal{C}_k$ of weighted focal sets or random sets. These focals are weighted by a map

$$m_k : \mathcal{A}_k \longrightarrow [0,1] : A \mapsto m_k(A)$$

with $\sum_{A \in \mathcal{A}_k} m_k(A) = 1$.

Then the upper probability or plausibility measure of a set $C_k \in \mathcal{C}_k$ is defined by

$$\overline{P}_k(C_k) = \mathrm{Pl}_k(C_k) = \sum_{A_k^i \cap C_k \neq \varnothing} m_k(A_k^i) \qquad (1)$$

and the lower probability or belief measure by

$$\underline{P}_k(C_k) = \mathrm{Bel}_k(C_k) = \sum_{A_k^i \subseteq C_k} m_k(A_k^i). \qquad (2)$$

If the focal sets are nested, e.g. $A_k^1 \supseteq A_k^2 \supseteq \cdots \supseteq A_k^{n_k}$, then the above plausibility measure is a possibility measure Pos_k with possibility density $\mu_k(\omega_k) = \mathrm{Pos}_k(\{\omega_k\})$ which is also the membership function of the corresponding fuzzy number.

The goal of this paper is to construct joint measures from marginals which are given by weighted focal sets. To do this we must know how the upper probability and lower probability can be obtained using sets of probability measures.

Therefore let $\mathcal{K}_k^i = \{P_k^i : P_k^i(A_k^i) = 1\}$ be the set of probability measures P_k^i "on" the corresponding focal set A_k^i and

$$\mathcal{K}_k = \left\{ P_k = \sum_{A_k^i \in \mathcal{A}_k} m_k(A_k^i) P_k^i : \ P_k^i \in \mathcal{K}_k^i \right\}$$

be the set of probability measures for the uncertain parameter λ_k generated by the weighted focal sets $A_k^1, \dots, A_k^n$.

Then the upper probability $\overline{P}_k(C_k)$ is obtained by solving the following optimization problem:

$$\begin{aligned} \overline{P}_k(C_k) &= \max\{P_k(C_k) : P_k \in \mathcal{K}_k\} \\ &= \sum_{A_k^i \in \mathcal{A}_k} m_k(A_k^i) P_k^{i*}(C_k) \end{aligned}$$

with certain $P_k^{i*} \in \mathcal{K}_k^i$.

Such an optimal P_k^{i*} can be chosen in the following way: $P_k^{i*} = \delta_{\omega_k^{i*}}$ with

$$\omega_k^{i*} \in \begin{cases} A_k^i \cap C_k & \text{if } A_k^i \cap C_k \neq \varnothing \\ A_k^i \text{ arbitrary} & \text{if } A_k^i \cap C_k = \varnothing. \end{cases}$$

δ_{ω_k} is the Dirac measure at $\omega_k \in \Omega_k$. Then $P_k^{i*}(C_k) = 1$ for $A_k^i \cap C_k \neq \varnothing$ and 0 otherwise which leads to the same result as in the defining formula (1).

The lower probability $\underline{P}_k(C_k)$ is obtained by:

$$\begin{aligned} \underline{P}_k(C_k) &= \min\{P_k(C_k) : P_k \in \mathcal{K}_k\} \\ &= \sum_{A_k^i \in \mathcal{A}_k} m_k(A_k^i) \delta_{\omega_k^{i**}}(C_k) \end{aligned}$$

with

$$\omega_k^{i**} \in \begin{cases} A_k^i \setminus C_k & \text{if } A_k^i \not\subseteq C_k \\ A_k^i \text{ arbitrary} & \text{otherwise.} \end{cases}$$

Then $\delta_{\omega_k^{i**}}(C_k) = 1$ for $A_k^i \subseteq C_k$ and 0 otherwise which leads to the same result as in formula (2).

3 Numerical Example

As a numerical example we consider a beam supported on both ends and additionally bedded on two springs, see Fig 1. The values of the beam rigidity $EI = 10$ kNm^2 and of the equally distributed load $f(x) = 1$ kN/m are assumed to be deterministic. But the values of the two spring constants λ_1 and λ_2 are uncertain.

