On the Identifiability of Overcomplete Dictionaries via the Minimisation Principle Underlying K-SVD

Karin Schnass

Abstract

This article gives theoretical insights into the performance of K-SVD, a dictionary learning algorithm that has gained significant popularity in practical applications. The particular question studied here is when a dictionary $\Phi \in \mathbb{R}^{d\times K}$ can be recovered as local minimum of the minimisation criterion underlying K-SVD from a set of N training signals $y_n = \Phi x_n$. A theoretical analysis of the problem leads to two types of identifiability results assuming the training signals are generated from a tight frame with coefficients drawn from a random symmetric distribution. First asymptotic results showing, that in expectation the generating dictionary can be recovered exactly as a local minimum of the K-SVD criterion if the coefficient distribution exhibits sufficient decay. This decay can be characterised by the coherence of the dictionary and the ℓ_1-norm of the coefficients. Based on the asymptotic results it is further demonstrated that given a finite number of training samples N, such that $N/\log N = O(K^3 d)$, except with probability $O(N^{-Kd})$ there is a local minimum of the K-SVD criterion within distance $O(KN^{-1/4})$ to the generating dictionary.

Index Terms

dictionary learning, sparse coding, K-SVD, finite sample size, sampling complexity, dictionary identification, minimisation criterion, sparse representation

1 Introduction

As the universe expands so does the information we are collecting about and in it. New and diverse sources such as the internet, astronomic observations, medical diagnostics etc. confront

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us with a flood of data in ever increasing dimensions and while we have a lot of technology at our disposal to acquire these data, we are already facing difficulties in storing and even more importantly interpreting them. Thus in the last decades high-dimensional data processing has become a very challenging and interdisciplinary field, requiring the collaboration of researchers capturing the data on one hand and researchers from computer science, information theory, electric engineering and applied mathematics, developing the tools to deal with the data on the other hand. One of the most promising approaches to dealing with high-dimensional data so far has proven to be through the concept of sparsity.

A signal is called sparse if it has a representation or good approximation in a dictionary, ie. a representation system like an orthonormal basis or frame, [7], such that the number of dictionary elements, also called atoms, with non-zero coefficients is small compared to the dimension of the space. Modelling the signals as vectors $y \in \mathbb{R}^d$ and the dictionary accordingly as a matrix collecting K normalised atom-vectors as its columns, ie. $\Phi = (\phi_1, \dots \phi_K), \phi_i \in \mathbb{R}^d, \|\phi\|_2 = 1$, we have

$$y \approx \Phi_I x_I = \sum_{i \in I} x(i) \varphi_i,$$

for a set I of size S, ie. $|I| = S$, which is small compared to the ambient dimension, ie. $S \ll d \leq K$.

The above characterisation already shows why sparsity provides such an elegant way of dealing with high-dimensional data. No matter the size of the original signal, given the right dictionary, its size effectively reduces to a small number of non-zero coefficients. For instance the sparsity of natural images in wavelet bases is the fundamental principle underlying the compression standard JPEG 2000.

Classical sparsity research studies two types of problems. The first line of research investigates how to perform the dimensionality reduction algorithmically, ie. how to find the sparse approximations of a signal given the sparsity inducing dictionary. By now there exists a substantial amount of theory including a vast choice of algorithms, e.g. [10], [6], [23], [3], [9], together with analysis about their worst case or average case performance, [30], [31], [28], [16]. The second line of research investigates how sparsity can be exploited for efficient data processing. So it has been shown that sparse signals are very robust to noise or corruption and can therefore easily be denoised, [12], or restored from incomplete information. This second effect is being exploited in the very active research field of *compressed sensing*, see [11], [5], [25].

However, while sparsity based methods have proven very efficient for high-dimensional data processing, they suffer from one common drawback. They all rely on the existence of a dictionary providing sparse representations for the data at hand.

The traditional approach to finding efficient dictionaries is through the careful analysis of the given data class, which for instance has led to the development of wavelets, [8], and curvelets, [4], for natural images. However when faced with a (possibly exotic) new signal class this analytic approach has the disadvantage of requiring too much time and effort. Therefore, more recently, researchers have started to investigate the possibilities of learning the appropriate dictionary directly from the new data class, ie. given N signals $y_n \in \mathbb{R}^d$, stored as columns in a matrix $Y = (y_1, \ldots, y_N)$ find a decomposition

$$Y \approx \Phi X$$

into a $d \times K$ dictionary matrix Φ with unit norm columns and a $K \times N$ coefficient matrix with sparse columns.

So far the research focus in dictionary learning has been on algorithmic development, meaning that by now there are several dictionary learning algorithms, which are efficient in practice and therefore popular in applications, see [13], [19], [1], [22], [34], [20], [29] or [26] for a more complete survey. On the other hand there is only a handful of dictionary learning schemes, for which theoretical results are available, [2], [15], [17], [14], [18]. While for these schemes there are known conditions under which a dictionary can be recovered from a given signal class, their practical applicability is severely limited by their computational complexity. In [2] the authors themselves state that the algorithm is only of theoretical interest and also the ℓ_1-minimisation principle, suggested in [35], [24] and studied in [17], [14], [18], is not suitable for very high-dimensional data.

In this paper we will start bridging the gap between practically efficient and provably efficient dictionary learning schemes, by providing identification results for the minimisation principle underlying K-SVD (K-Singular Value Decomposition), one of the most widely applied dictionary algorithms.

K-SVD was introduced by Aharon, Elad and Bruckstein in [1] as a generalisation of the K-means clustering process. The starting point for the algorithm is the following minimisation criterion.

Given some signals $Y = (y_1, \dots, y_N)$, $y_n \in \mathbb{R}^d$, find

$$\min_{\Phi \in \mathcal{D}, X \in \mathcal{X}_S} \|Y - \Phi X\|_F^2 \tag{1}$$

for $\mathcal{D} := \{\Phi = (\phi_1, \dots, \phi_K), \phi_i \in \mathbb{R}^d, \|\phi_i\|_2 = 1\}$ and $\mathcal{X}_S := \{X = (x_1, \dots, x_N), x_n \in \mathbb{R}^K, \|x_n\|_0 \leq S\}$, where $\|x\|_0$ counts the number of non-zero entries of x, and $\|\cdot\|_F$ denotes the Frobenius norm. In other words we are looking for the dictionary that provides on average the best S-term approximation to the signals in Y.

K-SVD aims to find the minimum of (1) by alternating two procedures, a) fixing the dictionary Φ and finding a new close to optimal coefficient matrix X^{new} column-wise, using a sparse approximation algorithm such as (Orthogonal) Matching Pursuit or Basis Pursuit, and b) updating the dictionary atom-wise, choosing the updated atom ϕ_i^{new} to be the left singular vector to the maximal singular value of the matrix having as its columns the residuals $y_n - \sum_{k \neq i} \phi_k x_n(k)$ of all signals y_n to which the current atom ϕ_i contributes, ie. $X_{ni} = x_n(i) \neq 0$. We will not go further into algorithmic details, but refer the reader to the original paper [1] as well as [2]. Instead we concentrate on the theoretical aspects of the posed minimisation problem.

First it will be convenient to rewrite the objective function using the fact that for any signal y_n the best S-term approximation using Φ is given by the largest projection onto a set of S atoms $\Phi_I = (\phi_{i_1} \dots \phi_{i_S})$, ie.,

$$\begin{aligned}
\min_{\Phi \in \mathcal{D}, X \in \mathcal{X}_S} \|Y - \Phi X\|_F^2 &= \min_{\Phi \in \mathcal{D}} \sum_i \min_{\|x_n\|_0 \leq S} \|y_n - \Phi x_n\|_2^2 \\
&= \min_{\Phi \in \mathcal{D}} \sum_i \min_{|I| \leq S} \|y_n - \Phi_I \Phi_I^\dagger y_n\|_2^2 \\
&= \|Y\|_F - \max_{\Phi \in \mathcal{D}} \sum_i \max_{|I| \leq S} \|\Phi_I \Phi_I^\dagger y_n\|_2^2,
\end{aligned}$$

where $\Phi_I^\dagger$ denotes the Moore-Penrose pseudo inverse of Φ_I. Abbreviating the projection onto the span of $(\phi_i)_{i \in I}$ by $P_I(\Phi) = \Phi_I \Phi_I^\dagger$, we can thus replace the minimisation problem in (1) with the following maximisation problem,

$$\max_{\Phi \in \mathcal{D}} \sum_i \max_{|I| \leq S} \|P_I(\Phi) y_n\|_2^2. \tag{2}$$

From the above formulation it is quite easy to see the motivation for the proposed learning criterion. Indeed assume that the training signals are all $\bar{S}$-sparse in an admissible dictionary

$\bar{\Phi} \in \mathcal{D}$, ie. $Y = \bar{\Phi}\bar{X}$ and $\|\bar{x}_i\|_0 \leq \bar{S}$, then clearly there is a global maximum[1] of (2) at $\bar{\Phi}$, respectively a global minimum of (1) at $(\bar{\Phi}, \bar{X})$, as long as $\bar{S} \leq S$. However in practice we will be facing something like,

$$y_n = \bar{\Phi}\bar{x}_n + r_n \quad \Leftrightarrow \quad Y = \bar{\Phi}\bar{X} + R, \tag{3}$$

where the coefficient vectors $\bar{x}_n$ in $\bar{X}$ are only approximately S-sparse or rapidly decaying and the pure signals are corrupted with noise $R = (r_1, \ldots, r_K)$. In this case it is no longer trivial or obvious that $\bar{\Phi}$ is a local maximum of (2), but we can hope for a result of the following type.

Theorem 1.1 (Goal): Assume that the signals y_n are generated as in (3), with x_n drawn from a distribution of approximately sparse or decaying vectors and r_n random noise. As soon as the number of signals N is large enough $N \geq C$, with high probability $p \approx 1$ there will be a local maximum of (2) within distance ε from $\bar{\Phi}$.

The rest of this paper is organised as follows. We first give conditions on the dictionary and the coefficients which allow for asymptotic identifiability by studying when $\bar{\Phi}$ is exactly at a local maximum in the limiting case, ie. replacing the sum in (2) with the expectation,

$$\max_{\Phi \in \mathcal{D}} \mathbb{E}_y \left(\max_{|I| \leq S} \|P_I(\Phi)y\|_2^2 \right). \tag{4}$$

Thus in Section 2 we will prove identification results for the case when in (4) we have $S = 1$, ie. $\mathcal{X}_S = \mathcal{X}_1$, assuming first a simple (discrete, noise-free) signal model and then progressing to a noisy, continuous signal model. In Section 3 we will extend these results to the case $S > 1$. Finally in Sections 4 and 5, we will go from asymptotic results to results for finite sample sizes and prove versions of Theorem 1.1 that quantify the sizes of the parameters ε, p in terms of the number of training signals N and the size of C in terms of the number of atoms K. In the last section we will discuss the implications of our results for practical applications, compare them to existing identification results and point out some directions for future research.

2 Asymptotic identification results for $S = 1$

2.1 Notation

Before we jump into the fray, a few words on notations; usually subscripted letters will denote vectors with the exception of c and ε where they are numbers, eg. $(x_1, \ldots, x_K) = X \in \mathbb{R}^{d \times K}$ vs.

1. $\bar{\Phi}$ is a global maximiser together with all $2^K K!$ dictionaries consisting of a permutation of the atoms in $\bar{\Phi}$ provided with a ± 1 sign. For a more detailed discussion on the uniqueness of the maximiser/minimiser see eg. [17].

$c = (c_1, \dots, c_K) \in \mathbb{R}^K$, however, it should always be clear from the context what we are dealing with. For a matrix M, we denote its (conjugate) transpose by $M^\star$ and its operator norm by $\|M\|_{2,2} = \max_{\|x\|_2=1} \|Mx\|_2$.

We consider a frame Φ a collection of $K \geq d$ vectors $\phi_i \in \mathbb{R}^d$ for which there exist two positive constants A, B such that for all $v \in \mathbb{R}^d$ we have

$$A\|v\|_2^2 \leq \sum_{i=1}^{K} |\langle \phi_i, v\rangle|^2 \leq B\|v\|_2^2. \tag{5}$$

If B can be chosen equal to A, ie. $B = A$, the frame is called tight and if all elements of a tight frame have unit norm we have $A = K/d$.

Finally we introduce the Landau symbols O, o to characterise the growth of a function. We write $f(\varepsilon) = O(g(\varepsilon))$ if $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = C < \infty$ and $f(\varepsilon) = o(g(\varepsilon))$ if $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = 0$.

2.2 The problem for $S = 1$

In case $S = 1$ the expression for which we have to maximise the expectation in (4) can be radically simplified, ie.

$$\max_{|I|\leq 1} \|P_I(\Phi)y\|_2^2 = \max_i |\langle \phi_i, y\rangle|^2 = \|\Phi^\star y\|_\infty^2,$$

and the maximisation problem we want to analyse reduces to,

$$\max_{\Phi\in\mathcal{D}} \mathbb{E}_y \left(\|\Phi^\star y\|_\infty^2\right). \tag{6}$$

As mentioned in the introduction if the signals y are all 1-sparse in a dictionary $\bar\Phi$ then clearly $\bar\Phi$ is a global maximiser of (6). However what happens if we do not have perfect sparsity? Let us start with a very simple negative example of a coefficient distribution for which the original generating dictionary is not at a local maximum.

