The influence of Spinoza’s concept of infinity on Cantor’s set theory

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1. Introduction

Georg Cantor (1845–1918) was not only the founder of set theory; he also cared very much about the philosophical tasks connected with it. Right from the beginning of Cantor’s university education, the works of Baruch de Spinoza (1632–1677) were among his major sources of philosophical inspiration. When Cantor completed his Habilitation, for example, he chose a ‘thesis’ referring to the appendix of the first part of Spinoza’s Ethics 1 for the public defense in order to get the license to teach at a university. 2 Subsequently, he continued to study Spinoza’s works intensively. 3 On several occasions, Cantor mentioned Spinoza’s philosophy in discussions with his colleagues and in his publications. 4 He called one of Spinoza’s letters ‘most important’ (höchst bedeutend) and found it ‘full of content’ (inhaltsvoll). 5 He claimed that a correct and perspicuous theory of infinity lacks due to the insufficient development of Spinozean thoughts onto Cantor’s; we discuss Spinoza’s philosophy of infinity, as it is contained in his Ethics; and we attempt to draw a parallel between Spinoza’s and Cantor’s ontologies. Our conclusion is that the study of Spinoza provides deepening insights into Cantor’s philosophical theory, whilst Cantor can not be called a ‘Spinozist’ in any stricter sense of that word.

1 We used the Latin edition of Spinoza’s Ethica ordine geometrico demonstrata (Spinoza, 1913), and the English translation (Spinoza, 2007). A collection of some of Cantor’s papers is Cantor (1932b). For a complete bibliography of Cantor’s works, see Tapp (2005), pp. 578–582.
2 This “thesis” reads: ‘hinc Spinoza mathesi (Eth. pars. I. prop. XXXVI, app.) ex eum vim tribuit, ut hominibus norma et regula veri in omnibus rebus indagandis sit’ (‘Rightly Spinoza attributed such a power to mathematics (Ethics, first part, prop. 36, appendix) that for mankind, it is the norm and rule of the true in all things of research’) See Cantor’s Habilitationsschrift ‘De transformatione formarum ternarium quadrarum’ (1869), reprinted in Cantor (1932b), p. 62. The translation is ours, as it will be in the following if nothing else is noted.
3 The most reliable evidence is Cantor’s notebook on the Ethics. Cantor began to study the Ethics in detail in 1871. He copied the text of Spinoza’s definitions, axioms, and propositions onto the left-hand sides of the pages of a notebook and added his own summaries and commentaries on the right-hand sides. See the Appendix to this paper for further description, an edition, and translation of the main text of this notebook (the original is in Latin). As far as we can see, the influence of Spinoza in Cantor’s thought has been studied only rudimentarily until now. Stauffer (1992) is primarily interested in interpreting Spinoza’s philosophy with the aid of the modern, Cantorian conception of infinity.
4 See, for example, Cantor (1932c [1883]), p. 175/[1896 [1883]], p. 890.
5 Cantor mentioned the letter no. 29 (addressed to L. Meyer) ibid. In the place cited, Cantor also refers to Spinoza’s Cognitata metaphysica.
6 Cantor (1932c, 1896 [1883]), §§.
7 At the end of the introduction to the ‘Mitteilungen zur Lehre vom Transfinite’ Cantor (1932d [1887–1888]) discusses Cardinal Franczelin’s (1816–1888) suspicion that Cantor himself was a Spinozist. In a letter to Kurd Leßwitz (1848–1910) dated 15 February 1884, Cantor seems to support Franczelin’s suspicion, saying that the correct parts of Hegel’s conception of infinity derive from Spinoza. However, at the end of the letter to Franczelin, which Cantor excluded from the publication in the ‘Mitteilungen’, he expresses his feeling that someday pantheism will be completely overcome. For a critial edition of the letter to Franczelin dated 22 January 1886, see Tapp (2005), pp. 268–271.

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2. Traces of Spinoza’s influence in Cantor’s conception of infinity

Cantor’s philosophy of infinity starts with the distinction of actual and potential infinity—a philosophical distinction also familiar to the mathematicians of his time. But he went on downright philosophically asking for the relation of actual infinity to God. Are they to be identified? Is God the actual infinity? And is God’s being the actual infinite the true reason for Aristotle’s famous infinitum actu non datur—interpreted in the sense of not being given in the world?

For Cantor it is clear that the actual infinite—in 1883 he called it ‘proper infinite’ (Eigentlich-Unendliches)—must not be simply identified with God. Firstly, he told apart two kinds of actual infinity, namely God’s infinity and non-divine infinity. Then, in 1885–1886, when he published Über die verschiedenen Standpunkte in Bezug auf das aktuelle Unendliche,8 he distinguished even three kinds of actual infinity, namely actual infinity in concreto, actual infinity in abstracto, and actual infinity in Deo. The actual infinite in concreto is what is realized if there are infinitely many concrete things. Cantor also calls it ‘the transfinite’. The actual infinity in abstracto is realized by the transfinite numbers, and God’s infinity is called ‘absolute infinity’. For Cantor, this distinction gave rise to a classification of philosophical positions according to their admitting or denying of the three kinds of actual infinity.9 Cantor counted himself in the group assuming the infinite in all three respects. Spinoza, in contrast, belongs to the group accepting the actual infinite in concreto, but refusing it in abstracto. In Cantor’s eyes, Spinoza was an advocate of the existence of infinite quantities and nonetheless an opponent to the existence of infinite numbers.10 Cantor (1932e [1886]) blames Spinoza for not distinguishing sufficiently between the absolute infinite and the transfinite, both being kinds of actual infinity. In his eyes, it was this mistake that led Spinoza to believe in the impossibility of actually infinite numbers and to assume infinite quantities which cannot be counted.

Indeed, Spinoza rejected the classical arguments against the existence of infinite quantities different from the Absolute, but he accepted the arguments against the existence of infinite numbers. To defeat traditional philosophical arguments against infinite numbers was one of Cantor’s main motives for doing philosophy. So he felt the necessity to defeat Spinoza’s conviction that actually infinite numbers are impossible. To that end he pointed out a petitio principii in Spinoza’s arguments as they employ a notion of number in which finiteness is tacitly presupposed.