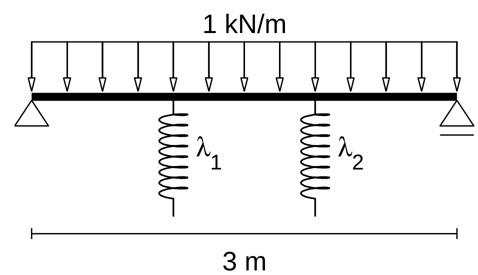

Figure 1: Beam bedded on two springs.

The uncertainty about the possible fluctuations of the spring constants λ_1 and λ_2 is modeled by the same three focal sets $A_k^1 = [20, 40]$, $A_k^2 = [30, 40]$ and $A_k^3 = \{30\}$, $k = 1, 2$ with weights $m_k(A_k^1) = 0.2$, $m_k(A_k^2) = 0.3$ and $m_k(A_k^3) = 0.5$, see Fig 2. The measurable spaces are $(\Omega_1, \mathcal{C}_1) = (\Omega_2, \mathcal{C}_2) = (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.

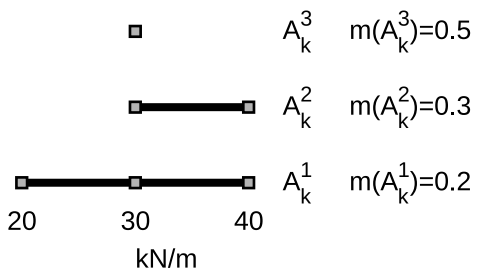

Figure 2: Uncertain spring constants λ_k.

We want to compute measures for the risk of failure of the beam. The criterion of failure is here

$$\max_{x \in [0,3]} |M(x)| \geq M_f,$$

where $M(x)$ is the bending moment at a point $x \in [0, 3]$ and M_f the moment of failure.

The bending moment M also depends on the two spring constants. We define a map

$$\mathcal{M} : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} : (\omega_1, \omega_2) \mapsto \max_{x \in [0,3]} |M(x, \omega_1, \omega_2)|,$$

which is the maximal bending moment of the beam depending on values ω_1 and ω_2 for the two spring constants. $M(x, \omega_1, \omega_2)$ is computed by the finite element method [7, 8, 13] for fixed parameter values ω_1 and ω_2.

4 Sets of joint probability measures

4.1 Preliminary definitions

Let $(\Omega, \mathcal{C})$ be the product measurable space with $\Omega = \Omega_1 \times \dots \times \Omega_n$ and σ-algebra $\mathcal{C} = \mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_n$. We want to write the set $\mathcal{K}$ of all joint probability measures which are generated by the marginal sets $\mathcal{K}_1$ and $\mathcal{K}_2$ as

$$\mathcal{K} = \left\{ P = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m(A_1^i \times A_2^j) P^{ij} \right\}.$$

$A_1^i \times A_2^j$ is a joint focal set in $\mathcal{A}_1 \times \mathcal{A}_2$ and P^{ij} is a probability measure on $A_1^i \times A_2^j$, which means again $P^{ij}(A_1^i \times A_2^j) = 1$ for all $i = 1, \dots, n_1$ and $j = 1, \dots, n_2$.

The marginals of P^{ij} are probability measures on A_1^i and A_2^j. We denote them by $P_1^{i,ij} \in \mathcal{K}_1^i$ and $P_2^{j,ij} \in \mathcal{K}_2^j$, see Fig. 3.

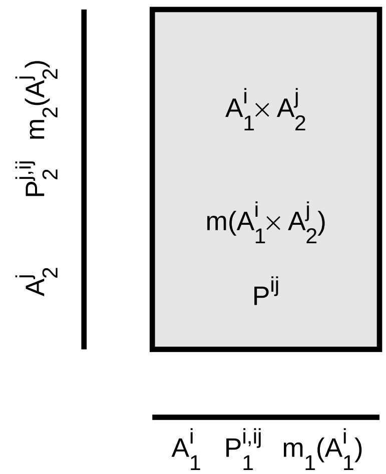

Figure 3: Focal set $A_1^i \times A_2^j$.

We have several possibilities to choose the weights $m(A_1^i \times A_2^j)$, the probability measures P^{ij}, and their marginals $P_1^{i,ij}$ and $P_2^{j,ij}$, respectively.

4.2 The choice of $m(A_1^i \times A_2^j)$

Case 1: The joint focal sets $A_1^i \times A_2^j$ are chosen in a stochastically independent way. Then we get

$$m(A_1^i \times A_2^j) = m_1(A_1^i) m_2(A_2^j)$$

for the weights of the joint focals.