Example 2.1: Let U be an orthonormal basis and x be randomly 2-sparse with 'flat' coeffcients, ie. pick two indices i, j choose $\sigma_{i/j} = \pm 1$ uniformely at random and set $x_k = \sigma_k$ for $k = i, j$ and zero else. Then U is not a local maximum of (6). Indeed since the signals are all 2-sparse the maximal inner product with all atoms in U is the same as the maximal inner product with only $d-1$ atoms. This degree of freedom we can use to construct an ascent direction. Choose $U_\varepsilon = (u_1, \dots, u_{d-1}, (u_d + \varepsilon u_1)/\sqrt{1+\varepsilon^2})$, then we have

$$\begin{aligned}\mathbb{E}_y \left(\|U_\varepsilon^\star y\|_\infty^2\right) &= \mathbb{E}_x \left(\|(x_1, \dots, x_{d-1}, \tfrac{x_d+\varepsilon x_1}{\sqrt{1+\varepsilon^2}})\|_\infty^2\right)\\ &= \mathbb{E}_x \max\left\{1, \tfrac{(x_d+\varepsilon x_1)^2}{(1+\varepsilon^2)}\right\} = 1 + \tfrac{1}{d(d-1)}\tfrac{\varepsilon}{1+\varepsilon^2} > 1 = \mathbb{E}_y \left(\|U^\star y\|_\infty^2\right)\end{aligned}$$

From the above example we see that in order to have a local maximum at the original dictionary we need a signal/coefficient model where the coefficients show some type of decay.

2.3 A simple model of decaying coefficients

To get started we consider a very simple coefficient model, constructed from a non-negative, non-increasing sequence $c \in \mathbb{R}^K$ with $\|c\|_2 = 1$, which we permute uniformly at random and provide with random $\pm$ signs. To be precise for a permutation $p : \{1, ..., K\} \to \{1, ..., K\}$ and a sign sequence σ, $\sigma_i = \pm 1$, we define the sequence $c_{p,\sigma}$ component-wise as $c_{p,\sigma}(i) := \sigma_i c_{p(i)}$, and set $y = \Phi x$ where $x = c_{p,\sigma}$ with probability $(2^K K!)^{-1}$.

The normalisation $\|c\|_2 = 1$ has the advantage that for dictionaries, which are an orthonormal basis, the resulting signals also have unit norm and for general dictionaries the signals have unit square norm in expectation, ie. $\mathbb{E}(\|y\|_2^2) = 1$. This reflects the situation in practical application, where we would normalise the signals in order to equally weight their importance.

Armed with this model we can now prove a first dictionary identification result for (6).

Theorem 2.1: Let Φ be a unit norm tight frame with frame constant $A = K/d$ and coherence μ. Let $x \in \mathbb{R}^K$ be a random permutation of a sequence c, where $c_1 \geq c_2 \geq c_3 \ldots \geq c_K \geq 0$ and $\|c\|_2 = 1$, provided with random $\pm$ signs, i.e. $x = c_{p,\sigma}$ with probability $\mathbb{P}(p, \sigma) = (2^K K!)^{-1}$. If c satisfies $c_1 > c_2 + 2\mu\|c\|_1$ then there is a local maximum of (6) at Φ. Moreover we have the following quantitative estimate for the basin of attraction around Φ. For all perturbations $\Psi = (\psi_1 \ldots \psi_K)$ of $\Phi = (\phi_1 \ldots \phi_K)$ with $0 < \max_i \|\psi_i - \phi_i\|_2 \leq \varepsilon$ we have $\mathbb{E}_x\|\Psi^\star \Phi x\|_\infty^2 < \mathbb{E}_x\|\Phi^\star \Phi x\|_\infty^2$ as soon as $\varepsilon < 1/5$ and

$$\varepsilon \leq \frac{\left(1 - 2\frac{c_2 + \mu\|c\|_1}{c_2 + c_1}\right)^2}{2A \log\left(2AK/(c_1^2 - \frac{1-c_1^2}{K-1})\right)}. \tag{7}$$

Proof: We start by calculating the expectation of the maximally recoverable energy using the original dictionary Φ.

$$\begin{aligned}
\mathbb{E}_x\|\Phi^\star \Phi x\|_\infty^2 &= \mathbb{E}_p \mathbb{E}_\sigma \|\Phi^\star \Phi c_{p,\sigma}\|_\infty^2 \\
&= \mathbb{E}_p \mathbb{E}_\sigma \left(\max_{i=1\ldots K} |\langle \phi_i, \Phi c_{p,\sigma} \rangle|^2 \right) \\
&= \mathbb{E}_p \mathbb{E}_\sigma \left(\max_{i=1\ldots K} \left| \left\langle \phi_i, \sum_{j=1}^K \sigma_j c_{p(j)} \phi_j \right\rangle \right|^2 \right).
\end{aligned}$$

To estimate the maximal inner product we first assume that p is fixed. Setting $i_p = p^{-1}(1)$ we get

$$|\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle| = \Big|\sigma_{i_p} c_1 + \sum_{j\neq i_p} \sigma_j c_{p(j)} \langle \phi_{i_p}, \phi_j\rangle\Big| > c_1 - \mu\|c\|_1, \tag{8}$$

while for all $i \neq i_p$ we have

$$|\langle \phi_i, \Phi c_{p,\sigma}\rangle| = \Big|\sigma_i c_i + \sum_{j\neq i} \sigma_j c_{p(j)} \langle \phi_i, \phi_j\rangle\Big| < c_2 + \mu\|c\|_1. \tag{9}$$

Together with the condition that $c_1 > c_2 + 2\mu\|c\|_1$ the above estimates ensure that the maximal inner product is attained by i_p, ie.

$$\|\Phi^\star \Phi c_{p,\sigma}\|_\infty = \max_{i=1\dots K} |\langle \phi_i, \Phi c_{p,\sigma}\rangle| = |\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle|.$$

Using the concrete expression for the maximal inner product we quickly[2] arrive at,

$$\begin{aligned}
\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 &= \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right)\\
&= \mathbb{E}_p\mathbb{E}_\sigma\left(\Big|\sum_i \sigma_i c_{p(i)}\langle \phi_{i_p}, \phi_i\rangle\Big|^2\right)\\
&= \mathbb{E}_p\left(\sum_i c_{p(i)}^2 \cdot |\langle \phi_{i_p}, \phi_i\rangle|^2\right)\\
&= \mathbb{E}_p\left(c_1^2 + \sum_{i\neq i_p} c_{p(i)}^2 \cdot |\langle \phi_{i_p}, \phi_i\rangle|^2\right)\\
&= c_1^2 + \frac{(1-c_1^2)}{K-1}(A-1).
\end{aligned}$$

To compute the expectation for a perturbation of the original dictionary we first note that we can parametrise all ε-perturbations Ψ of the original dictionary Φ with $\|\psi_i - \phi_i\|_2 = \varepsilon_i \leq \varepsilon$ as

$$\psi_i = \alpha_i\phi_i + \omega_i z_i,$$

for some z_i with $\langle \phi_i, z_i\rangle = 0, \|z_i\|_2 = 1$ and $\alpha_i := 1 - \varepsilon_i^2/2$ and $\omega_i := (\varepsilon_i^2 - \varepsilon_i^4/4)^{\frac{1}{2}}$. Expanding the expectation as before we get,

$$\begin{aligned}
\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 &= \mathbb{E}_p\mathbb{E}_\sigma\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2\\
&= \mathbb{E}_p\mathbb{E}_\sigma\left(\max_{i=1\dots K}|\langle \psi_i, \Phi c_{p,\sigma}\rangle|^2\right) \qquad (10)\\
&= \mathbb{E}_p\mathbb{E}_\sigma\left(\max_{i=1\dots K}|\alpha_i\langle \phi_i, \Phi c_{p,\sigma}\rangle + \omega_i\langle z_i, \Phi c_{p,\sigma}\rangle|^2\right)
\end{aligned}$$

2. More detailed computations of the expectation can be found in Appendix A.1.

The main idea for the proof is that for small perturbations and most sign patterns σ the maximal inner product is still attained by i such that $p(i) = 1$. For p fixed and $i_p = p^{-1}(1)$ we now have

$$\begin{aligned}
|\langle \psi_{i_p}, \Phi c_{p,\sigma}\rangle| &= |\alpha_{i_p}\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle + \omega_{i_p}\langle z_{i_p}, \Phi c_{p,\sigma}\rangle| \\
&= |\alpha_{i_p}\sigma_{i_p}c_1 + \alpha_{i_p}\sum_{j\neq i_p}\sigma_j c_{p(j)}\langle \phi_{i_p}, \phi_j\rangle + \omega_{i_p}\sum_{j\neq i_p}\sigma_j c_{p(j)}\langle z_{i_p}, \phi_j\rangle| \\
&\geq \alpha_{i_p}c_1 - \alpha_{i_p}\mu\|c\|_1 - \omega_{i_p}|\sum_{j\neq i_p}\sigma_j c_{p(j)}\langle z_{i_p}, \phi_j\rangle|.
\end{aligned}$$

Using Hoeffding's inequality we can estimate the typical size of the sum in the last expression,

$$\begin{aligned}
\mathbb{P}(|\langle z_{i_p}, \Phi c_{p,\sigma}\rangle| \geq t) &= P(|\sum_{j\neq i_p}\sigma_j c_{p(j)}\langle z_{i_p}, \phi_j\rangle| > t) \\
&\leq 2\exp\left(-\frac{t^2}{2\sum_{j\neq i_p} c_{p(j)}^2\langle z_{i_p}, \phi_j\rangle^2}\right) \leq 2\exp\left(-\frac{t^2}{2Ac_2^2}\right).
\end{aligned}$$

In case $\omega_{i_p} \neq 0$ or equivalently $\varepsilon_{i_p} \neq 0$, we set $t = sc_2/\omega_{i_p}$ to arrive at

$$\mathbb{P}(\omega_{i_p}|\langle z_{i_p}, \Phi c_{p,\sigma}\rangle| \geq sc_2) \leq 2\exp\left(-\frac{s^2}{2A\omega_{i_p}^2}\right) \leq 2\exp\left(-\frac{s^2}{2A\varepsilon_{i_p}^2}\right),$$

where we have used that $\omega_{i_p}^2 = \varepsilon_{i_p}^2 - \varepsilon_{i_p}^4/4 \leq \varepsilon_{i_p}^2$.
Similarly for $i \neq i_p$ we have

$$\begin{aligned}
|\langle \psi_i, \Phi c_{p,\sigma}\rangle| &= |\alpha_i\sigma_i c_i + \alpha_i\sum_{j\neq i}\sigma_j c_{p(j)}\langle \phi_i, \phi_j\rangle + \omega_i\sum_{j\neq i}\sigma_j c_{p(j)}\langle z_i, \phi_j\rangle| \\
&\leq \alpha_i c_2 + \alpha_i\mu\|c\|_1 + \omega_i|\sum_{j\neq i}\sigma_j c_{p(j)}\langle z_i, \phi_j\rangle|,
\end{aligned}$$

and, by Hoeffding's inequality,

$$\begin{aligned}
\mathbb{P}(|\langle z_i, \Phi c_{p,\sigma}\rangle| \geq t) &= P(|\sum_{j\neq i}\sigma_j c_{p(j)}\langle z_i, \phi_j\rangle| > t) \\
&\leq 2\exp\left(-\frac{t^2}{2\sum_{j\neq i} c_{p(j)}^2\langle z_i, \phi_j\rangle^2}\right) \leq 2\exp\left(-\frac{t^2}{2Ac_1^2}\right).
\end{aligned}$$

Thus in case $\omega_i, \varepsilon_i \neq 0$ we get

$$\mathbb{P}(\omega_i|\langle z_i, \Phi c_{p,\sigma}\rangle| \geq sc_1) \leq 2\exp\left(-\frac{s^2}{2A\omega_i^2}\right) \leq 2\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right).$$

Note that in case $\varepsilon_i = 0$ we trivially have that

$$\mathbb{P}(\omega_i|\langle z_i, \Phi c_{p,\sigma}\rangle| \geq sc_{1/2}) = 0$$

Summarising these findings we see that except with probability $\eta := 2\sum_{i|\varepsilon_i\neq 0}\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right)$,

$$|\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle| \geq \alpha_{i_p}c_1 - \alpha_{i_p}\mu\|c\|_1 - sc_2 \quad \text{and}$$
$$|\langle\psi_i,\Phi c_{p,\sigma}\rangle| \leq \alpha_i c_{p(i)} + \alpha_i\mu\|c\|_1 + sc_1 \quad \forall i\neq i_p.$$

