Boris Atadabeny

Assignment 4

Problem 1:

Find VC $Cov[\hat\beta, \hat\beta - b]$

$\hat\beta_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$ $\qquad y = X\beta + \varepsilon$

$\hat b_{OLS} = (X'X)^{-1}X'y$

$$\hat\beta_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X\beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon$$
$$= \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon$$

$$\hat b_{OLS} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon$$
$$= \beta + (X'X)^{-1}X'\varepsilon$$

$$\hat\beta_{GLS} - \hat b_{OLS} = \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon - \beta - (X'X)^{-1}X'\varepsilon$$
$$= [(X'\Omega^{-1}X)^{-1}X'\Omega^{-1} - (X'X)^{-1}X']\varepsilon$$

$$\left.\begin{aligned} E(\hat\beta_{GLS}) &= \beta \\ E(\hat b_{OLS}) &= \beta \end{aligned}\right\} \text{ both unbiased} \qquad \text{as } E[\varepsilon|X] = 0$$

$$E(\hat\beta_{GLS} - \hat b_{OLS}) = \beta - \beta = 0$$

$$Cov(\hat\beta, \hat\beta - b) = E[(\hat\beta - \beta)(\hat\beta - b)']$$
$$= E[(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\varepsilon\varepsilon'((X'\Omega^{-1}X)^{-1}X'\Omega^{-1} - (X'X)^{-1}X')'] =$$
$$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\sigma^2\Omega[(X'\Omega^{-1}X)^{-1}X'\Omega^{-1} - (X'X)^{-1}X']' =$$
$$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\sigma^2\Omega(\Omega^{-1}X(X'\Omega^{-1}X)^{-1} - X(X'X)^{-1})$$
$$= \sigma^2(X'\Omega^{-1}X)^{-1}X'\cancel{\Omega^{-1}\Omega}\Omega^{-1}X(X'\Omega^{-1}X)^{-1} -$$
$$-\sigma^2(X'\Omega^{-1}X)^{-1}X'\cancel{\Omega^{-1}\Omega}X(X'X)^{-1} =$$
$$= \sigma^2(X'\Omega^{-1}X)^{-1}\cancel{X'\Omega^{-1}X(X'\Omega^{-1}X)^{-1}} -$$
$$-\sigma^2(X'\Omega^{-1}X)^{-1}\cancel{X'X(X'X)^{-1}} =$$
$$= \sigma^2(X'\Omega^{-1}X)^{-1} - \sigma^2(X'\Omega^{-1}X)^{-1} = 0$$

$(AB)' = B'A'$
$(A+B)' = A'+$
Ω = symmetri

Problem 3:

random sampling from exponential distribution

$f(y) = \left(\frac{1}{\theta}\right) e^{-\frac{y}{\theta}} \qquad y \geq 0 \quad \theta > 0$

find the MLE of θ and its asymptotic variance

1.) log likelihood:

$\log L = -n \log \theta - \frac{\sum x_i}{\theta}$

$-\sum x_i \cdot \theta^{-1}$

2.) derivatives:

$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \qquad | \cdot \theta$

$\frac{\sum x_i}{\theta} = n \qquad | \cdot \theta \qquad | : n$

$\frac{\sum x_i}{n} = \theta$

$\hat{\theta}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$

3.) asymptotic variance $= \left(-E_0\left[\frac{\partial^2 \ln L(\theta_0)}{\partial \theta_0 \, \partial \theta_0'}\right]\right)^{-1}$

$\frac{\partial^2 \log L}{\partial \theta^2} = +\frac{n}{\theta^2} - 2\frac{\sum x_i}{\theta^3}$

asymptotic variance: $\left(-E\left[\frac{n}{\theta^2} - \frac{2\sum x_i}{\theta^3}\right]\right)^{-1}$

we know that $E\left[\frac{\sum x_i}{n}\right] = \theta$

$-\left(\frac{n}{\theta^2} - \frac{2E\sum x_i}{\theta^3}\right)^{-1} = -\left(\frac{n}{\theta^2} - \frac{2n\theta}{\theta^3}\right)^{-1} = -\left(-\frac{n}{\theta^2}\right)^{-1} =$

$= \underline{\underline{\frac{\theta^2}{n}}}$