Despite this considerable criticism, Cantor was nonetheless convinced that a proper theory of the mathematical infinite depends on developing certain ideas of Spinoza’s, especially the idea of modus infinitus. He held that it would be great scientific progress to bring out the exact relation between the finite and the infinite modi and to solve the difficult question how finite modi can have a being on their own and how they could possibly not be annullled by the infinite modi.11 At this point, a closer look on Spinoza’s theory is helpful (Sect. 3) before we further examine the role of infinite modi in Cantor’s theory (Sect. 4).

3. Spinoza’s conception of infinity

The Ethics contains Spinoza’s most complete discussion of the concept of infinity. Although this work is called ‘Ethics’ for its primary goals and intentions, it is nevertheless a piece of theoretical philosophy and maybe called ‘metaphysical’. In the following, some parts of Spinoza’s theory are discussed which concern the notion of infinity.

Spinoza distinguishes three different senses of infinity: the sense in which it is said of God, the sense in which it is said of God’s attributes, and the sense in which it is said of the modi.

In the first sense, infinity is predicated only of God or the Deus sive natura, that is, only of that being quod in se est et per se concipitur (which is in itself and is conceived through itself.)12
However, ‘being in itself’ and ‘being conceived through itself’ are not identical. So one might place more stress on one than the other, that is, one might interpret Spinoza more realistically underlining that God is a simple being or one might interpret Spinoza more ideally emphasizing that God is the being whose idea is simple.\(^{13}\)

We will come back to this alternative when we discuss the attributes in a minute. For Spinoza, *Deus sive natura* is however absolutely infinite, and that means infinite

\[
\text{non autem in suo genere; quicquid enim in suo genere tantum infinitum est, infinita de eo attributa negare possumus; quod autem absolute infinitum est, ad ejus essentiam pertinent, quicquid essentiam exprimit et negationem nullam involvit. (not after its kind: for, of a thing infinite only after its kind, infinite attributes may be denied; but that which is absolutely infinite, contains in its essence whatever expresses reality, and involves no negation.)}^{14}
\]

From an extensional point of view, God coincides with the totality of existing beings, both physical and mental. But from an intensional point of view, God is a whole which cannot be reduced to the sum of the single beings He extensionally comprises. The whole *natura* has a life of its own and this life is superstructured to the single lives of its parts.

Spinoza’s second sense of infinity applies to attributes, more precisely, to the attributes of God. The notion of an attribute is defined by Spinoza in the following way:

\[
\text{Per attributum intelligo id, quod intellectus de substantia percipit, tanquam ejusdem essentiam constitueat. (By attribute, I mean that which the intellect perceives as constituting the essence of substance.)}^{15}
\]

The attributes constitute the essence of the substance, that is, the essence of God. Here, this whole concept is put under the operator ‘that which the intellect perceives as …’, and again the realistic/idealistic alternative arises. Do the attributes constitute the essence of the substance in reality or in perception? Although the quoted definition, strictly read, supports the idealistic view, there is also good evidence in favor of the realistic one. So, for example, when Spinoza says:

\[
\text{Quo plus realitatis aut esse unaquaque res habet, eo plura attributa ipsi competent. (The more reality or being a thing has, the greater the number of its attributes.)}^{16}
\]

and

\[
\text{Deus sive substantia constans infinitis attributis, quorum unumquodque aeternam et infinitam essentiam exprimit, necessario existit. (God, or substance consisting of infinite attributes, of which each expresses eternal and infinite essentiality, necessarily exists.)}^{17}
\]

Anyway, Cantor embraced the realistic interpretation and so, for the aims of this paper, there is no need to decide this widely discussed question.

Anyhow, Spinoza states explicitly in which sense God’s attributes are infinite. He says:

\[
\text{Per Deum intelligo ens absolute infinitum, hoc est, substantiam constantem infinitis attributis, quorum unumquodque aeternam et infinitam essentiam exprimit. (By God, I mean a being absolutely infinite—that is, a substance consisting in infinite attributes of which each expresses eternal and infinite essentiality.)}^{18}
\]

So, there are two senses in which infinity is germane to God’s attributes: they are infinite in essence and they are infinite in quantity.

The first sense (infinity in essence) might also be characterized as ‘actually metrical’ for the essence of the attributes is something more ‘extended’ than every single object included in it. This becomes more comprehensible when one recalls that, for Spinoza, every attribute expresses an infinite essence which is part of the infinite divine essence.\(^{19}\) On the one hand, this implies that every attribute is a part of God’s attributes. On the other hand, this implies that God’s attributes are not reducible to other attributes. Furthermore, God’s attributes cannot be divided for a division of one of God’s attributes would lead to a division of the substance, which is unthinkable in the context of Spinoza’s metaphysics. Metaphorically speaking, an attribute of God is like an ‘organism’ which arises out of its ‘parts’ but is yet not reducible to its ‘parts’ for it precedes or supervenes what it contains. This can be understood in analogy to the famous slogan that a whole can be more than the sum of its parts, and in particular to the view—most prominently expressed by Aristotle—that points are part of a straight line without the line being composed of the points. Spinoza makes this even clearer by considering the *extensio* as an example of God’s attributes. Every possible natural object has a certain spatial extension. The *extensio* in general comprises all the possible particular extensions. Thus, the *extensio* is infinite in the ‘metrical’ sense. Like each other attribute of God, it is a *maximum in genere*, that is, an entity which belongs to a *genus*, but exceeds every other entity of this *genus*.