This means that as long as $\alpha_{i_p}c_1 - \alpha_{i_p}\mu\|c\|_1 - sc_2 \geq \alpha_i c_{p(i)} + \alpha_i\mu\|c\|_1 + sc_1$ for all $i\neq i_p$, which is for instance implied by setting $s = 1-\frac{\varepsilon^2}{2} - 2\frac{c_2+\mu\|c\|_1}{c_2+c_1}$ we have

$$\|\Psi^\star\Phi c_{p,\sigma}\|_\infty = \max_{i=1\dots K}|\langle\psi_i,\Phi c_{p,\sigma}\rangle| = |\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle|.$$

We now use this result for the calculation of the expectation over σ in (10). For any permutation p we define the set,

$$\Sigma_p := \bigcup_{i\neq i_p}\{\sigma \text{ s.t. } \omega_i|\langle z_i,\Phi c_{p,\sigma}\rangle| \geq sc_1\} \cup \{\sigma \text{ s.t. } \omega_{i_p}|\langle z_{i_p},\Phi c_{p,\sigma}\rangle| \geq sc_2\}.$$

We then have

$$\mathbb{E}_\sigma\left(\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2\right) = \sum_{\sigma\in\Sigma_p}\mathbb{P}(\sigma)\cdot\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2 + \sum_{\sigma\notin\Sigma_p}\mathbb{P}(\sigma)\cdot\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2.$$

The sum over Σ_p can be bounded as,

$$\sum_{\sigma\in\Sigma_p}\mathbb{P}(\sigma)\cdot\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2 \leq \mathbb{P}(\Sigma_p)\cdot\max_{\sigma\in\Sigma_p}\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2 \leq \eta\cdot A,$$

while for the complementary sum we get,

$$\begin{aligned}\sum_{\sigma\notin\Sigma_p}\mathbb{P}(\sigma)\cdot\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2 &= \sum_{\sigma\notin\Sigma_p}\mathbb{P}(\sigma)|\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle|^2 \\ &\leq \sum_{\sigma}\mathbb{P}(\sigma)|\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle|^2 = \mathbb{E}_\sigma\left(|\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle|^2\right)\end{aligned}$$

Re-substituting these estimates into (10) we get

$$\begin{aligned}\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 &= \mathbb{E}_p\mathbb{E}_\sigma\|\Psi^\star\Phi c_{p,\sigma}\|_\infty^2 \\ &\leq \mathbb{E}_p\left(A\eta + \mathbb{E}_\sigma\left(|\langle\psi_{i_p},\Phi c_{p,\sigma}\rangle|^2\right)\right) \\ &= A\eta + \frac{c_1^2}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2 + \frac{(1-c_1^2)}{(K-1)}\left(A - \frac{1}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2\right).\end{aligned}$$

Again more detailed calculations can be found in Appendix A.1. Recalling the definition that $\eta = 2\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right)$ with $s = 1-\frac{\varepsilon^2}{2} - 2\frac{c_2+\mu\|c\|_1}{c_2+c_1}$ and that $|\langle\psi_i,\phi_i\rangle| = \alpha_i = 1-\varepsilon_i^2/2$ leads us to,

$$\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 \le 2A\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right)+\frac{c_1^2}{K}\sum_{i=1}^K(1-\varepsilon_i^2/2)^2+\frac{1-c_1^2}{K-1}\left(A-\frac{1}{K}\sum_{i=1}^K(1-\varepsilon_i^2/2)^2\right)$$

$$\le c_1^2+\frac{1-c_1^2}{K-1}(A-1)$$

$$+\sum_{\varepsilon_i\neq 0}2A\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right)-\frac{c_1^2}{K}\sum_{i=1}^K(\varepsilon_i^2-\varepsilon_i^4/4)+\frac{1-c_1^2}{K(K-1)}\sum_{i=1}^K(\varepsilon_i^2-\varepsilon_i^4/4)$$

$$=\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2+\frac{1}{K}\sum_{\varepsilon_i\neq 0}\left(2AK\exp\left(-\frac{s^2}{2A\varepsilon_i^2}\right)-c_1^2(\varepsilon_i^2-\varepsilon_i^4/4)+\frac{1-c_1^2}{K-1}(\varepsilon_i^2-\varepsilon_i^4/4)\right).$$

Thus to prove that $\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2<\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2$ for all ε-perturbations Ψ, it suffices to show that for all $0<\varepsilon_i\le\varepsilon$ we have

$$2AK\exp\left(-\frac{(1-\frac{\varepsilon^2}{2}-2\frac{c_2+\mu\|c\|_1}{c_2+c_1})^2}{2A\varepsilon_i^2}\right)-c_1^2(\varepsilon_i^2-\varepsilon_i^4/4)+\frac{1-c_1^2}{K-1}(\varepsilon_i^2-\varepsilon_i^4/4)<0.$$

Since both e^{-c/ε^2} and ε^4 tend much faster to zero than ε^2 as ε goes to zero, this condition will be satisfied as soon as ε is small enough. Using some trickery that can be found in Appendix A.2 we can show that indeed all is fine if $\varepsilon\le 1/5$ and

$$\varepsilon\le\frac{\left(1-2\frac{c_2+\mu\|c\|_1}{c_2+c_1}\right)^2}{2A\log\left(2AK/(c_1^2-\frac{1-c_1^2}{K-1})\right)}.$$

□

Let us comment the result.

Remark 2.2: (i) First one may question why we chose the complicated approach above instead of doing a first order analysis using the the tangent space to the constraint manifold $\mathcal{D}$, as in [17]. The answer is simple, it fails. As can be seen during the proof, the first order terms $O(\varepsilon)$ are zero, requiring us to keep track also of the second order terms $O(\varepsilon^2)$.

(ii) Next note that in some sense Theorem 2.1 is sharp. Assume that Φ is an orthonormal basis (ONB) then $\mu=0$ and the condition to be a local minimum reduces to $c_1>c_2$. However from Example 2.1 we see that if $c_1=c_2$ we can again construct an ascent direction and so Φ is not a local maximum.

(iii) Similarly the condition that Φ is a tight frame is almost necessary in the non-trivial case where $|c_1|<1$.

Assume that Φ is not tight, ie. $A\|v\|_2^2\le\sum_i|\langle v,\phi_i\rangle|^2\le B\|v\|_2^2$, with $A<B$. Going through the

proof of Theorem 2.1 we see that using the same arguments, we again have

$$\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 = \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) \quad \text{and} \quad \mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 \geq \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right)$$

and by replacing A with B where relevant get the new upper bound,

$$\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 \leq \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) + 2B\tilde{\eta}$$

for

$$\tilde{\eta} = \sum_{\varepsilon_i\neq 0} \exp\left(-\frac{\left(1-\frac{\varepsilon^2}{2} - 2\frac{c_2+\mu\|c\|_1}{c_2+c_1}\right)^2}{2B\varepsilon_i^2}\right). \tag{11}$$

Since $B\tilde{\eta}$ is still of order $o(\varepsilon^2)$ to prove that Φ is a local maximum it suffices to show that up to second order $\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) > \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right)$. Conversely if we can find perturbation directions z_i such that the reversed inequality holds, Φ is not a local maximum. Using the explicit expressions for the expectations from the appendix, we get

$$\begin{aligned}
&\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) - \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right)\\
&\quad= c_1^2 + \frac{1-c_1^2}{K(K-1)}\left(\|\Phi^\star\Phi\|_F^2 - K\right) - \frac{c_1^2}{K}\sum_i |\langle\phi_i,\psi_i\rangle|^2 - \frac{1-c_1^2}{K(K-1)}\left(\|\Phi^\star\Psi\|_F^2 - \sum_i |\langle\phi_i,\psi_i\rangle|^2\right)\\
&\quad= \left(c_1^2 - \frac{1-c_1^2}{K-1}\right)\frac{1}{K}\sum_i(1-\alpha_i^2) + \frac{1-c_1^2}{K(K-1)}\left(\sum_i \|\Phi^\star\phi_i\|_2^2 - \sum_i \|\Phi^\star(\alpha_i\phi_i + \omega_i z_i)\|_2^2\right)\\
&\quad= \left(c_1^2 - \frac{1-c_1^2}{K-1}\right)\frac{1}{K}\sum_i\omega_i^2 + \frac{1-c_1^2}{K(K-1)}\left(\sum_i \omega_i^2(\|\Phi^\star\phi_i\|_2^2 - \|\Phi^\star z_i\|_2^2) - 2\sum_i \alpha_i\omega_i\langle\Phi\Phi^\star\phi_i, z_i\rangle\right).
\end{aligned}$$

Recalling that $\alpha_i = 1-\varepsilon_i^2/2$ and $\omega_i = (\varepsilon_i^2 - \varepsilon_i^4/4)^{\frac{1}{2}}$, we see that all terms in the above expression are of the order $O(\varepsilon^2)$ except for the last $\sum_i \alpha_i\omega_i\langle\Phi\Phi^\star\phi_i, z_i\rangle$ which is of order ε. Now assume that there exists an atom ϕ_{i_0} and an orthogonal perturbation direction z, such that $\langle\Phi\Phi^\star\phi_{i_0}, z\rangle \neq 0$, then for Ψ with $\psi_{i_0} = \alpha_i\phi_{i_0} + \sigma\omega z$, where $\sigma = \operatorname{sign}(\langle\Phi\Phi^\star\phi_{i_0}, z\rangle)$, and $\psi_i = \phi_i$ for all $i \neq i_0$, the expression above will be smaller than zero as soon as ε is small enough, meaning that Φ is not a local maximum.

Consequently a necessary condition for Φ to be a local maximum is that $\langle\Phi\Phi^\star\phi_i, z\rangle = 0$ whenever $\langle\phi_i, z\rangle = 0$, which is equivalent to every atom being an eigenvector of the frame operator of the dictionary, ie. $\Phi\Phi^\star\phi_i = \lambda_i\phi_i, \forall i$. While this condition is certainly fulfilled when Φ is a tight frame (corresponding to $\lambda_i = A$), it is sufficient for Φ to be a collection of m tight frames for m orthogonal subspaces of $\mathbb{R}^d$ (corresponding to the case $\Phi = (\Phi_{\lambda_1}, \ldots, \Phi_{\lambda_m})$ with $\Phi\Phi^\star\Phi_{\lambda_i} = \lambda_i\Phi_{\lambda_i}$).

Going through the same analysis as in the proof of Theorem 2.1 we see that in this second case Φ is again a local maximum under the additional condition that $c_1^2 > \frac{B-A+1}{B-A+K}$, where $A = \min_i \lambda_i$ and $B = \max_i \lambda_i$. However, for simplicity we will henceforth restrict our analysis to the situation where Φ is a tight frame.

2.4 A continuous model of decaying coefficients

After proving a recovery result for the simple coefficient model of the last section we would like to extend it to a wider range of coefficient distributions, especially continuous ones. To see which distributions are good candidates we will point out the properties of the simple model we needed for the proof to succeed.

- To see for which index the inner products $\langle \Phi_i, \Phi c_{p,\sigma} \rangle$ were maximal, cp. (8/9), we used the decay-condition $c_1 > c_2 + 2\mu \|c\|_1$.
- For the calculation of $\mathbb{E}_{p,\sigma} |\langle \Phi_{i_p}, \Phi c_{p,\sigma} \rangle|^2$ we used that the largest coefficient was equally likely to have any index, which was ensured by the fact that each permutation of the base sequence c was equally likely.
- Finally to bound the size of the inner products $\langle z_i, c_{p,\sigma} \rangle$ and thus the size of $\langle \psi_i, c_{p,\sigma} \rangle$ with high probability we needed the equal probability of all sign patterns.

Using these three observations we can now make the following definitions

Definition 2.1: A probability measure ν on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is called symmetric if for all measurable sets $\mathcal{X} \subseteq S^{d-1}$, for all sign sequences $\sigma \in \{-1, 1\}^d$ and all permutations p we have

$$\nu(\sigma \mathcal{X}) = \nu(\mathcal{X}), \quad \text{where} \quad \sigma \mathcal{X} := \{(\sigma_1 x_1, \ldots, \sigma_d x_d) : x \in \mathcal{X}\} \tag{12}$$

$$\nu(p(\mathcal{X})) = \nu(\mathcal{X}), \quad \text{where} \quad p(\mathcal{X}) := \{(x_{p(1)}, \ldots, x_{p(d)}) : x \in \mathcal{X}\} \tag{13}$$

Definition 2.2: A probability distribution ν on the unit sphere $S^{K-1} \subset \mathbb{R}^K$ is called (β, μ)-decaying if there exists a $\beta < 1/2$ such that for $c_1(x) \geq c_2(x) \geq \ldots \geq c_d(x) \geq 0$ a non increasing rearrangement of the absolute values of the components of x we have,

$$\nu \left(\frac{c_2(x) + \mu \|c(x)\|_1}{c_2(x) + c_1(x)} \leq \beta \right) = 1 \tag{14}$$

For the case $\mu = 0$ it will also be useful to define the following notion. A probability distribution ν on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is called f-decaying if there exists a function f such that

$$\exp\left(-\frac{f(\varepsilon)^2}{8\varepsilon^2} \right) = o(\varepsilon^2) \tag{15}$$

and

$$\nu\left(\frac{c_2(x)}{c_1(x)} \geq 1 - f(\varepsilon)\right) = o(\varepsilon^2). \tag{16}$$

Note that $(\beta, 0)$-decaying is a special case of f-decaying, ie. $f(\varepsilon)$ can be chosen constant β. To illustrate both concepts we give simple examples for (β, μ)- and f-decaying distributions on S^1.