Case 2: If there is no information on how to choose the joint focal sets we allow arbitrary weights $m(A_1^i \times A_2^j)$, but with the restriction that

$$m_1(A_1^i) = \sum_{j=1}^{n_2} m(A_1^i \times A_2^j)$$

and

$$m_2(A_2^j) = \sum_{i=1}^{n_1} m(A_1^i \times A_2^j)$$

must hold.

If the marginal focal sets are nested we also will use a special correlation of these weights which leads to the joint possibility measure.

4.3 The choice of P^{ij}

Case A: The measures P^{ij} on the joint focals $A_1^i \times A_2^j$ are chosen as product measures

$$P^{ij} = P_1^{i,ij} \otimes P_2^{j,ij}$$

with $P_1^{i,ij} \in \mathcal{K}_1^i$ and $P_2^{j,ij} \in \mathcal{K}_2^j$.

Case B: Now arbitrary dependencies are allowed for the measures on $A_1^i \times A_2^j$. Then the only restrictions on the P^{ij} are:

$$P^{ij}(\cdot \times A_2^j) = P_1^{i,ij} \text{ and } P^{ij}(A_1^i \times \cdot) = P_2^{j,ij}.$$

4.4 The choice of $P_1^{i,ij}$ and $P_2^{j,ij}$

Case a: We use always the same marginal probability measures in $\mathcal{K}_1^i$ and $\mathcal{K}_2^j$ respectively. We denote this by:

$$P_1^i := P_1^{i,i1} = P_1^{i,i2} = \cdots = P_1^{i,in_2}$$

and

$$P_2^j := P_2^{j,1j} = P_2^{j,2j} = \cdots = P_2^{j,n_1j}.$$

Case b: We allow arbitrary marginal probability measures $P_1^{i,ij} \in \mathcal{K}_1^i$ and $P_2^{j,ij} \in \mathcal{K}_2^j$ respectively.

4.5 Case 1Aa

Let $\mathcal{K}_{1\text{Aa}}$ be the set of all probability measures generated according to case 1Aa. A probability measure $P \in \mathcal{K}_{1\text{Aa}}$ is written for a set $C \in \mathcal{C}$ as

$$\begin{aligned} P(C) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m(A_1^i \times A_2^j)P^{ij}(C) = \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m_1(A_1^i)m_2(A_2^j)(P_1^i \otimes P_2^j)(C) = \\ &= \left(\sum_{i=1}^{n_1} m_1(A_1^i)P_1^i\right) \otimes \left(\sum_{j=1}^{n_2} m_2(A_2^j)P_2^j\right)(C) = \\ &= (P_1 \otimes P_2)(C) \end{aligned}$$

with $P_1 \in \mathcal{K}_1$ and $P_2 \in \mathcal{K}_2$. So we have $\mathcal{K}_{1\text{Aa}} = \{P_1 \otimes P_2 : P_1 \in \mathcal{K}_1, P_2 \in \mathcal{K}_2\}$.

This is the case of *strong independence* or *type-1 extension*, cf. [1, 16] where the outcomes of two uncertain parameters are always stochastically independent. We denote this set by $\mathcal{K}_\text{S}$ and the upper and lower probability by $\overline{P}_\text{S}$ and $\underline{P}_\text{S}$ respectively.

We introduce the following computational method to obtain $\overline{P}_\text{S}(C)$ and $\underline{P}_\text{S}(C)$. $\overline{P}_\text{S}(C)$ ($\underline{P}_\text{S}(C)$) is the optimal value of the objective function of the optimization problem

$$\begin{aligned} &\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m(A_1^i \times A_2^j)(\delta_{\omega_1^i} \otimes \delta_{\omega_2^j})(C) = \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m(A_1^i \times A_2^j) I_C(\omega_1^i, \omega_2^j) = \max\ (\min) \end{aligned}$$

subject to

$$\begin{aligned} \omega_1^i &\in A_1^i,\ i = 1, \ldots, n_1 \\ \omega_2^j &\in A_2^j,\ j = 1, \ldots, n_2 \end{aligned}$$

where I_C is the indicator function of the set C. Note that the objective function takes only a finite set of values.