The second sense in which infinity applies to God’s attributes (infinity in quantity), might also be called ‘actually quantitative’ for God’s attributes form an actually infinite totality whose quantity exceeds every finite number. In short: God’s attributes are infinitely many, that is, when one counted the single thoughts falling under the *cogitatio*, for example, this counting would never come to an end.\(^{20}\) For Spinoza, however, this does not entail the existence of an infinite number. He opposed the general supposition that every quantity is measurable by a number. So he writes in his letter to Meyer in April 1663:

\[
\text{Nam, praeter quam multa invenerunt, quae nullo Numero explicare possunt; quod satis numerorum defectum ad omnia determinandum patefacit; multa enim habent, quae nullo numero adaequari possunt, sed omnem, qui dari potest, numerum superat. (For, apart from that they [the mathematicians] have discovered many things not expressible by any number—which illuminates sufficiently the inadequacy of numbers for determining every-}
\]

\(^{13}\) Cantor characterizes his own philosophical position as ‘thoroughly realistic but, at the same time, no less idealistic’ (‘durchaus realistisch, zugleich aber nicht weniger idealistisch’). See Cantor (1996 [1833], §8, p. 896; 1932c [1883], §8, p. 181).

\(^{14}\) Spinoza (1913, 2007), first part, explanation to def. 6.

\(^{15}\) Ibid., def. 4.

\(^{16}\) Ibid., prop. 9.

\(^{17}\) Ibid., prop. 11.

\(^{18}\) Ibid., def. 6.

\(^{19}\) See, for example, ibid., notes to prop. 10 and prop. 11.

\(^{20}\) One might say that a discrete *genus* is infinite in the sense that it can not be counted by a finite number, but only by an infinite number. But this is not a good way to put it because Spinoza explicitly denies the possibility of actually infinite numbers.
thing—they know many things which can not be sufficiently expressed by any number because they exceed every possible number.)\textsuperscript{21}

Thus, Spinoza held that infinite quantities do not have to be measurable by a number. Infinite numbers for him had something absurd, leading to contradictions like:

\begin{quote}
ac propterea unus numeros infinitus erit duodecies major alio infinito.
\end{quote}

(one infinity would be twelve times as great as the other.)\textsuperscript{22}

Apparently, Spinoza held infinite quantities and numbers as strictly apart as to that these contradictions do not transfer to infinite quantities.

The third and final sense in which infinity is used in Spinoza’s Ethics is as applicable to modi. Spinoza defines modi as

\begin{quote}
substantiae affectiones, sive id, quod in alio est, per quod etiam concipitur.
\end{quote}

(the modifications of substance, or that which exists in, and is conceived through, something other than itself.)\textsuperscript{23}

Therefore the modi are not self-subsistent, but depend ontologically on something else—the substance. In a corollary to proposition 25, the notion of a modus is made clearer, when Spinoza says about the individual things that they

\begin{quote}
nihil sunt, nisi Dei attributorum affectiones, sive modi, quibus Dei attributa certo et determinato modo exprimuntur.
\end{quote}

(are nothing but modifications of the attributes of God, or modes by which the attributes of God are expressed in a fixed and definite manner.)\textsuperscript{24}

Thus, the finite modi are nothing but the individual creatures, minds, and thoughts (will, cupidity, love, and so on)—or their individual manifestations. For example, according to Spinoza, the human intellect is a modus. In a certain sense, the modi are just the phenomenal expressions of the attributes of God.

The infinite modi need a closer examination. The fundamental propositions concerning them are propositions 21 and 22 of the Ethics. Spinoza says:

\begin{quote}
Omnia, quae ex absoluta natura alicujus attributi Dei sequuntur, semper et infinita existere debuerunt, sive per idem attributum aeterna et infinita sunt.
\end{quote}

(All things which follow from the absolute nature of any attribute of God must always exist and be infinite, or, in other words, are eternal and infinite through the said attribute.)\textsuperscript{25}

Hence, the main difference between attributes and modi is that an attribute is in itself, while a modus is ontologically dependent on attributes for it exists only as something like a property of an attribute. For example, the cogitatio is a being in itself which includes every thinking being (even when the cogitatio is not the ‘sum’ of the thinking beings, but has its own proper ‘life’), and God’s infinite intellect is a modus of this cogitatio. This modus is infinite and eternal, but it is not a being in itself, and it includes only part of the cogitatio. Furthermore, the attributes belong to the natura naturans, while the modi belong to the natura naturata. The infinite intellect and the cogitatio are also different as the cogitatio is more extended having the intellect as one of its modi.\textsuperscript{26} The first point can be clarified by considering an analogy: a man is a being in itself, but an arm or a leg are not beings in themselves, they do not have an essence which is independent from the one of the man. Furthermore God’s intellect is not as extended as the cogitatio because, for example, it does not include God’s infinite and eternal will, God’s infinite and eternal love, and so on. These last are other infinite modi. In this way, Proposition 21 supplies Spinoza’s first concept of a modus of God as directly deriving from the nature of God’s attributes.

Proposition 22 presents Spinoza’s second characterization of an infinite modus:

\begin{quote}
Quicquid ex aliquo Dei attributo, quatenus modificatum est tali modificatione, quae et necessario et infinita per idem existit, sequitur, debet quoque et necessario et infinitum existere.
\end{quote}

(Whatssoever follows from any attribute of God, in so far as it is modified by a modification, which exists necessarily and as infinite, through the said attribute, must also exist necessarily and as infinite.)\textsuperscript{27}

Those are ‘infinite modi of second kind’ because they do not derive directly from God’s attributes, but from their modifications. Spinoza gave an example of an infinite modus of second kind in a letter to Schiller on 29 July 1675.\textsuperscript{28} Here Spinoza calls the aspect (facies) of the universe an infinite modus of second kind. The aspect changes in infinite ways, but it remains always the same (as a person can change her position remaining the same person). Therefore the aspect of the universe is an infinite modus, which is a modification of the extensio.

So, in the Ethics, Spinoza distinguishes three senses of infinity: the infinity of God, the infinity of God’s attributes, and the infinity of modi. The attribute and the modi case have in common that infinity means a maximum in generis. A common feature of all three senses of infinity is that they are actualizations of a potentially infinite process.

4. An analogy and its limits

Having surveyed a portion of Spinoza’s ontology, we can now come back to Cantor’s view that Spinoza admitted the actual infinite in concreto, but refused it in abstracto. This was indeed so, for Spinoza was convinced of the existence of the infinite attributes and modi, and so he can be said to have accepted the actual infinite in concreto. And, on the other hand, he refused it in abstracto for he did not believe in the existence of infinite numbers.