Example 2.3: • Let ν be the symmetric distribution on S^1 defined by $c_2(x)$ being uniformly distributed on $[0, \frac{1}{\sqrt{2}} - \theta]$ for $\theta > 0$ (and accordingly $c_1(x) = \sqrt{1 - c_2^2(x)}$), then ν is (β, μ)-decaying for all $\mu < \frac{\theta}{\sqrt{2}}$.

- Let ν be the symmetric distribution on S^1 defined by $c_2(x)$ being distributed on $[0, \frac{1}{\sqrt{2}}]$ with density $20\sqrt{2}(\frac{1}{\sqrt{2}} - x)^4$, then ν is f-decaying for e.g. $f(\varepsilon) = \sqrt{\varepsilon}$.
- Let ν be the symmetric distribution on S^1 defined by $c_2(x)$ being distributed on $[0, \frac{1}{\sqrt{2}}]$ with density $4(\frac{1}{\sqrt{2}} - x)$, then ν is not f-decaying.

While the decay properties for the first two examples follow from basic integrations, we will elaborate shortly on this last example. For any function f we have the lower bound,

$$\begin{aligned}\nu\left(\frac{c_2(x)}{c_1(x)} \geq 1 - f(\varepsilon)\right) &= 4\int_{\frac{1-f(\varepsilon)}{\sqrt{2-2f(\varepsilon)+f(\varepsilon)^2}}}^{\frac{1}{\sqrt{2}}} \left(\frac{1}{\sqrt{2}} - x\right) dx \\ &= \left(1 - \frac{1 - f(\varepsilon)}{\sqrt{1 - f(\varepsilon) + f(\varepsilon)^2/2}}\right)^2 \geq \frac{f(\varepsilon)^2}{4}.\end{aligned}$$

This means that we need $f(\varepsilon)^2 = o(\varepsilon^2)$ at the same time as $\exp\left(-\frac{f(\varepsilon)^2}{8\varepsilon^2}\right) = o(\varepsilon^2)$, which is impossible, so ν cannot be f-decaying.

An important group of probability distributions expected to be (β, μ)-decaying are the distributions introduced in [33] to model strongly compressible, ie. nearly sparse vectors.

With these examples of suitable probability distributions in mind we can now turn to proving a continuous version of Theorem 2.1.

Theorem 2.2: (a) Let Φ be a unit norm tight frame with frame constant $A = K/d$ and coherence μ. If x is drawn from a symmetric (β, μ)-decaying probability distribution ν on the unit sphere S^{K-1}, then there is a local maximum of (6) at Φ and we have the following quantitative estimate for the basin of attraction around Φ. Define $\bar{c}_1^2 := \mathbb{E}_x\|x\|_\infty^2$. For all perturbations $\Psi = (\psi_1 \ldots \psi_K)$ of $\Phi = (\phi_1 \ldots \phi_K)$ with $0 < \max_i \|\psi_i - \phi_i\|_2 \leq \varepsilon$ we have $\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 < \mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2$ as soon as

$\varepsilon < 1/5$ and

$$\varepsilon \le \frac{(1-2\beta)^2}{2A\log\left(2AK/(\bar{c}_1^2 - \frac{1-\bar{c}_1^2}{K-1})\right)}.$$

(b) If Φ is an orthonormal basis, there is a local maximum of (6) at Φ whenever x is drawn from a symmetric f-decaying probability distribution ν on the unit sphere S^{d-1}.

Proof: (a) Let c denote the mapping that assigns to each $x \in S^{d-1}$ the non increasing rearrangement of the absolute values of its components, i.e. $c_i(x) = |x_{p(i)}|$ for a permutation p such that $c_1(x) \ge c_2(x) \ge \ldots \ge c_d(x) \ge 0$. Then the mapping c together with the probability measure ν on S^{K-1} induces a pull-back probability measure ν_c on $c(S^{K-1})$, by $\nu_c(\Omega) := \nu(c^{-1}(\Omega))$ for any measurable set $\Omega \subseteq c(S^{K-1})$. With the help of this new measure we can rewrite the expectations we need to calculate as,

$$\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 = \int_x \|\Phi^\star\Phi x\|_\infty^2 d\nu(x) = \int_{c(x)} \mathbb{E}_p\mathbb{E}_\sigma\|\Phi^\star\Phi c_{p,\sigma}(x)\|_\infty^2 d\nu_c(x).$$

The expectation inside the integral should seem familiar. Indeed we have calculated it already in the proof of Theorem 2.1 for $c(x)$ a fixed decaying sequence satisfying $c_1(x) > c_2(x) + 2\mu\|c(x)\|_1$. This property is satisfied almost surely since ν is (β,μ)-decaying and so we have,

$$\begin{aligned}
\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 &= \int_{c(x)} \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\phi_{i_p}, \Phi c_{p,\sigma}(x)\rangle|^2\right) d\nu_c(x)\\
&= \int_{c(x)} c_1^2(x) + \frac{1-c_1^2(x)}{K-1}(A-1) d\nu_c(x)\\
&= \int_x c_1^2(x) + \frac{1-c_1^2(x)}{K-1}(A-1) d\nu(x).
\end{aligned}$$

Note that the integral term $\int_{c(x)} c_1(x)^2 d\nu_c(x)$ is simply $\mathbb{E}_x\|x\|_\infty^2 = \bar{c}_1^2$, leading to the concise expression for the expectation,

$$\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 = \bar{c}_1^2 + \frac{1-\bar{c}_1^2}{K-1}(A-1).$$

For the expectation of a perturbed dictionary Ψ we get in analogy

$$\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 \le \int_{c(x)} A\eta(x) + \mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p}, \Phi c_{p,\sigma}(x)\rangle|^2\right) d\nu_c(x), \tag{17}$$

where

$$\eta(x) := 2\sum_{\varepsilon_i \ne 0} \exp\left(-\frac{\left(1 - \frac{\varepsilon^2}{2} - 2\frac{c_2(x)+\mu\|c(x)\|_1}{c_2(x)+c_1(x)}\right)}{2A\varepsilon_i^2}\right).$$

Define,

$$\eta_\beta := 2\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{(1-\frac{\varepsilon^2}{2}-2\beta)^2}{2A\varepsilon_i^2}\right),$$

then since ν is (β,μ)-decaying $\eta(x)\leq\eta_\beta$ almost surely. Continuing the estimate in (17) we get

$$\begin{aligned}\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 &\leq A\eta_\beta+\int_{c(x)}\frac{c_1^2(x)}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2+\frac{1-c_1^2(x)}{K-1}\left(A-\frac{1}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2\right)d\nu_c(x)\\ &= A\eta_\beta+\frac{\bar{c}_1^2}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2+\frac{1-\bar{c}_1^2}{K-1}\left(A-\frac{1}{K}\sum_i|\langle\psi_i,\phi_i\rangle|^2\right).\end{aligned}$$

Following the same argument as in the proof of Theorem 2.1 we see that $\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 < \mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2$ once we have $\varepsilon\leq 1/5$ and

$$\varepsilon\leq\frac{(1-2\beta)^2}{2A\log\left(2AK/(\bar{c}_1^2-\frac{1-\bar{c}_1^2}{K-1})\right)}.$$

(b) If Φ is actually an orthonormal basis, ie. $A=1$, we simply have $\mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2=\mathbb{E}_x\|x\|_\infty^2=\bar{c}_1^2$. However if ν is only f-decaying we need to be more careful in our estimation of $\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2$. Let ι denote the index for which ε_i is maximal. We have,

$$\begin{aligned}\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 &= \int_{x:\frac{c_2(x)}{c_1(x)}\geq 1-f(\varepsilon_\iota)}\|\Psi^\star\Phi x\|_\infty^2 d\nu(x)+\int_{x:\frac{c_2(x)}{c_1(x)}<1-f(\varepsilon_\iota)}\|\Psi^\star\Phi x\|_\infty^2 d\nu(x)\\ &\leq \nu\left(\frac{c_2(x)}{c_1(x)}\geq 1-f(\varepsilon_\iota)\right)+\int_{c(x):\frac{c_2(x)}{c_1(x)}<1-f(\varepsilon_\iota)}\mathbb{E}_p\mathbb{E}_\sigma\|\Phi^\star\Phi c_{p,\sigma}(x)\|_\infty^2 d\nu_c(x)\end{aligned}$$

For convenience we write $\Omega:=\{c(x):\frac{c_2(x)}{c_1(x)}<1-f(\varepsilon_\iota)\}$, leading to

$$\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2\leq\nu\left(\frac{c_2(x)}{c_1(x)}\geq 1-f(\varepsilon_\iota)\right)+\int_\Omega\eta_\iota(x)+\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p},\Phi c_{p,\sigma}(x)\rangle|^2\right)d\nu_c(x)$$

where

$$\eta_\iota(x):=2\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{\left(1-\frac{\varepsilon_\iota^2}{2}-\frac{2c_2(x)}{c_2(x)+c_1(x)}\right)}{2\varepsilon_i^2}\right).$$

As long as $c(x)\in\Omega$ we have $\eta_\iota(x)\leq 2\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{(f(\varepsilon_\iota)-\varepsilon_\iota^2)^2}{8\varepsilon_i^2}\right)$, so we can further bound

$$\begin{aligned}\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 &\leq \nu\left(\frac{c_2(x)}{c_1(x)}\geq 1-f(\varepsilon_\iota)\right)\\ &\qquad +2\sum_{\varepsilon_i\neq 0}\exp\left(-\frac{(f(\varepsilon_\iota)-\varepsilon_\iota^2)^2}{8\varepsilon_i^2}\right)+\int\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle\psi_{i_p},\Phi c_{p,\sigma}(x)\rangle|^2\right)d\nu_c(x)\\ &\leq \nu\left(\frac{c_2(x)}{c_1(x)}\geq 1-f(\varepsilon_\iota)\right)+2d\exp\left(-\frac{(f(\varepsilon_\iota)-\varepsilon_\iota^2)^2}{8\varepsilon_\iota^2}\right)\\ &\qquad +\frac{\bar{c}_1^2}{d}\sum_i|\langle\psi_i,\phi_i\rangle|^2+\frac{1-\bar{c}_1^2}{d-1}\left(1-\frac{1}{d}\sum_i|\langle\psi_i,\phi_i\rangle|^2\right),\end{aligned}$$

leading to the following estimate,

$$\mathbb{E}_x\|\Psi^\star\Phi x\|_\infty^2 - \mathbb{E}_x\|\Phi^\star\Phi x\|_\infty^2 = \nu\left(\frac{c_2(x)}{c_1(x)} \geq 1 - f(\varepsilon_\iota)\right) + 2d\exp\left(-\frac{(f(\varepsilon_\iota)-\varepsilon_\iota^2)^2}{8\varepsilon_\iota^2}\right) - \left(\bar{c}_1^2 - \frac{1-\bar{c}_1^2}{d-1}\right)\frac{1}{d}\sum_i(\varepsilon_i^2 - \varepsilon_i^4/4).$$

The terms in ε_i in the above estimates are clearly smaller than zero for $\varepsilon_i \leq \varepsilon_\iota \leq 1$ so to finish the proof all that remains to be shown is that

$$\nu\left(\frac{c_2(x)}{c_1(x)} \geq 1 - f(\varepsilon_\iota)\right) + 2d\exp\left(-\frac{(f(\varepsilon_\iota)-\varepsilon_\iota^2)^2}{8\varepsilon_\iota^2}\right) < \left(\bar{c}_1^2 - \frac{1-\bar{c}_1^2}{d-1}\right)(\varepsilon_\iota^2 - \varepsilon_\iota^4/4).$$

This, however, is guaranteed by ν be f-decaying, which ensures that the first two terms in the above expression are of order $o(\varepsilon_\iota^2)$ and therefore smaller than the third term of order $O(\varepsilon_\iota^2)$, as soon as ε_ι is close enough to zero. □

Remark 2.4: It would of course be possible to extend the notion of f-decaying to (f,μ)-decaying. However, for $\mu > 0$ the condition $c_1 > c_2 + \mu\|c\|$ is only sufficient but not necessary for Φ to be a local minimum. It is merely the result of using the simple but crude bounds in (8) and (9) and could for instance be replaced by $(1+\mu)c_1 > (1-\mu)c_2 + \mu\|c\|_1$. Thus unless we have a sharp bound on the coefficient sequence for $|\langle\phi_i, \Phi c_{p,\sigma}\rangle|$ to take its maximum uniquely at $i = i_p$ it is quite useless to try to approach this bound in probability.