For our example we have the set

$$C = \{(\omega_1, \omega_2) \in \Omega : \mathcal{M}(\omega_1, \omega_2) \geq M_f\}.$$

The upper probability $\overline{P}_\text{S}(\mathcal{M} \geq M_f)$ of failure for strong independence is depicted in Fig. 4 as a function of M_f.

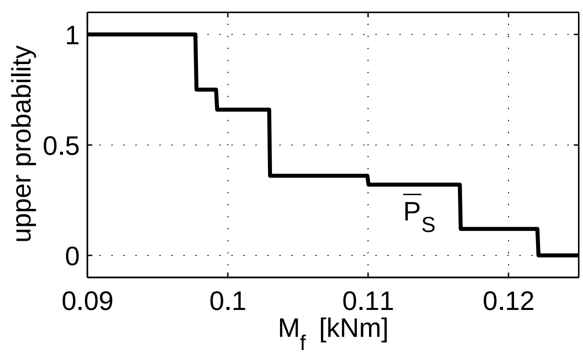

Figure 4: Upper probability of failure $\overline{P}_\text{S}(\mathcal{M} \geq M_f)$.

4.6 Case 1Bb

Let $\mathcal{K}_{1\text{Bb}}$ be the set of all probability measures generated according to case 1Bb. A probability measure $P \in \mathcal{K}_{1\text{Bb}}$ is written as

$$\begin{aligned} P(C) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m(A_1^i \times A_2^j)P^{ij}(C) = \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} m_1(A_1^i)m_2(A_2^j)P^{ij}(C) \end{aligned}$$

with

$$P^{ij}(\cdot \times A_2^j) = P_1^{i,ij} \in \mathcal{K}_1^i$$

and

$$P^{ij}(A_1^i \times \cdot) = P_2^{j,ij} \in \mathcal{K}_2^j.$$

Here the sets $A_1^i \times A_2^j$ are selected stochastically independent, but for the measures on $A_1^i \times A_2^j$ dependent selections are allowed. This is the case of *random set independence* [1, 2]. Here we use the notation $\mathcal{K}_\text{R}$, $\overline{P}_\text{R}$ and $\underline{P}_\text{R}$.

The upper probability $\overline{P}_\text{R}(C)$ is obtained by

$$\overline{P}_\text{R}(C) = \text{Pl}(C) = \sum_{A_1^i \times A_2^j \cap C \neq \varnothing} m_1(A_1^i)m_2(A_2^j),$$

which is the formula for the joint plausibility measure. Alternatively it is given as in the one-dimensional case by

$$\overline{P}_{\mathrm{R}}(C) = \max\{P(C) : P \in \mathcal{K}_{\mathrm{R}}\} = \\ = \sum_{A_1^i \times A_2^j} m_1(A_1^i) m_2(A_2^j) P^{ij*}(C)$$

where the P^{ij*} are again Dirac measures on $A_1^i \times A_2^j$. Then we also have $\overline{P}_{1\mathrm{Bb}}(C) = \overline{P}_{1\mathrm{Ab}}(C)$, because for Dirac measures the condition in case A holds. But $\mathcal{K}_{1\mathrm{Ab}}$ is a subset of $\mathcal{K}_{1\mathrm{Bb}}$.

The lower probability $\underline{P}_{\mathrm{R}}(C)$ is the joint belief measure Bel. Here also $\underline{P}_{1\mathrm{Bb}}(C) = \underline{P}_{1\mathrm{Ab}}(C)$ holds.

Computational method to obtain $\overline{P}_{\mathrm{R}}$ for our example:

$$\overline{P}_{\mathrm{R}}(C) = \sum_{A_1^i \times A_2^j \cap C \neq \varnothing} m(A_1^i \times A_2^j) = \\ = \sum_{\mathcal{M}(A_1^i \times A_2^j) \cap [M_f, \infty) \neq \varnothing} m(A_1^i \times A_2^j) = \\ = \sum_{\mathcal{M}_R \geq M_f} m(A_1^i \times A_2^j)$$

with

$$\mathcal{M}_R = \max_{(\omega_1, \omega_2) \in A_1^i \times A_2^j} \mathcal{M}(\omega_1, \omega_2).$$

The upper probability of failure for random set independence is depicted in Fig. 5.