In contrast, Cantor held that actually infinite numbers are possible and that their type of infinity has to be especially distinguished from the absolute infinity of God. When he wrote to Rudolph Lipschitz (1832–1903)\textsuperscript{29} in November 1883, Cantor underlined this point. Concerning the sort of actual infinity that can be treated mathematically by a theory of infinite numbers, he added a consideration that provides a basis for a structural analogy between Spinoza’s theory and his own one. He said:

Das Vollendetunendliche findet sich allerdings in gewissem Sinne in den Zahlen $\omega$, $\omega + 1$, $\ldots$, $\omega^n$, $\ldots$; sie sind Zeichen für gewisse Modi des Vollendetunendlichen und weil daher das

\textsuperscript{21} Spinoza to L. Meyer, 20 April 1663, see Spinoza (1914), p. 41.

\textsuperscript{22} Spinoza (1913, 2007), first part, note to prop. 15.

\textsuperscript{23} Ibid., def. 5.

\textsuperscript{24} Ibid., cor. to prop. 25.

\textsuperscript{25} Ibid., prop. 21.

\textsuperscript{26} Ibid., note to prop. 29, prop. 31, and demonstration.

\textsuperscript{27} Ibid., prop. 22.

\textsuperscript{28} Spinoza (1914), pp. 205–206.

\textsuperscript{29} Cf. Cantor’s letter to Rudolph Lipschitz of 19 November 1883 in Lipschitz (1886), p. 33.
The word *Modus* is hard to understand if one takes the German text as it is. But as Cantor had the philosophy of Spinoza in his intellectual background, the passage may be read as referring to Spinoza's technical term *modus*. Further evidence for this reading is supplied by Cantor's statement that the development of a mathematical theory of the infinite hinges upon a further development of Spinoza's idea of *modus infinitus*, as was mentioned above.

The transfinite ordinal numbers (\(\omega, \omega + 1, \ldots, \omega^n, \ldots\)) are one of the most important inventions of Cantor's set theory. Transfinite numbers have the following two properties (among others):

- (A) Each one is larger than every finite number.
- (B) Each one can be represented by the set of its infinitely many (ordinal) predecessors.

These two properties can be interpreted in Spinozean terms. When the finite numbers are interpreted as finite *modi* then (A) means that every transfinite number represents a metrical infinite and (B) means that every transfinite number represents a quantitative infinite. So the transfinite numbers have both properties of Spinoza's infinite *modi*. Hence, it is plausible that Cantor used Spinoza's notion of *modus* in his assertion, that the numbers represent signs for the *modi* of the completed infinity.

Further evidence for this similarity between Spinoza's *modus* and Cantor's transfinite numbers can be found in Cantor's analysis of a classical objection to infinite numbers. In the Grundlagen, Cantor dealt with the argument that, if infinite numbers existed, they would annul the finite numbers. To answer this objection, Cantor's reference point was just the Spinozean theory of the finite and infinite *modi*. Cantor wrote:

Ein besonders schwieriger Punkt in dem Systeme des Spinoza ist das Verhältnis der endlichen Modii zu den unendlichen Modii; es bleibt dort unaufgeklärt, wieso und unter welchen Umständen sich das Endliche gegenüber dem Unendlichen oder das Unendliche gegenüber dem noch stärker Unendlichen in seiner Selbständigkeit behaupten könne … Ist \(\omega\) die erste Zahl der zweiten Zahlenklasse, so hat man \(1 + \omega = \omega\), dagegen ist \(\omega + 1 = (\omega + 1)\), wo \((\omega + 1)\) eine von \(\omega\) durchaus verschiedene Zahl ist. Auf die Stellung des Endlichen zum Unendlichen kommt also, wie man hier deutlich sieht, alles an; tritt das erstere vor, so geht es in dem Unendlichen auf und verschwindet darin; beschiedet es sich aber und nimmt seinen Platz hinter dem Unendlichen, so bleibt es erhalten und verbindet sich mit jenem zu einem neuen, weil modifizierten Unendlichen.

(An especially difficult point in Spinoza’s system is the relationship of the finite modes to the infinite one; it remains unexplained how and under what circumstances the finite can maintain its independence with respect to the infinite, or the infinite with respect to still higher infinities. … If \(\omega\) is the first number of the second number class, then one has \(1 + \omega = \omega\), but \(\omega + 1 = (\omega + 1)\), where \((\omega + 1)\) is a number entirely distinct from \(\omega\). Therefore, as one here clearly sees, everything depends on the *placement* of the finite relative to the [in]finite; if the former comes first, it merges into the infinite and vanishes therein; but if it contents itself to take its place after the infinite it is preserved and unites with it to form a new, because modified, infinite.\(^{30}\)

Cantor held that a correct theory of infinity can take its origin in Spinoza and has to overcome the limits that Spinoza himself had imposed to his own theory. Cantor showed that the finite is not necessarily annulled by the infinite. The quotation clearly confirms that Cantor saw an interconnection between the transfinite numbers and the infinite *modi*.

This line of interpretation can be elongated. Spinoza’s ontology has (at least) four major levels: the *modi* of second kind, the *modi* of first kind, the attributes of God and finally God himself, the absolute infinity. To a certain extent, there is a structural parallel to this Spinozean hierarchy in Cantor’s set theory.

In the considerations above, we have left open both, whether we speak about transfinite ordinal or transfinite cardinal numbers, and whether we speak about infinite *modi* of first or of second kind. Both conditions, (A) and (B), are fulfilled by ordinals as well as cardinals. Can the two borderlines, the one between ordinals and cardinals, and the one between infinite *modi* of first and of second kind, be made coincide?