2.5 Bounded white noise

With the tools used to prove the two noiseless identification results in the last two subsections it is also possible to analyse the case of (very small) bounded white noise.

Theorem 2.3: Let Φ be a unit norm tight frame with frame constant $A = K/d$ and coherence μ. Assume that the signals y are generated from the following model

$$y = \Phi x + r, \tag{18}$$

where r is a bounded random white noise vector, ie. there exist two constants $\rho, \rho_{\max}$ such that $\|r\|_2 \leq \rho_{\max}$ almost surely, $\mathbb{E}(r) = 0$ and $\mathbb{E}(rr^\star) = \rho^2 I$. If x is drawn from a symmetric decaying probability distribution ν on the unit sphere S^{K-1} with $\mathbb{E}_x\|x\|_\infty^2 = \bar{c}_1^2$ and the maximal size of the noise is small compared to the size and decay of the coefficients c_1, c_2, meaning there exists $\beta < 1/2$, such that

$$\nu\left(\frac{c_2(x) + \mu\|c(x)\|_1 + \rho_{\max}}{c_1(x) - c_2(x)} \leq \beta\right) = 1 \tag{19}$$

then there is a local maximum of (6) at Φ and we have the following quantitative estimate for the basin of attraction around Φ. For all perturbations $\Psi = (\psi_1 \dots \psi_K)$ of $\Phi = (\phi_1 \dots \phi_K)$ with $0 < \max_i \|\psi_i - \phi_i\|_2 \leq \varepsilon$ we have $\mathbb{E}_y \|\Psi^\star y\|_\infty^2 < \mathbb{E}_y \|\Phi^\star y\|_\infty^2$ as soon as $\varepsilon < 1/5$ and

$$\varepsilon \leq \frac{(1-2\beta)^2}{2A \log\left(2AK/(\bar{c}_1^2 - \frac{1-\bar{c}_1^2}{K-1})\right)}.$$

Proof: We just sketch the proof, since it relies on the same ideas as those of Theorem 2.1 and Theorem 2.2. Condition (19) ensures that with probability 1 $\max_i |\langle \phi_i, y\rangle| = \max_i |\langle \phi_i, \Phi x + r\rangle|$ is attained for $i = i_p$, so we have

$$\begin{aligned}\mathbb{E}_y \|\Phi^\star y\|_\infty^2 &= \mathbb{E}_{x,r} |\langle \phi_{i_p}, \Phi x + r\rangle|^2 \\ &= \mathbb{E}_x |\langle \phi_{i_p}, \Phi x\rangle|^2 + \mathbb{E}_r |\langle \phi_{i_p}, r\rangle|^2 = \mathbb{E}_x |\langle \phi_{i_p}, \Phi x\rangle|^2 + \rho^2.\end{aligned}$$

Similarly $\max_i |\langle \psi_i, y\rangle| = \max_i |\langle \psi_i, \Phi x + r\rangle|$ is attained for $i = i_p$ except with probability at most

$$\eta_\beta := 2 \sum_{\varepsilon_i \neq 0} \exp\left(-\frac{(1 - \frac{\varepsilon^2}{2} - 2\beta)^2}{2A\varepsilon_i^2}\right),$$

leading to

$$\begin{aligned}\mathbb{E}_y \|\Psi^\star y\|_\infty^2 &\leq A\eta_\beta + \mathbb{E}_{x,r} |\langle \psi_{i_p}, \Phi x + r\rangle|^2 \\ &= A\eta_\beta + \mathbb{E}_x |\langle \psi_{i_p}, \Phi x\rangle|^2 + \mathbb{E}_r |\langle \psi_{i_p}, r\rangle|^2 = A\eta_\beta + \mathbb{E}_x |\langle \psi_{i_p}, \Phi x\rangle|^2 + \rho^2.\end{aligned}$$

The result then follows from the usual arguments. □

3 Asymptotic identification results for $S \geq 1$

In this section we extend the identification results from the last section to the case where $S \geq 1$, ie. we study the problem

$$\max_{\Psi \in \mathcal{D}} \mathbb{E}_y \left(\max_{|I| \leq S} \|P_I(\Psi) y\|_2^2\right). \tag{20}$$

We use essentially the same tools as for the 1-sparse case. However, since the problem does not reduce, the proofs become more technical - for instance we need to estimate the difference between $P_I(\Phi)$ and $P_I(\Psi)$ instead of ϕ_i and ψ_i and need a vector version of Hoeffding's inequality to estimate the typical size of $P_I(\Phi)\Phi c_{p,\sigma}$. So to keep the presentation concise we rely heavily on the O, o notation. Also the results are in a different spirit. We trade concreteness, such as explicit conditions on the coefficient sequence for Φ to be a local maximum or an estimate for the basin of attraction, for sharpness by formulating our results as tight as the available tools permit.

We start by proving a general version of Theorem 2.1 for the simple coefficient model introduced in Section 2.3, which will again lay the ground work for the more complicated signal models.

Theorem 3.1: Let Φ be a unit norm tight frame with frame constant $A = K/d$ and coherence μ. Let x be a random permutation of a positive, nonincreasing sequence c, where $c_1 \geq c_2 \geq c_3 \ldots \geq c_K \geq 0$ and $\|c\|_2 = 1$, provided with random $\pm$ signs, i.e. $x = c_{p,\sigma}$ with probability $\mathbb{P}(p,\sigma) = (2^K K!)^{-1}$. Assume that the signals are generated as $y = \Phi x$. If we have

$$\forall \sigma, p: \ \|P_{I_p}(\Phi)\Phi c_{p,\sigma}\|_2 > \max_{|I|\leq S, I\neq I_p} \|P_I(\Phi)\Phi c_{p,\sigma}\|_2, \ \text{where } I_p := p^{-1}\left(\{1,\ldots S\}\right), \tag{21}$$

then there is a local maximum of (20) at Φ.

Proof: We first calculate the expectation using the original dictionary Φ. Condition (21) quite obviously (and artlessly) guarantees that the maximum is always attained for the set I_p, so setting $\gamma^2 := c_1^2 + \ldots + c_S^2$ we get[3],

$$\begin{aligned}
\mathbb{E}_y\left(\max_{|I|\leq S} \|P_I(\Phi)y\|_2^2\right) &= \mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Phi)\Phi c_{p,\sigma}\|_2^2\right) \\
&= \frac{A(1-\gamma^2)S}{(K-S)} + \left(\frac{\gamma^2}{S} - \frac{1-\gamma^2}{K-S}\right)\binom{K}{S}^{-1}\sum_J \|\Phi_J\|_F^2 \\
&= \gamma^2 + \frac{(A-1)(1-\gamma^2)S}{K-S}.
\end{aligned}$$

We use the same parametrisation for all ε-perturbations as in the last section. Since we have to calculate with projections $P_I(\Psi)$ we also define $A_I = \operatorname{diag}(\alpha_i)_{i\in I}$ and $W_I = \operatorname{diag}(\omega_i)_{i\in I}$ to get $\Psi_I = \Phi_I A_I + Z_I W_I$.

As in the case $S = 1$ our strategy will be to show that with high probability for a fixed permutation p the maximal projection is still onto the atoms indexed by I_p.

For any index set I of size S we can bound the difference between the projection using the corresponding atoms in Ψ or Φ using the reversed triangular inequality,

$$\left|\|P_I(\Psi)\Phi y\|_2 - \|P_I(\Phi)\Phi y\|_2\right| \leq \|\big(P_I(\Psi) - P_I(\Phi)\big)\Phi y\|_2. \tag{22}$$

3. for a detailed calculation see Appendix B.1

To estimate the typical size of the right hand side in the above equation we need a vector valued version of Hoeffding's inequality. We take the following convenient if not optimal concentration inequality for Rademacher series from [21], Chapter 4.

Corollary 3.2 (of Theorem 4.7 in [21]): For a vector-valued Rademacher series $V = \sum_i \sigma_i v_i$, ie. for σ_i independent Bernoulli variables with $\mathbb{P}(\sigma_i = \pm 1) = 1/2$ and $v_i \in \mathbb{R}^n$, and $t > 0$ we have,

$$\mathbb{P}(\|V\|_2 > t) \leq 2 \exp\left(\frac{-t^2}{32\mathbb{E}(\|V\|_2^2)}\right). \tag{23}$$

Applied to $v_i = c_{p(i)}\big(P_I(\Psi) - P_I(\Phi)\big)\phi_i$ this leads to the following estimate,

$$\begin{aligned}\mathbb{P}\left(\|\big(P_I(\Psi) - P_I(\Phi)\big)\Phi c_{p,\sigma}\|_2 > t\right) &\leq 2\exp\left(\frac{-t^2}{32\sum_i c_{p(i)}^2 \|\big(P_I(\Psi) - P_I(\Phi)\big)\phi_i\|_2^2}\right)\\ &\leq 2\exp\left(\frac{-t^2}{32\sum_i c_{p(i)}^2 \|P_I(\Psi) - P_I(\Phi)\|_{2,2}^2}\right)\\ &\leq 2\exp\left(\frac{-t^2}{32\|P_I(\Psi) - P_I(\Phi)\|_F^2}\right),\end{aligned}$$

whenever $P_I(\Psi) \neq P_I(\Phi)$ (otherwise we trivially have $\mathbb{P}\left(\|\big(P_I(\Psi) - P_I(\Phi)\big)\Phi c_{p,\sigma}\|_2 > t\right) = 0$). From Appendix B.2 we know that $\|P_I(\Psi) - P_I(\Phi)\|_F^2 = O(\|Q_I(\Phi) Z_I W_I A_I^{-1}\|_F^2)$, where $Q_I(\Phi)$ is the projection onto the orthogonal complement of the span of Φ_I, so we finally get,

$$\mathbb{P}\left(\|\big(P_I(\Psi) - P_I(\Phi)\big)\Phi c_{p,\sigma}\|_2 > t\right) \leq 2\exp\left(\frac{-t^2}{O\left(\|Q_I(\Phi) Z_I W_I A_I^{-1}\|_F^2\right)}\right).$$

Define $\kappa := \frac{1}{2}\min_{p,\sigma}\left(\|P_{I_p}(\Phi)\Phi c_{p,\sigma}\|_2 - \max_{|I|\leq S, I\neq I_p}\|P_I(\Phi)\Phi c_{p,\sigma}\|_2\right)$, then by Condition 21 we have $\kappa > 0$ and

$$\begin{aligned}\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2 &\geq \|P_{I_p}(\Phi)\Phi c_{p,\sigma}\|_2 - \kappa\\ &\geq \max_{I:I\neq I_p}\|P_I(\Phi)\Phi c_{p,\sigma}\|_2 + \kappa \geq \max_{I:I\neq I_p}\|P_I(\Psi)\Phi c_{p,\sigma}\|_2,\end{aligned}$$

with probability at least $\eta_S = 2\sum_{I:Q_I(\Phi)Z_I W_I A_I^{-1}\neq 0}\exp\left(\frac{-\kappa^2}{O\left(\|Q_I(\Phi)Z_I W_I A_I^{-1}\|_F^2\right)}\right)$. To calculate the expectation $\mathbb{E}_\sigma\left(\max_{|I|\leq S}\|P_I(\Phi)c_{p,\sigma}\|_2^2\right)$ we again define a set Σ_p,

$$\Sigma_p = \bigcup_{I:|I|=S}\{\sigma : \|\big(P_I(\Psi) - P_I(\Phi)\big)\Phi c_{p,\sigma}\|_2 > \kappa\}.$$

Splitting the expectation in a sum over the sign sequences contained in Σ_p and its complement, we can estimate,

$$\begin{aligned}\mathbb{E}_\sigma\left(\max_{|I|\leq S}\|P_I(\Psi)c_{p,\sigma}\|_2^2\right) &= \sum_{\sigma\in\Sigma_p}\max_{|I|\leq S}\|P_I(\Psi)\Phi c_{p,\sigma}\|_2^2 + \sum_{\sigma\notin\Sigma_p}\max_{|I|\leq S}\|P_I(\Psi)\Phi c_{p,\sigma}\|_2^2\\ &\leq \mathbb{P}(\Sigma_p)\max_{\sigma\in\Sigma_p}\|\Phi c_{p,\sigma}\|_2^2 + \sum_{\sigma\notin\Sigma_p}\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\\ &\leq \eta_S A + \mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right).\end{aligned}$$

Using the expression for $\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right)$ derived in Appendix B.1 we get the following bound for the expectation of the maximal projection using a perturbed dictionary,

$$\mathbb{E}_p\mathbb{E}_\sigma\left(\max_{|I|\leq S}\|P_I(\Psi)\Phi c_{p,\sigma}\|_2^2\right) \leq \eta_S A + \frac{A(1-\gamma^2)S}{(K-S)} + \left(\frac{\gamma^2}{S} - \frac{1-\gamma^2}{K-S}\right)\binom{K}{S}^{-1}\sum_J \|P_J(\Psi)\Phi_J\|_F^2.$$