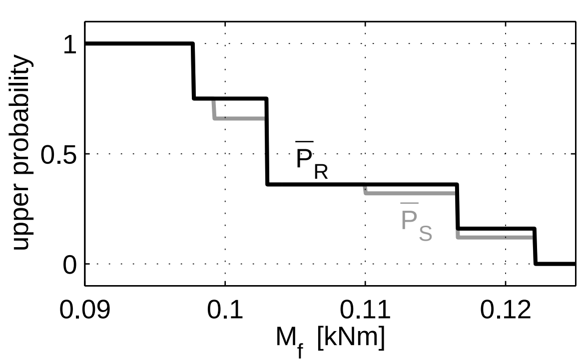

Figure 5: Upper probability of failure $\overline{P}_{\mathrm{R}}(\mathcal{M} \geq M_f)$.

$\overline{P}_{\mathrm{R}}(C)$ is always greater than or equal to $\overline{P}_{\mathrm{S}}(C)$ (and $\mathcal{K}_{\mathrm{S}} \subseteq \mathcal{K}_{\mathrm{R}}$), because the conditions for the P^{ij} are less restrictive.

4.7 Case 2Bb

Let $\mathcal{K}_{2\mathrm{Bb}}$ be the set of probability measures generated according to case 2Bb. Then we have

$$\overline{P}_{2\mathrm{Bb}}(C) = \max\{P(C) : P \in \mathcal{K}_{2\mathrm{Bb}}\} = \\ = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m^{ij*}(A_1^i \times A_2^j) P^{ij*}(C) = \\ = \sum_{A_1^i \times A_2^j \cap C \neq \varnothing} m^{ij*}(A_1^i \times A_2^j)$$

where P^{ij*} is an appropriate Dirac measure as for $\overline{P}_{\mathrm{R}}$ and where m^{ij*} is the solution of the following optimization problem:

$$\sum_{A_1^i \times A_2^j \cap C \neq \varnothing} m^{ij*}(A_1^i \times A_2^j) = \max$$

subject to

$$m_1(A_1^i) = \sum_{j=1}^{n_2} m(A_1^i \times A_2^j) \quad (3)$$

$$m_2(A_2^j) = \sum_{i=1}^{n_1} m(A_1^i \times A_2^j). \quad (4)$$

$\mathcal{K}_{2\mathrm{Bb}}$ is the set of probability measures generated by the least restrictive conditions on m and P^{ij}. We will show that

$$\mathcal{K}_{2\mathrm{Bb}} = \mathcal{K}_{\mathrm{U}} := \{P : P(\cdot \times \Omega_2) \in \mathcal{K}_1,\ P(\Omega_1 \times \cdot) \in \mathcal{K}_2\}$$

holds.

$\mathcal{K}_{\mathrm{U}}$ is the set of joint probability measures whose marginal probability measures belong to $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. In this case the interactions between the two marginals are completely unknown [1]. The following theorem will show us that every $P_{\mathrm{U}} \in \mathcal{K}_{\mathrm{U}}$ belongs also to $\mathcal{K}_{2\mathrm{Bb}}$.

But first we need some definitions: Let $\mathcal{B}_k = \{B_k^1, \ldots, B_k^{N_k}\}$ be a partition of $\bigcup_i A_k^i$ such that either $B_k^r \subseteq A_k^i$ or $B_k^r \cap A_k^i = \varnothing$ holds.

Example: Let $A_1^1 = [0, 2]$ and $A_1^2 = [1, 3]$. Then the partition $\mathcal{B}_1 = \{[0, 1), [1, 2], (2, 3]\}$ of $[0, 3]$ has the above property.

Further we define for convenience: $A^{ij} := A_1^i \times A_2^j$, $B^{rs} := B_1^r \times B_2^s$ and $m^{ij} := m(A_1^k \times A_2^j)$.

Theorem 1. *A probability measure*

$$P_{\mathrm{U}} \in \mathcal{K}_{\mathrm{U}} = \{P : P(\cdot \times \Omega_2) \in \mathcal{K}_1,\ P(\Omega_1 \times \cdot) \in \mathcal{K}_2\}$$

can be written as

$$P_{\mathrm{U}} = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m^{ij} P^{ij}$$

with weights

$$m^{ij} = \sum_{B^{rs} \subseteq A^{ij}} M_1^{ir} M_2^{js} P_{\mathrm{U}}(B^{rs})$$

and probability measures

$$P^{ij}(C) = \frac{1}{m^{ij}} \sum_{B^{rs} \subseteq A^{ij}} M_1^{ir} M_2^{js} P_{\mathrm{U}}(C \cap B^{rs})$$

on the focal sets A^{ij}.