A decisive suggestion can be found again in Cantor’s letter to Lipschitz, when Cantor says that the ordinal numbers are *modi* of the actual infinite and that in this way the completed infinite occurs in *modifications* conceivable by the finite human mind. What can it mean to speak of ‘*modifications*’ in the context of transfinite ordinal numbers? Transfinite ordinal numbers can indeed be conceived as following from modifications, namely, from modifications of a set by taking that to be a (re-) (well-) ordering of the set. For transfinite ordinals are the order types of (well-founded) sets, they can be said to follow from a set’s modification in the sense of a specific well-ordering of it.\(^{31}\) But is this interpretation of ‘*modifications*’ not too remote from Spinoza’s views? We do not think so. Spinoza described the infinite *modi* of second kind as ‘following in so far as it is modified by a modification’ (sequitur quatenus *modificatum est tali *modificatione).\(^{32}\) And this is true on the level of the transfinite ordinal numbers when they are said to stem from modifications of the ordering of an infinite set.

So, one can fix the first element of the structural analogy between Spinoza’s and Cantor’s philosophy of the infinite by taking the transfinite ordinal numbers to play the role of infinite *modi* of second kind. And in Cantor’s theory an ‘object’ which has the same functional role as the *modi* of the first kind is directly at hand. In Spinoza, infinite *modi* of the second kind derive from a divine attribute by a modification, while the infinite *modi* of the first kind follow directly from a divine attribute, without any modification. Having interpreted the modifications as re-orderings of an infinite set, it seems just consistent to draw the parallel between the infinite *modi* of first kind and the cardinal numbers. In fact, the cardinal number of a set is not modified by any permutation of the elements of the set.

The next step in a parallelization of Cantor’s and Spinoza’s theory has to deal with the central concept in Cantor’s theory—the concept of a set. For finding an analogue in Spinoza’s theory, there are two options.

\(^{30}\) Cantor (1932c [1883]), §5, p. 177/1996 [1883], §5, p. 892.

\(^{31}\) The natural ordering of the set of natural numbers \(0, 1, 2, 3, \ldots\) has the transfinite ordinal number \(\omega\) as its order type, while a modified ordering like 2, 3, 4, \ldots, 0, 1 has order type \(\omega + 2 \neq \omega\).

\(^{32}\) Spinoza (2007), prop. 22.
4.1. Sets as modi of third kind

One could try to interpret the sets as belonging to the same ontological level as the transfinite numbers (that is, as the modi). In this case, they must be considered as a new sort of modi, as modi of third kind.

A cardinal number is not modified by any change or any permutation of the elements. An ordinal number is not modified by any change of the elements, but it is modified by permutations of the elements. A set is modified both by a change of the elements and by a permutation of the elements (according to one of Cantor’s two notions of set).33 The difference between sets and modi of second kind can thus be that modi of second kind can be modified by one kind of modification, while sets can be modified by two kinds of modifications. Hence, it seems consistent to interpret sets as modi of third kind, more specific and modifiable in more ways than the ordinal and cardinal numbers. This interpretation goes along well with Cantor’s abstraction account of transfinite numbers: one abstraction process leads from a set to its ordinal number, a second one from the ordinal number to its cardinal number.34

But on the other hand, the cardinal and the ordinal numbers are numbers of sets, they can be said to refer to sets or to be attached to sets. Therefore, it is doubtful whether numbers and sets really belong to the same ontological level. If numbers depend on the sets and express some of their important properties, then the ontological level of the sets shall be superior to the one connoting the numbers. So, we consider the following alternative.

4.2. Sets as attributes

According to this second option, sets are interpreted as attributes of God, while the transfinite numbers are modi of these attributes.

Sets and numbers are strictly connected and Cantor himself stresses their connection. In Cantor’s eyes, a set has always a cardinal number and an ordinal number (or, at least, an order type) and vice versa every cardinal and ordinal number is associated to the set of its (ordinal) predecessors. (In modern terms, that comes down to the fact that Cantor held the Axiom of Choice to be true.) We read indeed:

Undenliche Zahl und Menge sind unlösbar miteinander verknüpft; gibt man die eine auf, so hat man kein Recht mehr auf die andere. (Infinite number and set are indissolubly tied; if one abandons the former, one loses the right to the latter.)35

It seems to us that in Cantorian theory the sets are the real first basic element. This does not imply that a set can be conceived without its own cardinal number and without its order type; in this sense, sets and numbers are logically co-original. But the numbers represent properties of the sets. They are well-defined closed unities in themselves, but ontologically dependent from the sets—just like Spinoza’s modi with respect to the attributes.

The second option (to take sets as attributes) seemingly avoids the problems of the interpretation as modi of third kind. But when we try to extend this interpretation to Cantor’s notion of inconsistent multiplicities, this option faces other problems.

33 Cantor operates with two different conceptions of sets. Here, we rely on the conception according to which sets have an internal ordering. The other one, which is used in today’s mathematics exclusively, is that the ordering is distinct from the set which is ordered; indeed, the ordering is itself a (different) set.


35 Cantor (1932d [1887–1888]), p. 394.

36 Cantor (1932f [1879–1884]), Nr. 5, p. 205.

37 For these letters see Cantor (1932b), pp. 443–450; Ewald (1996), pp. 926–940.

38 Cp. Hallett’s view of set theory as limitation of size (Hallett, 1984).

39 In his early works, Cantor took for granted that for two cardinal numbers a and b, either a > b, a = b, or a < b. Afterwards he realized that this trichotomy has to be proved. He attempted a proof in Cantor (1932a [1895–1897]), but a successful proof would have required to make use of the axiom of choice (AC).

4.3. Absolute infinity and inconsistent multiplicities

In 1883 Cantor made the famous remark that the endless series of infinite numbers can be seen as an adequate symbol of the Absolute.36 This thought is expressed more clearly and in more detail in the letters to Dedekind in July–August 1899.37 While Burali-Forti, who had discovered the paradox of the biggest ordinal number, concluded that set theory leads to contradictions, Cantor gave a completely different interpretation which holds also for the paradoxes of the system of all cardinal numbers and of the system of all sets. For the proofs of the paradoxes make use of the principle that every totality can be considered as a set; the proofs are nothing but reductio ad absurdum that this principle is false. There are multiplicities which lead to contradictions if they are taken as sets, as mathematical unities, and therefore can not be conceived as sets. Cantor called these totalities inconsistent multiplicities.