Finally we are ready to compare the above expression to the corresponding one for the original dictionary. We abbreviate $\lambda = \frac{\gamma^2}{S} - \frac{1-\gamma^2}{K-S}$ and $B_I = Z_I W_I A_I^{-1}$. Employing $\|P_I(\Psi)\Phi_I\|_F^2 = \|\Phi_I\|_F^2 - \|Q_J(\Phi)B_I\|_F^2 + O(\|Q_I(\Phi)B_I\|_F^2\|B_I\|_F$ from Appendix B.2 we get,

$$\begin{aligned}&\mathbb{E}_y\left(\max_{|I|\leq S}\|P_I(\Psi)y\|_2^2\right) - \mathbb{E}_y\left(\max_{|I|\leq S}\|P_I(\Phi)y\|_2^2\right)\\ &\quad\leq 2A\sum_{P_I(\Psi)\neq P_I(\Phi)}\exp\left(\frac{-\kappa^2}{32\|P_I(\Psi)-P_I(\Phi)\|_F^2}\right) + \lambda\tbinom{K}{S}^{-1}\sum_I\left(\|P_I(\Psi)\Phi_I\|_F^2 - \|\Phi_I\|_F^2\right) \qquad (24)\\ &\quad\leq \sum_{I:...}2A\exp\left(\frac{-\kappa^2}{O\left(\|Q_I(\Phi)B_I\|_F^2\right)}\right) - \lambda\tbinom{K}{S}^{-1}\left(\|Q_I(\Phi)B_I\|_F^2 + O(\|Q_I(\Phi)B_I\|_F^2\|B_I\|_F)\right).\end{aligned}$$

Using the usual arguments we see that for $\varepsilon \neq 0$ the above expression is strictly smaller than zero as soon as ε and consequently $\|Q_I(\Phi)B_I\|_F^2 \leq \|B_I\|_F^2 \leq S\varepsilon^2/(1-\varepsilon^2)$ are small enough, showing that there is a local maximum of (20) at Φ. □

Remark 3.1: To make the above theorem more applicable it would be nice to have a concrete condition in terms of the coherence of the dictionary rather than the abstract condition in (21). Indeed it can be shown, see [27] Appendix C, that Condition (21) is implied by the following decay of the coefficients

$$c_S > \frac{1-S\mu}{1-2S\mu}c_{S+1} + \frac{4\mu}{1-2S\mu}\sum_{i>S+1}|c_i|, \qquad (25)$$

for $S\mu < 1/2$. Up to a factor this corresponds to the decay condition for the case $S = 1$.

We will now state a version of Theorem 3.1 for a continuous coefficient model, analogue to Theorem 2.2(a). However we will omit the proof since no new insights can be gained from it.

Theorem 3.3: Let Φ be a unit norm tight frame with frame constant K/d and coherence μ. Let x be drawn from a symmetric probability distribution ν on the unit sphere and assume that the signals are generated as $y = \Phi x$. If there exists $\kappa > 0$ such that for $c(x)$ a non-increasing rearrangement of the absolute values of x and $I_p := p^{-1}(\{1, \dots S\})$ we have,

$$\nu\left(\min_{p,\sigma}\left(\|P_{I_p}(\Phi)\Phi c_{p,\sigma}(x)\|_2 - \max_{|I|\leq S, I\neq I_p} \|P_I(\Phi)\Phi c_{p,\sigma}(x)\|_2\right) \geq 2\kappa\right) = 1 \tag{26}$$

then there is a local maximum of (20) at Φ.

Proof: Apply the technique used to prove Theorem 2.2 to the results derived in the proof of Theorem 3.1. □

Remark 3.2: (a) Again the abstract condition in (26) can be replaced by a decay-condition on the coefficients involving the coherence, ie. analogue to (25) we have for $S\mu < 1/2$,

$$\nu\left(c_S > \frac{1 - S\mu}{1 - 2S\mu} c_{S+1} + \frac{4\mu}{1 - 2S\mu} \sum_{i > S+1} |c_i| + 2\kappa\right) = 1. \tag{27}$$

(b) Note that with the available tools it is also be possible to extend Theorem 3.3 to signal models with coefficient distributions approaching the limit in (26), ie. $\kappa = 0$, or including bounded white noise. However, to keep the presentation concise, we leave both the formulation and the proof of generalisations corresponding to Theorems 2.2(b) and 2.3 to the interested reader, and instead turn to the analysis of the practically relevant case when we have a finite sample size.

4 FINITE SAMPLE SIZE RESULTS FOR $S = 1$

Finally make the step from the asymptotic identification results derived in the last two sections to identification results for a finite number of training samples. Again we start with the simple case when $S = 1$, ie. we consider the maximisation problem,

$$\max_{\Psi \in \mathcal{D}} \frac{1}{N} \sum_{n=1}^{N} \|\Psi^\star y_n\|_\infty^2. \tag{28}$$

The main idea is that whenever Ψ is near to Φ we have

$$\frac{1}{N} \sum_{n=1}^{N} \|\Psi^\star y_n\|_\infty^2 \approx \mathbb{E}\|\Psi^\star y\|_\infty^2 < \mathbb{E}\|\Phi^\star y\|_\infty^2 \approx \frac{1}{N} \sum_{n=1}^{N} \|\Phi^\star y_n\|_\infty^2.$$

Concretising the sharpness of $\approx$ quantitatively and making sure that it is valid for all possible ε-perturbations at the same time, leads to the following theorem.

Theorem 4.1: Let Φ be a unit norm tight frame with frame constant $A = K/d$ and coherence μ. Assume that the signals y_n are generated as $y_n = \Phi x_n + r_n$, where r_n is a bounded random white noise vector, ie. there exist two constants $\rho, \rho_{\max}$ such that $\|r_n\|_2 \leq \rho_{\max}$ almost surely, $\mathbb{E}(r_n) = 0$ and $\mathbb{E}(r_n r_n^\star) = \rho^2 I$. Further let x_n be drawn from a symmetric decaying probability distribution ν on the unit sphere S^{K-1} with $\mathbb{E}_x\|x\|_\infty^2 = \bar{c}_1^2$ and the maximal size of the noise be small compared to the size and decay of the coefficients c_1, c_2, meaning there exists $\beta < 1/2$, such that

$$\nu\left(\frac{c_2(x) + \mu\|c(x)\|_1 + \rho_{\max}}{c_1(x) - c_2(x)} \leq \beta\right) = 1. \tag{29}$$

Abbreviate $\lambda := \bar{c}_1^2 - \frac{1-\bar{c}_1^2}{K-1}$ and $C_L := (\sqrt{A} + \rho_{\max})^2$. If for some $0 < q < 1/4$ the number of samples N satisfies

$$N^{-q} + N^{-2q}/K \leq \frac{(1-2\beta)^2}{4A\log(4AK/\lambda)} \tag{30}$$

then except with probability

$$\exp\left(\frac{-N^{1-4q}\lambda^2}{4K^2 C_L} + Kd\log(NKC_L/\lambda)\right),$$

there is a local maximum of (28) resp. local minimum of (1) with $S = 1$ within distance at most $2N^{-q}$ to Φ, ie. for the local maximum $\tilde{\Psi}$ we have $\max_k \|\tilde{\psi}_k - \phi_k\|_2 \leq 2N^{-q}$.

Proof: Conceptually we need to show that for some $\varepsilon_{\min}(N) < \varepsilon_{\max}(N)$ and with probability $p(N)$ for all perturbations Ψ with $\varepsilon_{\min}(N) \leq \max_k \|\phi_k - \psi_k\| \leq \varepsilon_{\max}(N)$ we have

$$\frac{1}{N}\sum_{n=1}^{N} \|\Phi^\star y_n\|_\infty^2 > \frac{1}{N}\sum_{n=1}^{N} \|\Psi^\star y_n\|_\infty^2. \tag{31}$$

To do this we need to add three ingredients to the asymptotic results of Theorem 2.3, 1) that with high probability for fixed perturbation Φ the sum of signal responses concentrates around its expectation, 2) a dense enough net for the space of all perturbations and 3) that the mapping $\Psi \longrightarrow \|\Phi^\star y\|_\infty^2$ is Lipschitz. Then we can argue that an arbitrary perturbation will be close to a perturbation in the net, for which the sum concentrates around its expectation. This expectation is in turn is smaller than the expectation of the generating dictionary, around which the sum for the generating dictionary concentrates. We start by showing that $\Psi \longrightarrow \|\Phi^\star y\|_\infty^2$ is Lipschitz on the set of all perturbations Ψ with $\max_k \|\psi_k - \phi_k\|_2 < 1/2$. For simplicity we will write from now on $d(\Psi,\bar\Psi) := \max_k \|\psi_k - \bar\psi_k\|_2$. We have,

$$\begin{aligned}
&\big|\|\Psi^\star y\|_\infty^2 - \|\bar\Psi^\star y\|_\infty^2\big| \\
&\qquad = \left|\max_k |\langle \bar\psi_k + (\psi_k - \bar\psi_k), y\rangle|^2 - \max_k |\langle \bar\psi_k, y\rangle|^2\right| \\
&\qquad \le \left|\max_k \left(|\langle \bar\psi_k, y\rangle|^2 + 2|\langle(\psi_k - \bar\psi_k), y\rangle||\langle \bar\psi_k, y\rangle| + |\langle(\psi_k - \bar\psi_k), y\rangle|^2\right) - \max_k |\langle \bar\psi_k, y\rangle|^2\right| \\
&\qquad \le 2\|y\|_2^2 \max_k \|\psi_k - \bar\psi_k\|_2 + \|y\|_2^2 \max_k \|\psi_k - \bar\psi_k\|_2^2 \\
&\qquad \le 3\|y\|_2^2 \cdot d(\Psi,\bar\Psi)
\end{aligned}$$

Since the signals $y_n = \Phi x_n + r_n$ are generated from a tight frame with unit norm coefficients and a bounded white noise vector, we have $\Psi \longrightarrow \frac{1}{N}\sum_{n=1}^N \|\Psi^\star y_n\|_\infty^2$ is Lipschitz with constant $3(\sqrt{A}+\rho_{\max})^2$.

Next we use Hoeffding's inequality to estimate the probability that for a fixed dictionary Ψ, the sum of responses $\frac{1}{N}\sum_{n=1}^N \|\Psi^\star y_n\|_\infty^2$ deviates from its expectation. Set $Y_n = \|\Psi^\star y_n\|_\infty^2$, then we have $Y_n \in [0, (\sqrt{A}+\rho_{\max})^2]$ and get the estimate,

$$P\left(\left|\frac{1}{N}\sum_{n=1}^N \|\Psi^\star y_n\|_\infty^2 - \mathbb{E}(\|\Psi^\star y_1\|_\infty^2)\right| \ge t\right) < \exp\left(\frac{-Nt^2}{(\sqrt{A}+\rho_{\max})^2}\right).$$

The last ingredient is a δ-net for all perturbations Ψ with $d(\Psi,\Psi) \le \varepsilon_{\max}$, ie. a finite set of perturbations $\mathcal{N}$ such that for every Ψ we can find $\bar\Psi \in \mathcal{N}$ with $d(\Psi,\bar\Psi) < \delta$. Remembering the parametrisation of all ε-perturbations from the proof of Theorem 2.1 we see that the space we need to cover is the product of K balls with radius $\varepsilon_{\max}$ in $\mathbb{R}^{d-1}$. Following e.g. the argument in Lemma 2 of [32] we know that for the m-dimensional ball of radius $\varepsilon_{\max}$ we can find a δ net $\mathcal{N}_m$ with

$$\sharp\mathcal{N}_m \le \left(\varepsilon_{\max} + \frac{2\varepsilon_{\max}}{\delta}\right)^m.$$

Thus for the product of K balls in $\mathbb{R}^{d-1}$ we can construct a δ-net $\mathcal{N}$ as the product of K δ-nets $\mathcal{N}_{d-1}$. Assuming that $\delta < 1$ we then have,

$$\sharp\mathcal{N} \le \left(\varepsilon_{\max} + \frac{2\varepsilon_{\max}}{\delta}\right)^{K(d-1)} \le \left(\frac{3\varepsilon_{\max}}{\delta}\right)^{K(d-1)}.$$

Using a union bound we can now estimate the probability that for all perturbations in the net the sum of responses concentrates around its expectation, as

$$P\left(\exists\bar\Psi \in \mathcal{N} : \left|\frac{1}{N}\sum_{n=1}^N \|\bar\Psi^\star y_n\|_\infty^2 - \mathbb{E}(\|\bar\Psi^\star y_1\|_\infty^2)\right| \ge t\right) \le \left(\frac{3\varepsilon_{\max}}{\delta}\right)^{K(d-1)} \exp\left(\frac{-Nt^2}{(\sqrt{A}+\rho_{\max})^2}\right).$$