The weights M_1^{ir} *and* M_2^{js} *are defined by*

$$M_1^{ir} = \frac{m_1(A_1^i) R_1^i(B_1^r)}{P_1(B_1^r)} \text{ and } M_2^{js} = \frac{m_2(A_2^j) R_2^j(B_2^s)}{P_2(B_2^s)},$$

where R_1^i *and* R_2^j *are some probability measures such that* $P_1 = P_{\mathrm{U}}(\cdot \times \Omega_2)$ *and* $P_2 = P_{\mathrm{U}}(\Omega_1 \times \cdot)$ *can be written as* $P_1 = \sum_i m_1(A_1^i) R_1^i$ *and* $P_2 = \sum_j m_2(A_2^j) R_2^j$.

In the case of $P_1(B_1^r) = 0$ *or* $P_2(B_2^s) = 0$ *the above weights can be chosen arbitrary, because then only* $P_{\mathrm{U}}(C \cap B^{rs}) = 0$ *is weighted.*

Proof. First we observe that $\sum_i M_k^{ir} = 1$ holds:

$$\sum_{i=1}^{n_1} M_k^{ir} = \sum_{i=1}^{n_1} \frac{m_1(A_1^i) R_1^i(B_1^r)}{P_1(B_1^r)} = \frac{P_1(B_1^r)}{P_1(B_1^r)} = 1.$$

Now we show that for all B^{rs}

$$\sum_i \sum_j m^{ij} P^{ij}(C \cap B^{rs}) = P_{\mathrm{U}}(C \cap B^{rs})$$

holds:

$$\begin{aligned}
&\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} m^{ij} P^{ij}(C \cap B^{rs}) = \\
&\quad = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} M_1^{ir} M_2^{js} P_{\mathrm{U}}(C \cap B^{rs}) = \\
&\quad = P_{\mathrm{U}}(C \cap B^{rs}) \left(\sum_{i=1}^{n_1} M_1^{ir} \right) \left(\sum_{j=1}^{n_2} M_2^{js} \right) \\
&\quad = P_{\mathrm{U}}(C \cap B^{rs}) \cdot 1 \cdot 1.
\end{aligned}$$

We have to prove conditions (3) and (4) as well:

$$\begin{aligned}
&\sum_{j=1}^{n_2} m^{ij} = \sum_{j=1}^{n_2} \sum_{r,s: B^{rs} \subseteq A^{ij}} M_1^{ir} M_2^{js} P_{\mathrm{U}}(B^{rs}) = \\
&\quad = \sum_{r=1}^{N_1} \sum_{s=1}^{N_2} \sum_{j: A^{ij} \supseteq B^{rs}} M_1^{ir} M_2^{js} P_{\mathrm{U}}(B^{rs}) = \\
&\quad = \sum_{r=1}^{N_1} \sum_{s=1}^{N_2} M_1^{ir} P_{\mathrm{U}}(B^{rs}) = \sum_{r=1}^{N_1} M_1^{ir} P_{\mathrm{U}}(B_1^r \times \Omega_2) = \\
&\quad = \sum_{r=1}^{N_1} \frac{m_1(A_1^i) R_1^i(B_1^r)}{P_1(B_1^r)} P_1(B_1^r) = \\
&\quad = m_1(A_1^i) \sum_{r=1}^{N_1} R_1^i(B_1^r) = m_1(A_1^i).
\end{aligned}$$

The proof of (4) is analogous.

We have shown that $\mathcal{K}_{\mathrm{U}} \subseteq \mathcal{K}_{2\mathrm{Bb}}$. Since $\mathcal{K}_{\mathrm{U}}$ is the biggest possible set of joint probability measures we get $\mathcal{K}_{\mathrm{U}} = \mathcal{K}_{2\mathrm{Bb}}$. □

Using the same arguments as in case 1Bb leads to $\overline{P}_{2\mathrm{Ab}}(C) = \overline{P}_{2\mathrm{Bb}}(C)$ and $\underline{P}_{2\mathrm{Ab}}(C) = \underline{P}_{2\mathrm{Bb}}(C)$.

Computational method to obtain $\overline{P}_{\mathrm{U}}(C)$: The set C determines the objective function. The conditions (3) and (4) are always the same and have to be generated only once.