In an intuitive sense one could say that these inconsistent multiplicities are too big to be sets.38 This gives rise to the aforementioned extension of our interpretation, namely, to take an inconsistent multiplicity of the kind mentioned to be a maximum in genere. All three examples—the multiplicity of all ordinal numbers, the one of all cardinal numbers, and the one of all sets—are the extensions of notions that can be conceived of as genera. So it is obvious to us to interpret the system of all ordinal numbers as the maximum in genere of the genus ‘ordinal number’, and accordingly for cardinals and sets.

Hence, it seems that those inconsistent multiplicities have a role which is quite similar to Spinoza’s attributes of God. First of all, they are infinities which are a maximum in genere—they can not be increased. In a certain view of mathematics, it is this character which excludes the inconsistent multiplicities from the field of the ‘objects’ which can be treated mathematically while, on the contrary, ordinal numbers, cardinal numbers, and sets are mathematical entities which can always be increased and compared.39 Furthermore, the Spinozean attributes contain all entities of a determined kind—for example the cogitatio contains all thoughts—as do the inconsistent multiplicities. Finally, we have seen that, in Spinoza's ontology, God’s attributes represent the fundamental ontological essence of the substance and, therefore, can not be separated from the substance.

This interpretation may provide an important aspect in the understanding of Cantor’s assertion that the absolutely infinite series of cardinal numbers represents a symbol of the Absolute. But it is difficult to insert the notion of set into this interpretative scheme. The problem is that, if the inconsistent multiplicities have the same ontological status as Spinoza’s attributes, then the sets can not be called ‘attributes’ in the same sense. Sets are well determined mathematical unities and there is no contradiction in taking them as a whole. Sets are measured by numbers, while this is not the case for the inconsistent multiplicities.

Perhaps, one could try to tackle the problem of sets by taking them to be ‘attributes of second kind’, while the Spinozean attributes are ‘attributes of first kind’. Or, the sets could be interpreted as modi of third kind, as was discussed above (Sect. 4.1). But in either way of interpreting Cantor’s philosophy of the infinite in this ‘Spinozistic’ manner, it is necessary to introduce a new ontological level which does not exist in Spinoza’s theory.
To sum up: We tried to draw a parallel between Spinoza’s and Cantor’s theories of the infinite. The ordinal numbers are aligned with infinite modi of second kind, the cardinal numbers with the infinite modi of first kind, and some maximal inconsistent multiplicities with God’s attributes. For the sets, there is no proper place in this schema, although they are indispensable in a Cantorian ontology of infinite numbers and inconsistent multiplicities. This shows the limit of our proposed parallelization of Cantor and Spinoza.

5. Existence and pantheism

Another connection between Cantor and Spinoza goes far beyond the structural parallel we discussed in the last section. It can be characterized by the keywords ‘existence’ and ‘pantheism’.

5.1. Existence

In §8 of his ‘Grundlagen einer allgemeinen Mannigfaltigkeitslehre’ (Cantor, 1932c [1883]), Cantor makes some remarks about his elaborated view on the reality of mathematical entities. In general, he discerns two senses of existence, namely ‘immanent’ and ‘transient’ existence. The immanent reality of numbers is their reality as mental objects, as constituents of thoughts, and as instruments of the mind. The transient reality is somewhat more complicated. Cantor says that numbers have transient reality als sie für einen Ausdruck oder ein Abbild von Vorgängen und Beziehungen in dem Intellekt gegenüberstehenden Außenwelt gehalten werden müssen. (to the extent that they must be taken as an expression or copy of the events and relationships in the external world which confronts the intellect.)

The existence of classes of objects which can be counted by the abstract numbers is one kind of their (possible) transient reality. The notion of immanent reality is related to Spinoza’s term ada-equatus. Spinoza used this term to characterize true ideas independently from their relation to an object which is imagined by the idea. So adequate ideas comprise all ‘internal’ or ‘immanent’ properties true ideas have, irrespective of their relation to objects in an ‘external’ world.

Cantor held the thesis that—under certain circumstances with which we do not want to deal now—mathematical notions do have a transient reality, but that discovering this reality is an ambitious task of metaphysical research. In note 6 to §§, Cantor (1932c [1883]) points out that this immanent–transient connection unites him with Spinoza, as Spinoza issued the famous maxim:

Ordo et connexio idearum idem est ac ordo et connexio rerum. (The order and connection of ideas is the same as the order and connection of things.)

The reason for this close connection between immanent and transient reality is, for Cantor, to be seen in the

Einheit des Alls, zu welchem wir selbst mitgehören. (unity of the all to which we ourselves belong.)

5.2. Pantheism

Cantor’s examination of Spinoza’s philosophy is not limited to the problems concerning mathematics and its foundations. In Cantor (1932e [1886]), he asserts that the Achilles heel of Spinoza’s Ethics—its pantheism—depends on the fact that Spinoza did not distinguish precisely between absolute and transfinite infinity, although he had recognized the existence of different actual infinities. And in fact, from a certain perspective, Spinoza’s admitting of more than one actual infinity gives rise to an accusation of pan- and of polytheism.

To begin with the latter, one has to recall that Spinoza admitted the existence of three kinds of actual infinity: God, God’s attributes, and God’s infinite modi. Attributes and modi are infinite in a metrical and in a quantitative sense. The quantitative sense seems to be theoretically unproblematic, for Christian belief goes well with assumptions such as that God has an infinite quantity of attributes, that God has created an infinite quantity of creatures, and so on. The problems arise with the metrical conception of actual infinity. To consider the example of the attributes: admitting that an attribute is a maximum in genere can be held to imply a substantialization of the attributes; then, in case of an infinite maximum, we would have another infinite substance. God would not be the only actually infinite substance and his attributes may appear as ‘Gods’ themselves—albeit Gods on a ‘minor level’. This ‘mistake of polytheism’ is unacceptable from a theological point of view.