Finally we are ready for the triangle inequality argument. For any Ψ with $d(\Psi,\Phi)=\varepsilon<\varepsilon_{\max}$ we can find $\bar{\Psi}\in\mathcal{N}$ with $d(\bar{\Psi},\Psi)\leq\delta$ and assuming wlog that $\Phi\in\mathcal{N}$ we have that

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\|\Phi^\star y_n\|_\infty^2-\frac{1}{N}\sum_{n=1}^{N}\|\Psi^\star y_n\|_\infty^2 \\ &=\frac{1}{N}\sum_{n=1}^{N}\|\Phi^\star y_n\|_\infty^2-\mathbb{E}\|\Phi^\star y\|_\infty^2+\mathbb{E}\|\Phi^\star y\|_\infty^2-\mathbb{E}\|\bar{\Psi}^\star y\|_\infty^2\\ &\quad+\mathbb{E}\|\bar{\Psi}^\star y\|_\infty^2-\frac{1}{N}\sum_{n=1}^{N}\|\bar{\Psi}^\star y_n\|_\infty^2+\frac{1}{N}\sum_{n=1}^{N}\|\bar{\Psi}^\star y_n\|_\infty^2-\frac{1}{N}\sum_{n=1}^{N}\|\Psi^\star y_n\|_\infty^2\\ &\geq\mathbb{E}\|\Phi^\star y\|_\infty^2-\mathbb{E}\|\bar{\Psi}^\star y\|_\infty^2-2t-3\delta C_L\\ &\geq\sum_{k:\bar{\psi}_k\neq\phi_k}\left(\frac{\lambda}{K}(1-|\langle\bar{\psi}_k,\phi_k\rangle|^2)-2A\exp\left(\frac{(1-\frac{d(\bar{\Psi},\Phi)^2}{2}-2\beta)^2}{2A(1-|\langle\bar{\psi}_k,\phi_k\rangle|^2)}\right)\right)-2t-3\delta C_L.\end{aligned}$$

Next we identify $\varepsilon_{\max}$ up to δ by showing that for $d(\bar{\Psi},\Phi)=\bar{\varepsilon}\leq\varepsilon_{\max}$ we can lower bound the sum in the last equation by $\frac{\lambda}{K}\bar{\varepsilon}^2/2$. Following the argument in Appendix A.2 with the necessary changes we see that for $\bar{\varepsilon}<1/5$ and

$$\bar{\varepsilon}\leq\frac{(1-2\beta)^2}{2A\log(4AK/\lambda)}\,:,$$

we have

$$\exp\left(\frac{(1-\frac{\bar{\varepsilon}^2}{2}-2\beta)^2}{2A\bar{\varepsilon}^2)}\right)<\frac{\lambda}{4AK}(\bar{\varepsilon}^2-\bar{\varepsilon}^4/2).$$

Thus as soon as $\varepsilon\leq\frac{(1-2\beta)^2}{2A\log(4AK/\lambda)}-\delta:=\varepsilon_{\max}$ we have

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\|\Phi^\star y_n\|_\infty^2-\frac{1}{N}\sum_{n=1}^{N}\|\Psi^\star y_n\|_\infty^2&>\frac{\lambda}{K}\frac{\bar{\varepsilon}^2}{2}-2t-3\delta C_L\\ &>\frac{\lambda}{K}\frac{(\varepsilon-\delta)^2}{2}-2t-3\delta C_L\geq\frac{\lambda}{K}\frac{\varepsilon^2}{2}-2t-4\delta C_L.\end{aligned}$$

If for $q<1/4$ we choose $t=N^{-2q}\lambda/(2K)$ and $\delta=N^{-2q}\lambda/(4KC_L)$ then except with probability

$$\exp\left(\frac{-N^{1-4q}\lambda^2}{4K^2C_L}+K(d-1)\log(12\varepsilon_{\max}C_LKN^{2q}/\lambda)\right)$$

we have

$$\frac{1}{N}\sum_{n=1}^{N}\|\Phi^\star y_n\|_\infty^2>\frac{1}{N}\sum_{n=1}^{N}\|\Psi^\star y_n\|_\infty^2$$

whenever $\varepsilon\geq2N^{-q}:=\varepsilon_{\min}$. The statement then follows from the simplification that $\varepsilon_{\max}<1/5$ together with $N^{1-4q}\geq4K^2$ implies $12\varepsilon_{\max}N^{2q}\leq N$ and from verifying that $\varepsilon_{\min}<\varepsilon_{\max}$. □

Remark 4.1: Note that the above theorem is not only a result for the K-SVD minimisation principle but actually for K-SVD. While for $S > 1$ the decay-condition is not strong enough to ensure that the sparse approximation algorithm used for K-SVD always finds the best approximation as soon as we are close enough to the generating dictionary, in the case $S = 1$ any simple greedy algorithm, e.g. thresholding, will always find the best 1-term approximation to any signal given any dictionary. Thus given the right initialisation and sufficiently many training samples K-SVD can recover the generating dictionary up to the prescribed precision with high probability. To make the theorem more applicable we quickly concretise how the distance between the generating dictionary Φ and the local minimum output by K-SVD $\tilde{\Psi}$ decreases with the sample size. If we want the success probability to be of the order $1 - N^{-Kd}$ we need

$$\frac{-N^{1-4q}\lambda^2}{4K^2C_L} + Kd\log(NKC_L/\lambda) \approx -Kd\log N,$$

or $N^{1-4q} \approx K^3 d \log N$ meaning that $-q \approx -\frac{1}{4} + \frac{\log K}{\log N}$. Thus we have

$$\log\Big(d(\Phi, \tilde{\Psi})\Big) = -q\log N \approx -\frac{\log N}{4} + \log K \qquad \text{or} \qquad d(\Phi, \tilde{\Psi}) \approx KN^{-1/4} \tag{32}$$

5 Finite sample size results for $S \geq 1$

Let us now turn to the analysis of the problem with $S \geq 1$, ie.

$$\max_{\Phi\in\mathcal{D}} \frac{1}{N}\sum_{n=1}^{N} \max_{|I|\leq S} \|P_I(\Phi)y_n\|_2^2. \tag{33}$$

As for the asymptotic case we will be less concrete but more precise and instead of using the coherence will give the results in terms of the lower isometry constant of the generating dictionary, which is defined as the largest distance of the smallest eigenvalue $\lambda_{\min}$ of $\Phi_I^\star\Phi_I$ to 1, ie. $\delta_S := \max_{|I|\leq S}(1 - \lambda_{\min}(\Phi_I^\star\Phi_I))$. For simplicity we again state only the noise-free version.

Theorem 5.1: Let Φ be a unit norm tight frame with frame constant $A = K/d$, coherence μ and lower isometry constant $\delta_S \leq \mu S$. Assume that the signals y_n are generated as $y_n = \Phi x_n$, where x_n is drawn from a symmetric decaying probability distribution ν on the unit sphere S^{K-1}, and that there exists $\kappa > 0$ such that for $c(x)$ a non-increasing rearrangement of the absolute values of x, ie. $c_1(x) \geq c_2(x) \ldots \geq c_K(x)$ and $I_p := p^{-1}(\{1, \ldots S\})$ we have,

$$\nu\left(\min_{p,\sigma}\left(\|P_{I_p}(\Phi)\Phi c_{p,\sigma}(x)\|_2 - \max_{|I|\leq S, I\neq I_p}\|P_I(\Phi)\Phi c_{p,\sigma}(x)\|_2\right) \geq 2\kappa\right) = 1. \tag{34}$$

Define γ_S^2 as the expected energy of the S largest coefficients, i.e. $\gamma_S^2 := \mathbb{E}_x\big(c_1^2(x) + \dots c_S^2(x)\big)$ and abbreviate $\lambda_S := \frac{\gamma_S^2}{S} - \frac{1-\gamma_S^2}{K-S}$ and $C_S := \Big(1 - \frac{S}{d(1-\delta_S)}\Big)^2$. If for some $0 < q < 1/4$ the number of samples N satisfies

$$N^{-q} + N^{-2q}/K \leq \frac{\kappa^2(1-\delta_S)}{68\sqrt{S}\log\left(5AK^S/\lambda\right)}, \tag{35}$$

then except with probability

$$\exp\left(-\frac{N^{1-4q}\lambda^2 S^2 C_S}{4K^2 A} + Kd\log\left(\frac{NKA}{\lambda S}\right)\right),$$

there is a local maximum of (33) resp. local minimum of (1) within distance at most $2N^{-q}$ to Φ, ie. for the local maximum $\tilde{\Psi}$ we have $\max_k \|\tilde{\psi}_k - \phi_k\|_2 \leq 2N^{-q}$.

Proof: The proof follows the same strategy as in the simple case. However since we now have to deal with projections instead of simple inner products we have to suffer a bit more. Again we first show that the mapping $\Psi \longrightarrow \max_{|I|\leq S}\|P_I(\Psi)y_n\|_2^2$ is Lipschitz on the set of perturbations with $d(\Psi,\Phi) \leq \varepsilon_{\max}$. We have,

$$\begin{aligned}
&\left|\max_{|I|\leq S}\|P_I(\Psi)y_n\|_2^2 - \max_{|I|\leq S}\|P_I(\bar\Psi)y_n\|_2^2\right| \\
&\qquad = \left|\max_{|I|\leq S}\|P_I(\bar\Psi)y_n - (P_I(\Psi)-P_I(\bar\Psi))y_n\|_2^2 - \max_{|I|\leq S}\|P_I(\bar\Psi)y_n\|_2^2\right| \\
&\qquad = 2\max_{|I|\leq S}\|(P_I(\Psi)-P_I(\bar\Psi))y_n\|_2 \max_{|I|\leq S}\|P_I(\bar\Psi)y_n\|_2 + \max_{|I|\leq S}\|(P_I(\Psi)-P_I(\bar\Psi))y_n\|_2^2 \\
&\qquad \leq 3A\max_{|I|\leq S}\|P_I(\Psi)-P_I(\bar\Psi)\|_{2,2}.
\end{aligned}$$

Following the line of argument in Appendix B.2 we know that

$$\|P_I(\Psi)-P_I(\bar\Psi)\|_{2,2}^2 \leq \|P_I(\Psi)-P_I(\bar\Psi)\|_F^2 \leq \frac{2S\frac{d(\Psi,\bar\Psi)^2}{1-d(\Psi,\bar\Psi)^2}}{\|\Psi_I^\dagger\|_{2,2}^{-1}\left(\|\Psi_I^\dagger\|_{2,2}^{-1} - 2\sqrt{S}\frac{d(\Psi,\bar\Psi)}{\sqrt{1-d(\Psi,\bar\Psi)^2}}\right)}.$$

Now note that $\|\Psi_I^\dagger\|_{2,2}^{-1}$ is simply the minimal singular value of Ψ_I. Remembering that 26 implies $\delta_S < 1$ we therefore have,

$$\begin{aligned}
\|\Psi_I^\dagger\|_{2,2}^{-1} = \sigma_{\min}(\Psi_I) = \sigma_{\min}(\Phi_I A_I + Z_I W_I) &\geq \sigma_{\min}(\Phi_I)\sigma_{\min}(A_I) - \sigma_{\max}(Z_I W_I) \\
&\geq \sqrt{1-\delta_S}\sqrt{1-\varepsilon^2} - \sqrt{S}\varepsilon.
\end{aligned}$$

The combination of the last three estimates, together with some simplifications, using the fact that both ε and $d(\Psi,\bar\Psi)$ are smaller than $\varepsilon_{\max} \leq \frac{1-\delta_S}{64\sqrt{S}}$, leads us to the final Lipschitz bound,

$$\left|\max_{|I|\leq S}\|P_I(\Psi)y_n\|_2^2 - \max_{|I|\leq S}\|P_I(\bar\Psi)y_n\|_2^2\right| \leq d(\Psi,\bar\Psi)\cdot\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}}. \tag{36}$$

Next for $Y_n = \max_{|I|\leq S} \|P_I(\Psi)y_n\|_2^2$ we have $Y_n \in [0, A]$ and therefore by Hoeffding's inequality,

$$P\left(\left|\frac{1}{N}\sum_{n=1}^N \max_{|I|\leq S} \|P_I(\Psi)y_n\|_2^2 - \mathbb{E}(\max_{|I|\leq S} \|P_I(\Psi)y_1\|_2^2)\right| \geq t\right) \leq e^{-Nt^2/A}.$$

By a union bound we can estimate that the above holds for all, at most $(3\varepsilon_{\max}/\delta)^{K(d-1)}$, elements of a δ-net $\mathcal{N}$ for the set of perturbations with $d(\Psi, \Phi) \leq \varepsilon_{\max}$. We can now turn to the triangle inequality argument. For a perturbation Ψ with $d(\Psi, \Phi) = \varepsilon \leq \varepsilon_{\max}$ we can find $\bar{\Psi} \in \mathcal{N}$ with $d(\Psi, \bar{\Psi}) \leq \delta$ and $d(\bar{\Psi}, \Phi) = \bar{\varepsilon}$. Analogue to the case $S = 1$ we then have