The upper probability $\overline{P}_{\mathrm{U}}(\mathcal{M} \geq M_f)$ for unknown interaction is depicted in Fig. 6.

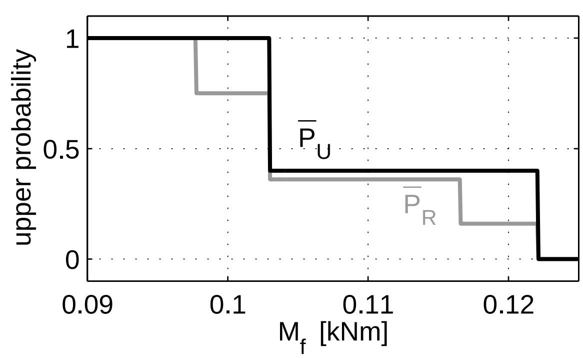

Figure 6: Upper probability of failure $\overline{P}_{\mathrm{U}}(\mathcal{M} \geq M_f)$.

4.8 The joint possibility measure

Now we want to compute the joint possibility measure given by weighted marginal focal sets.

Let $A_1^1 \supsetneq A_1^2 \supsetneq \cdots \supsetneq A_1^{n_1}$ and $A_2^1 \supsetneq A_2^2 \supsetneq \cdots \supsetneq A_2^{n_2}$ be given nested focal sets with weights $m_k^i = m_k(A_k^i)$.

Further we need points $\omega_1^1, \ldots, \omega_1^{n_1} \in \Omega_1$ and $\omega_2^1, \ldots, \omega_2^{n_2} \in \Omega_2$ with the following property

$$\omega_k^i \in \begin{cases} A_k^i \setminus A_k^{i+1} & \text{if } i < n_k, \\ A_k^{n_1} & \text{if } i = n_k. \end{cases}$$

The possibility measures of these points ω_k^i are

$$\mathrm{Pos}_k(\{\omega_k^i\}) = \sum_{s=1}^{i} m_k^s$$

and the joint possibility measure

$$\mathrm{Pos}(\{(\omega_1^i, \omega_2^j)\}) = \min\{\mathrm{Pos}_1(\{\omega_1^i\}), \mathrm{Pos}_2(\{\omega_2^j\})\}.$$

On the other hand we want to have

$$\mathrm{Pos}(\{(\omega_1^i, \omega_2^j)\}) = \sum_{(\omega_1^i,\omega_2^j)\in A_1^r \times A_2^s} m^{rs} = \sum_{r=1}^{i}\sum_{s=1}^{j} m^{rs}$$

with $m^{rs} = m(A_1^r \times A_2^s)$.

We get a system of linear equations for the weights m^{ij}:

$$\sum_{r=1}^{i}\sum_{s=1}^{j} m^{rs} = \min\left\{\sum_{r=1}^{i} m_1^r, \sum_{s=1}^{j} m_2^s\right\}$$

for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. The left hand side is a binary matrix, exactly a lower triangular matrix with ones in the diagonal if we use an appropriate numbering. So the weights of the joint focals for the joint possibility measure are uniquely determined. Then the joint possibility measure Pos for a set $C \in \mathcal{C}$ can be obtained by

$$\mathrm{Pos}(C) = \sum_{A_1^i \times A_2^j \cap C \neq \varnothing} m^{ij}$$

with the weights m^{ij} computed by the above procedure. The joint focal sets are not nested in general, but here the sets with weights $m^{ij} > 0$ are nested, because: The nested α-cuts of the density function of the joint possibility measure are among the joint focal sets. The weights of these sets, needed for the formula for Pos, can also be obtained directly from the density function. Then for these weights the above equations must also hold. Since the solution is unique they coincide with the ones computed above.

We say here that there is *fuzzy set independence* and denote the set of joint probability measures for this choice of the weights m^{ij} by $\mathcal{K}_\mathrm{F}$ and the upper and lower probability by $\overline{P}_\mathrm{F}$ and $\underline{P}_\mathrm{F}$ respectively.

Remark: If we replace min by the product on the right hand side, we get the weights for the joint plausibility measure.