Similarly, pantheism is a stumbling block in Spinoza’s theory. It consists in an identification of God, God’s attributes, and God’s modi with nature. Cantor thought that Spinoza’s mistake depends on the fact that Spinoza did not provide a clear qualitative distinction between God and the other kinds of metrical infinity. In Cantor’s opinion, the accusation of pantheism makes sense only because Spinoza had not clearly distinguished between an actual infinite which can be increased (transfinite) and an actual infinite which can not be increased (absolute). Thus, Cantor believed to have found the qualitative difference between actual infinities in the dichotomic pair increasable/non-increasable. According to this view, there is no problem anymore: the increasable actual infinite is not the Absolute; it cannot be confused with God. Once the transfinite and the Absolute are clearly distinguished and a mathematical treatment of the transfinite is made possible, doctrines which admit the existence of more than one actual infinity can not simply be identified with pantheism. The infinite modi, when characterized mathematically, were completely different from the Absolute. Hence, according to Cantor, there was no longer any danger of pantheism.

6. Conclusions

In Spinoza’s philosophy, infinity plays a fundamental role. He was one of the main authors in Cantor’s philosophico-theological background. But Spinoza did not construct a speculation on infinity as detailed as Cantor’s. From Cantor’s perspective this was a serious lack in Spinoza’s theory. The fact that Spinoza did not believe in the existence of the infinite numbers was one reason for that. This is not to mean that Cantorian set theory is a necessary development which departed directly from Spinoza. Freely, Cantor constructed a mathematical theory and, in the course of time, he felt more and more the need to justify it from a philosophical and a theological point of view. In contrast, Spinoza’s aims were not mathematical at all. He wanted to construct more geometrico a ‘system of the world’ in which mathematics played no
important role, while in Cantor's thought, mathematics belongs to the core of a 'system of the world'. It seems to be consistent with Spinoza's philosophical views to admit the infinite in quantity but no infinite numbers, even if this is a mistake from Cantor's perspective. The fact that Cantor's philosophy rests—to a certain extent—on Spinoza must not hide the differences between the two thinkers.

Another reason for the lack of an advanced theory of infinity in Spinoza's thoughts are, however, the conceptual difficulties with which Spinoza scholars have to deal when they try to understand Spinoza. Historically one can sustain the thesis that one of the factors which made the successful invention of infinite numbers before Cantor impossible was that the concepts were not clear enough and contained too much non-explicated presuppositions.

A major difference between Cantor's and Spinoza's thoughts on infinity concerns background and context. Cantor presented a theory of infinity with two sides, the mathematical treatment of transfinite numbers and the philosophical discussion of the concept of infinity. The dialectic between the mathematical and the philosophical-theological aspect was continuously present in Cantor's work and it is important to be kept in mind in order to understand Cantor's philosophical attitudes in all their dimensions. Set theory was born in an environment which reached from mathematical things like Fourier series to philosophico-theological topics like the Absolute.

Spinoza's philosophy does not have a whole and complete theory of infinity, although it has many elements of it. The inspirational power of his theory was as big as to influence Cantor's invention of the first reliable mathematical theory of infinity. Moreover, it was—to a certain extent—possible to draw a parallel between Spinoza's general philosophy and Cantor's philosophy of set theory. Although this parallel clearly has its limits, we hold that there is more than a mere formal analogy between Cantor's thought and Spinoza's. Surely, Cantor was not a 'Spinozist' in a stricter sense. But his investigations of Spinoza's philosophy influenced deeply the philosophical conception of Cantor's main mathematical achievement—set theory.

Appendix: A Cantorian notebook on Spinoza's Ethics

The Cantor Nachlaß kept by the Staats- und Universitätsbibliothek Göttingen contains a sixty-two-page notebook in a blue card jacket (SUB Göttingen: Cod. Ms. G. Cantor 27). The entries start with the headline 'Ethica Benedicti de Spinoza,' and the remark 'Wintersemester 1871–1872.' Each page is divided into a left-hand and a right-hand side, with the left-hand sides dedicated to Latin quotations from Spinoza's Ethics. The right columns contain: on page 1 the table of contents of the Ethica:

5. Pars quinta. De potentia intellectus seu de libertate humana. 87.

and then—as on the following pages—presumably Cantor's annotations to the text. This commentary is written with black ink while the Spinoza quotations are written with blue. Unfortunately, Cantor's main notes end with item 11 on page 7.

In the following, we present Cantor's notes in the original Latin version opposite to an English translation. The single paragraphs are preceded by remarks in square brackets which concern the arrangement of Cantor's notices. Passages Cantor underscored are reproduced in italics.

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| 1. | Unica solummodo est substantia, nempe subst. divina; quaecumque vero praeter unicum hanc substantiam existunt, non sunt nisi variae ejusdem affectiones sive modi: est autem substantia divina aeterna atque infinita; ejusque essentia formaliter identificatur tum cum ejus existentia tum etiam cum ejus potentia. P.I, pr. 7,8,14,15,34. |
| 2. | Quia autem quo plus realitatis alicui rei inest, eo plura attributa ci competunt, (P. I pr. 9) substantia absoluta infinitis constat attributis. (I, pr. 11). Porro haec attributa non procedunt ex efficacia substantiae absolutae, nec proinde sunt ipsa modi aliqui qui sint in hac substantia tamquam in alio, verum eorum unumquodque essentiam absolutae substantiae constituat atque exprimit. (I def. 4 et 6 et pr. 11). Ex quo fit tum ut unumquodque attributum sit aeternum (I pr. 19) et in suo genere infinitum (I def. 6 et demonstr. pr. 16), tum ut ipsum consciatur per se, hoc est independerit a conceptu alius cujusvis attributi (I. pr. 10) [1/2] |
| 3. | The substance is only one, that is the divine substance; but whatever exists besides this one substance, is nothing but different affectiones or modi of it: the divine substance is eternal and infinite; its essence is formally identified both with its existence and with its power. P.I, pr. 7.8,14,15,34. |
| 4. | But since the more reality is in a thing, the more attributes come together in it, (P. I pr. 9) the absolute substance is composed of infinite attributes. (I, pr. 11). Furthermore, these attributes do not spring from the efficiency (power) of the absolute substance, and consequently they are not modi which are in this substance as if they were in something different, but truly every one of them constitutes and expresses the essence of the absolute substance. (I def. 4 and 6 and pr. 11) From this it follows firstly that every attribute is eternal (1 pr. 19) and infinite in its kind (I def. 6 and demonstr. pr. 16), and secondly that it is conceived through itself, i.e., independently from the concept of any other attribute (I, pr. 10) [1/2] |

In addition, only two among the infinite attributes of God are clearly perceived by us: surely the infinite and absolute cogitatio and the infinite and absolute extensio. (II pr. 1 and 2). The reason of this is that all the single things which we perceive are either bodies, and hence determined modi of the absolute extensio, or they are finite modi of the cogitatio, and thus determined modi of the absolute cogitatio. (II ax. 5).
3. Quia autem ex infinito infinita infinitis modis sequi necesse est (I, pr. 16), hinc ex unoqueque attributo divino sequuntur infiniti modi, quorum unusquisque est quidem aeternus et in suo genere infinitus (I, pr. 21), qui tamen non constituent essentiam substantiae, sed sunt in substantia et efficientier procedunt et substantia. (I sch. pr. 29). Quae etiam ratio est cur non dicatur attributa sed modi attributorum sive substantiae.

4. Hinc singulare quodcumque, hoc est, res quaevis, quae finita est et temporalis determinatur a causa quae et ipsa finita est et temporalis, et haec rursus ab alia ejusdemmodi, et sic in infinitum. Sive, quod idem est, quilibet modus substantiae divinae omnino finitus et determinatus non procedit ab attributo aliquo divino, nisi quatenus modificatum est modificacione quae et ipsa omnino finita sit et determinata; et haec rursus modificatio debet procercare ab ali modos omnino finito, et sic semper, quin possibile sit devenir ad primum in serie horum modorum finitorum (I, pr. 28).

5. Absoluta substantia atque infinita attributa quibus constituitur et exprimitur, sive aliter Deus, secundum quod per se concipitur et est causa omnium quae fiunt, dicitur natura naturans. Omnes vero modi attributorum, sive infiniti sint, sive finiti, secundum modum quod in Deo esse et ab efficiente Deus virtute procedere intelligiuntur, dicuntur natura naturata. (sch. pr. 29) [3/4]

Ut tamen patet ex praedictis, non funditus inter se differunt natura naturans et natura naturata, verum ipsae sunt una eademque substantia atque natura, quae secundum est natura naturans sese active evolvit in naturam naturatam.

Et quidem secundum quod ad naturam naturatam pertinent modi illi aeterni atque in suo genere infiniti qui infinita serie sive ex attributis divinis sive aliis aliis procedunt, evolutio naturae naturantis in naturam naturatam stricto sensu est aeterna, hoc est, ipsa est simultanea, tota simul atque immobialis. Secundum quod vero ad naturam naturatam pertinent infinita series modorum finitorum sive rerum singularium, evolutio naturae naturantis in naturam [originali e naturatam] naturatam est quidem sempiterna, temporaria tamen, successiva atque mobili.
6. Verumtamen omnia quae modo quocumque fiunt, sive sint modi aeterni et in suo genere infiniti, sive sint singularia, hoc est modi transitorii atque finiti; omnia haec, inquam, ex absoluta necessitate divinae naturae determinata sunt et ad existendum, quando et quamdiu existunt, et ad operandum, quando et quamdiu operantur (I, pr. 29). Quamobrem in rerum natura nullam datur contingens, neque possibile fuit res aliud modo alioque ordine a Deo produci atque ipsae reversa productae (I, pr. 29 et 33 et sch. 2 pr. 33).

Ratio hujus est—tum a) quod, ut dictum est, ex infinito infinitis modis, hoc est omnia possibilia, sequi necesse est—tum b) quod essentia naturae divinae identificatur cum potentia divina, quoque subinde sicuti natura divina per essentiam suam est determinata ad existendum, sic et per essentiam suam ipsa est determinata ad operandum quidquid ipsi est possibile operari (I, sch. pr. 17 et pr. 34, 35).

7. Hinc Deus est causa libera, non quod ipse non absoluta necessitate operetur, sed quia ipse ex sola sua naturae determinazione et a nemine coactus operatur (I def. 7, pr. 17); aliae vero omnes res sunt causae necessariae sive coactae, quia ab alio determinantur ad operandum. (I def. 7, pr. 17).

8. Hence God does not act for any purposes; and there is no inner distinction between good and bad, merit and sin, beauty and ugliness in the nature of things. Therefore, as said, even those things which are commonly called bad, sinful and ugly, must emerge from the infinite perfection of the divine nature in the very way in which they emerge. These and other such notions arose from the prejudices of men who ignore the true causes of the things and who imagine God acting for some own advantage, like they do. (I app. and IV pref.) [5/6]

9. Furthermore the evolution of the substance in so far as it is the absolute cogitatio proceeds in the same order as its evolution in so far as it is the absolute extensio and generally in so far as it is conceived under whatsoever one of its attributes (II, pr. 7, note). The reason for this is that by each attribute of the absolute substance its one and the same infinite essence is expressed, and thus also one and the same law of self-evolution inherent in it. Here, as to every modus of the extensio there corresponds a modus of the cogitatio which is in every part commensurate to it—the order and connection of ideas is the same as the order and connection of things. (II, pr. 7)

10. Like the extended substance and the thinking substance are one and the same substance comprehended now through one attribute and now through the other; so, also, a modus of the extensio and the idea of that modus are also one and the same thing, though expressed in two ways. For example, a circle existing in nature and its idea, when considered in themselves, are one and the same thing presenting two different aspects. The reason of this is that of things as they are in themselves God is the true cause not inasmuch as he consists of only one attribute but inasmuch as he consists of infinite attributes. (II, note to pr. 7) [6/7]

Cantor, G. (1932d). Mitteilungen zur Lehre vom Transfiniten. In idem,

Cantor, G. (1932b).

Cantor, G. (1932f). Über unendliche, lineare Punktmannigfaltigkeiten. In idem,

Cantor, G. (1932a). Beiträge zur Begründung der transfiniten Mengenlehre. In idem,

References


[From the next page (no. 11) on, Cantor copied Spinoza’s propositions without further annotations.]