For our example we get the linear system $Am = b$ with

$$A = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0\\ 1&1&1&1&1&1&0&0&0\\ 1&0&0&1&0&0&1&0&0\\ 1&1&0&1&1&0&1&1&0\\ 1&1&1&1&1&1&1&1&1 \end{pmatrix},$$

$$m = (m^{11}, m^{21}, m^{31}, m^{12}, m^{22}, m^{32}, m^{13}, m^{23}, m^{33})^T$$

and

$$b = (0.2, 0.2, 0.2, 0.5, 0.5, 0.5, 1, 1, 1)^T.$$

The solution is $m^{11} = 0.2$, $m^{22} = 0.3$, $m^{33} = 0.5$ and 0 otherwise.

The upper probability of failure $\overline{P}_\mathrm{F}(\mathcal{M} \geq M_f)$ for fuzzy set independence is depicted in Fig. 7. $\overline{P}_\mathrm{F}$ is sometimes greater and sometimes less than $\overline{P}_\mathrm{S}$ and $\overline{P}_\mathrm{R}$ respectively, but of course it is always less or equal to $\overline{P}_\mathrm{U}$.

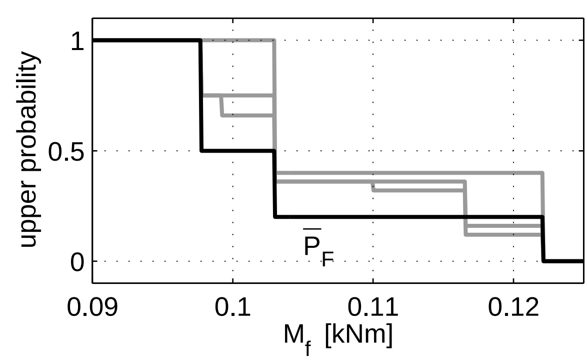

Figure 7: Upper probability of failure $\overline{P}_\mathrm{F}(\mathcal{M} \geq M_f)$.

5 Summary and Conclusion

We have investigated the five cases 1Aa, 1ba, 1Bb, 2Ba and 2Bb and get for the sets of joint probability measures the results

$$\begin{array}{ccccccc} \mathcal{K}_{1\mathrm{Aa}} & \subseteq & \mathcal{K}_{1\mathrm{Ab}} & \subseteq & \mathcal{K}_{1\mathrm{Bb}} & \subseteq & \mathcal{K}_{2\mathrm{Bb}} \\ \shortparallel & & & & \shortparallel & & \shortparallel \\ \mathcal{K}_\mathrm{S} & & & & \mathcal{K}_\mathrm{R} & & \mathcal{K}_\mathrm{U} \end{array}$$

and

$$\mathcal{K}_\mathrm{F} \subseteq \mathcal{K}_\mathrm{U},$$

where $\mathcal{K}_\mathrm{S}$ is the set for strong independence, $\mathcal{K}_\mathrm{R}$ the set for random set independence, $\mathcal{K}_\mathrm{U}$ the set for unknown interaction and $\mathcal{K}_\mathrm{F}$ the set for fuzzy set independence.

For the upper probabilities for a set $C \in \mathcal{C}$ we have

$$\begin{array}{ccccc} \overline{P}_{1\mathrm{Aa}}(C) & \le & \overline{P}_{1\mathrm{Bb}}(C) & \le & \overline{P}_{2\mathrm{Bb}}(C) \\ \| & & \| & & \| \\ \overline{P}_{\mathrm{S}}(C) & & \overline{P}_{1\mathrm{Ab}}(C) & & \overline{P}_{2\mathrm{Ab}}(C) \\ & & \| & & \| \\ & & \overline{P}_{\mathrm{R}}(C) & & \overline{P}_{\mathrm{U}}(C) \end{array} .$$

The joint possibility measure $\mathrm{Pos}(C) = \overline{P}_{\mathrm{F}}(C)$ does not fit into this ordering. Here only

$$\mathrm{Pos}(C) = \overline{P}_{\mathrm{F}}(C) \le \overline{P}_{\mathrm{U}}(C)$$

holds.

Which of the above methods is preferable depends on the type of independence. The choice of the type of independence has to be part of the modelling of the joint uncertainty of the parameters. If nothing is known about the dependence, unknown interaction ($\overline{P}_{\mathrm{U}}(C)$) is the most preferable method to be on the safe side in reliability analysis. In the case of strong independence the computational effort is in general very high, so the upper bound $\overline{P}_{\mathrm{R}}(C) \ge \overline{P}_{\mathrm{S}}(C)$ can be used as a first approximation.

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