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^N \max_{|I|\leq S} \|P_I(\Phi)y_n\|_2^2 - \frac{1}{N}\sum_{n=1}^N \max_{|I|\leq S} \|P_I(\Psi)y_n\|_2^2 \\
&\quad\geq \mathbb{E}\left(\max_{|I|\leq S} \|P_I(\Phi)y_n\|_2^2\right) - \mathbb{E}\left(\max_{|I|\leq S} \|P_I(\bar{\Psi})y_n\|_2^2\right) - 2t - \delta\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}} \\
&\quad\geq \lambda\tbinom{K}{S}^{-1}\sum_I \left(\|\Phi_I\|_F^2 - \|P_I(\bar{\Psi})\Phi_I\|_F^2\right) \\
&\qquad\quad - 2A\sum_{I:P_I(\Phi)\neq P_I(\bar{\Psi})} \exp\left(\frac{-\kappa^2}{32\|P_I(\Phi)-P_I(\bar{\Psi})\|_F^2}\right) - 2t - \delta\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}},
\end{aligned}$$

where we have used the continuous equivalent of the estimate in (24). From Appendix B.2 we know that for $\bar{\varepsilon} \leq \varepsilon_{\max} \leq \frac{1-\delta_S}{64\sqrt{S}}$ we have

$$\|\Phi_I\|_F^2 - \|P_I(\bar{\Psi})\Phi_I\|_F^2 \geq \frac{29}{30}\|Q_I(\Phi)\bar{B}_I\|_F^2 \quad \text{and} \quad 32\|P_I(\Phi) - P_I(\bar{\Psi})\|_F^2 \leq \frac{67}{1-\delta_S}\|Q_I(\Phi)\bar{B}_I\|_F^2,$$

so we can continue the estimate above as,

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^N \max_{|I|\leq S} \|P_I(\Phi)y_n\|_2^2 - \frac{1}{N}\sum_{n=1}^N \max_{|I|\leq S} \|P_I(\Psi)y_n\|_2^2 \\
&\quad\geq \sum_{I\ldots}\left(\frac{29\lambda}{30}\tbinom{K}{S}^{-1}\|Q_I(\Phi)\bar{B}_I\|_F^2 - 2A\exp\left(\frac{-\kappa^2(1-\delta_S)}{67\|Q_I(\Phi)\bar{B}_I\|_F^2}\right)\right) - 2t - \delta\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}}.
\end{aligned}$$

As in the case $S = 1$ we now identify $\varepsilon_{\max}$ up to δ by checking when the expressions in the sum above are larger than $\frac{\lambda}{2}\binom{K}{S}^{-1}\|Q_I(\Phi)\bar{B}_I\|_F^2$. Following again the line of argument in Appendix A.2 we get that

$$2A\exp\left(\frac{-\kappa^2(1-\delta_S)}{67\|Q_I(\Phi)\bar{B}_I\|_F^2}\right) \leq \frac{14\lambda}{30}\tbinom{K}{S}^{-1}\|Q_I(\Phi)\bar{B}_I\|_F^2,$$

as soon as

$$\|Q_I(\Phi)\bar{B}_I\|_F \leq \frac{\kappa^2(1-\delta_S)}{67\log\left(5\binom{K}{S}A/\lambda\right)},$$

which is in turn implied by

$$\bar{\varepsilon}/\sqrt{1-\bar{\varepsilon}^2} \leq \frac{\kappa^2(1-\delta_S)}{67\sqrt{S}\log\left(5\binom{K}{S}A/\lambda\right)} \quad \text{or} \quad \bar{\varepsilon} \leq \frac{\kappa^2(1-\delta_S)}{68\sqrt{S}\log\left(5AK^S/\lambda\right)} := \varepsilon_{\max} + \delta.$$

Thus as soon as $\varepsilon \leq \frac{\kappa^2(1-\delta_S)}{68\sqrt{S}\log(5AK^S/\lambda)} - \delta := \varepsilon_{\max}$ we have,

$$\frac{1}{N}\sum_{n=1}^{N}\max_{|I|\leq S}\|P_I(\Phi)y_n\|_2^2 - \frac{1}{N}\sum_{n=1}^{N}\max_{|I|\leq S}\|P_I(\Psi)y_n\|_2^2 \geq \frac{\lambda}{2}\binom{K}{S}^{-1}\sum_I\|Q_I(\Phi)\bar{B}_I\|_F^2 - 2t - \delta\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}}.$$

To estimate the size of the sum over all possible supports we remember that $\bar{b}_i = \frac{\bar{\omega}_i}{\bar{\alpha}_i}\bar{z}_i$ where $\bar{\psi}_i = \bar{\alpha}_i\phi_i + \bar{\omega}_i\bar{z}_i$ with $\langle\Phi_i, \bar{z}_i\rangle = 0$ and that $\max_i\|\bar{\psi}_i - \phi_i\|_2 = \bar{\varepsilon}$. We have

$$\begin{aligned}
\binom{K}{S}^{-1}\sum_I\|Q_I(\Phi)\bar{B}_I\|_F^2 &= \binom{K}{S}^{-1}\sum_I\left(\|\bar{B}_I\|_F^2 - \|P_I(\Phi)\bar{B}_I\|_F^2\right) \\
&= \binom{K}{S}^{-1}\binom{K-1}{S-1}\|\bar{B}\|_F^2 - \binom{K}{S}^{-1}\sum_I\|(\Phi_I^\dagger)^\star\Phi_I^\star\bar{B}_I\|_F^2 \\
&\geq \frac{S}{K}\|\bar{B}\|_F^2 - \binom{K}{S}^{-1}\sum_I\|\Phi_I^\dagger\|_{2,2}^2\|\Phi_I^\star\bar{B}_I\|_F^2 \\
&\geq \frac{S}{K}\|\bar{B}\|_F^2 - (1-\delta_S)^{-1}\binom{K}{S}^{-1}\sum_I\|\Phi_I^\star\bar{B}_I\|_F^2 \\
&\geq \frac{S}{K}\|\bar{B}\|_F^2 - \binom{K}{S}^{-1}\binom{K-2}{S-2}(1-\delta_S)^{-1}\|\Phi^\star\bar{B}\|_F^2 \\
&\geq \frac{S}{K}\left(1 - \frac{A}{1-\delta_S}\frac{S-1}{K-1}\right)\|\bar{B}\|_F^2 \geq \frac{S}{K}\left(1-\frac{S}{d(1-\delta_S)}\right)\bar{\varepsilon}^2,
\end{aligned}$$

where in the last inequality we have used that

$$\|\bar{B}\|_F^2 \geq \frac{\bar{\varepsilon}^2 - \bar{\varepsilon}^4/4}{1-\bar{\varepsilon}^2+\bar{\varepsilon}^4/4} \geq \bar{\varepsilon}^2.$$

With this last simplification we finally arrive at an estimate, which suggests the correct sizes for t and δ, ie.

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}\max_{|I|\leq S}\|P_I(\Phi)y_n\|_2^2 - \frac{1}{N}\sum_{n=1}^{N}\max_{|I|\leq S}\|P_I(\Psi)y_n\|_2^2 \\
&\qquad\geq \frac{\lambda S}{2K}\left(1-\frac{S}{d(1-\delta_S)}\right)\bar{\varepsilon}^2 - 2t - \delta\frac{5A\sqrt{S}}{\sqrt{1-\delta_S}} \\
&\qquad\geq \frac{\lambda S}{2K}\left(1-\frac{S}{d(1-\delta_S)}\right)\varepsilon^2 - 2t - \delta\frac{6A\sqrt{S}}{\sqrt{1-\delta_S}}.
\end{aligned}$$

We now choose $t = N^{-2q}\frac{\lambda S}{2K}\left(1 - \frac{S}{d(1-\delta_S)}\right)$ and $\delta = N^{-2q}\frac{\lambda\sqrt{S(1-\delta_S)}}{6AK}\left(1 - \frac{S}{d(1-\delta_S)}\right)$ to get, that except with probability,

$$\exp\left(-\frac{N^{1-4q}\lambda^2 S^2\left(1 - \frac{S}{d(1-\delta_S)}\right)^2}{4K^2 A} + K(d-1)\log\left(\frac{18\varepsilon_{\max} AKN^{2q}}{\lambda\sqrt{S(1-\delta_S)}}\right)\right),$$

we have

$$\frac{1}{N}\sum_{n=1}^{N}\max_{|I|\le S}\|P_I(\Phi)y_n\|_2^2 - \frac{1}{N}\sum_{n=1}^{N}\max_{|I|\le S}\|P_I(\Psi)y_n\|_2^2 \ge \frac{\lambda S}{2K}\left(1 - \frac{S}{d(1-\delta_S)}\right)(\varepsilon^2 - 4N^{-2q}),$$

which is larger than zero as long as $\varepsilon > 2N^{-q} := \varepsilon_{\min}$. The statement again follows from simplifications using $\varepsilon_{\max} \le \frac{(1-\delta_S)}{68S\sqrt{S}}$ and verifying that $\varepsilon_{\min} < \varepsilon_{\max}$. □

Note that in order to get a more explicit result the abstract condition in (34) can again be replaced by a concrete condition in terms of the coherence (27), and also the lower isometry constant can be estimated by $\delta_S \le (S-1)\mu$.

Let us now turn to a discussion of our results.

6 Discussion

We have shown that the minimisation principle underlying K-SVD can identify a tight frame with arbitrary precision from signals generated from a wide class of decaying coefficients distributions, provided that the training sample size is large enough. For the case $S = 1$ in particular this means that K-SVD in combination with a greedy algorithm can recover the generating dictionary up to prescribed precision. To illustrate our results we conducted two experiments.

The first experiment demonstrates that the requirement on the dictionary to be tight in order to be identifiable translates to the case of finitely many training samples. For simplicity and to allow for a visual representation of the outcome it was conducted in $\mathbb{R}^2$. We generated 1000 coefficients by drawing c_2 uniformly at random from the interval $[0, 0.6]$, setting $c_1 = \sqrt{1-c_2^2}$, randomly permuting the resulting vector and providing it with random $\pm$ signs. We then generated four sets of signals, using four bases with increasing coherence and the same coefficients, and for each set of signals found the minimiser of the K-SVD criterion (1) with $S = 1$. Figure 1 shows the objective function for the case of an orthonormal basis, while Figure 2 shows the four signal sets, the generating bases and the recovered bases. As predicted by our theoretical results when the generating basis is orthogonal it is also the minimiser of the K-SVD criterion, while for an oblique generating basis the minimiser is distorted towards the maximal eigenvector of the basis. Since for a 2-dimensional basis in combination with our coefficient distribution the abstract condition in (26) is always fulfilled, this effect can only be due to the violation of the tightness-condition.

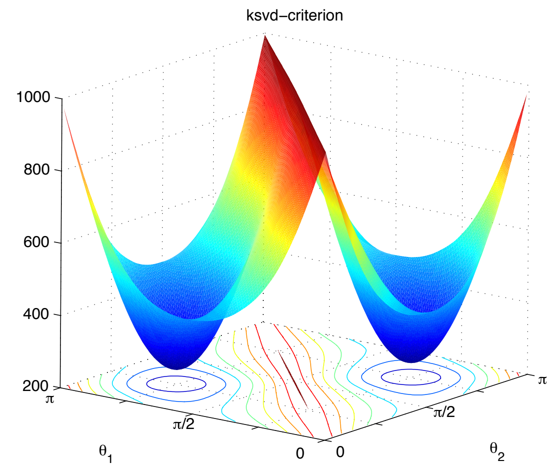


Fig. 1. The K-SVD-criterion for the signals created from the decaying coefficients and an orthonormal basis, the admissible dictionaries are parametrised by two angles (θ_1, θ_2), ie. $\phi_i = (\cos\theta_i, \sin\theta_i)$.

The second experiment illustrates how the local minimum near the generating dictionary approaches the generating dictionary as the number of signals increases. As generating dictionary we choose the union of two orthonormal bases, the Hadamard and the Dirac basis, in dimension $d = 4, 8, 16$, ie. $K = 2d$. We then generated 2-sparse signals by first drawing c_1 uniformly at random from the interval $[0.99, 1]$, setting $c_2 = \sqrt{1 - c_1^2}$, meaning $c_2 \in [0, 0.1]$, and $c_i = 0$ for $i \geq 3$ and then setting $y = \Phi c_{\sigma,p}$ for a uniformly at random chosen sign sequence σ and permutation p. We then run the original K-SVD algorithm as described in [1], with a greedy algorithm, and sparsity parameter $S = 1$, using both an oracle initialisation (ie. the generating dictionary) and a random initialisation, on training sets containing $128 \cdot 2^n$ signals for n increasing from 0 to 7. Figure 3 (a) plots the maximal distance between two corresponding atoms of the generating and the learned dictionary, $d(\Phi, \tilde{\Psi}) = \max_i \|\phi_i - \psi_i\|_2$, averaged over 10 runs. Figure 3 (b) is designed to be comparable to the experiment conducted for the noisy

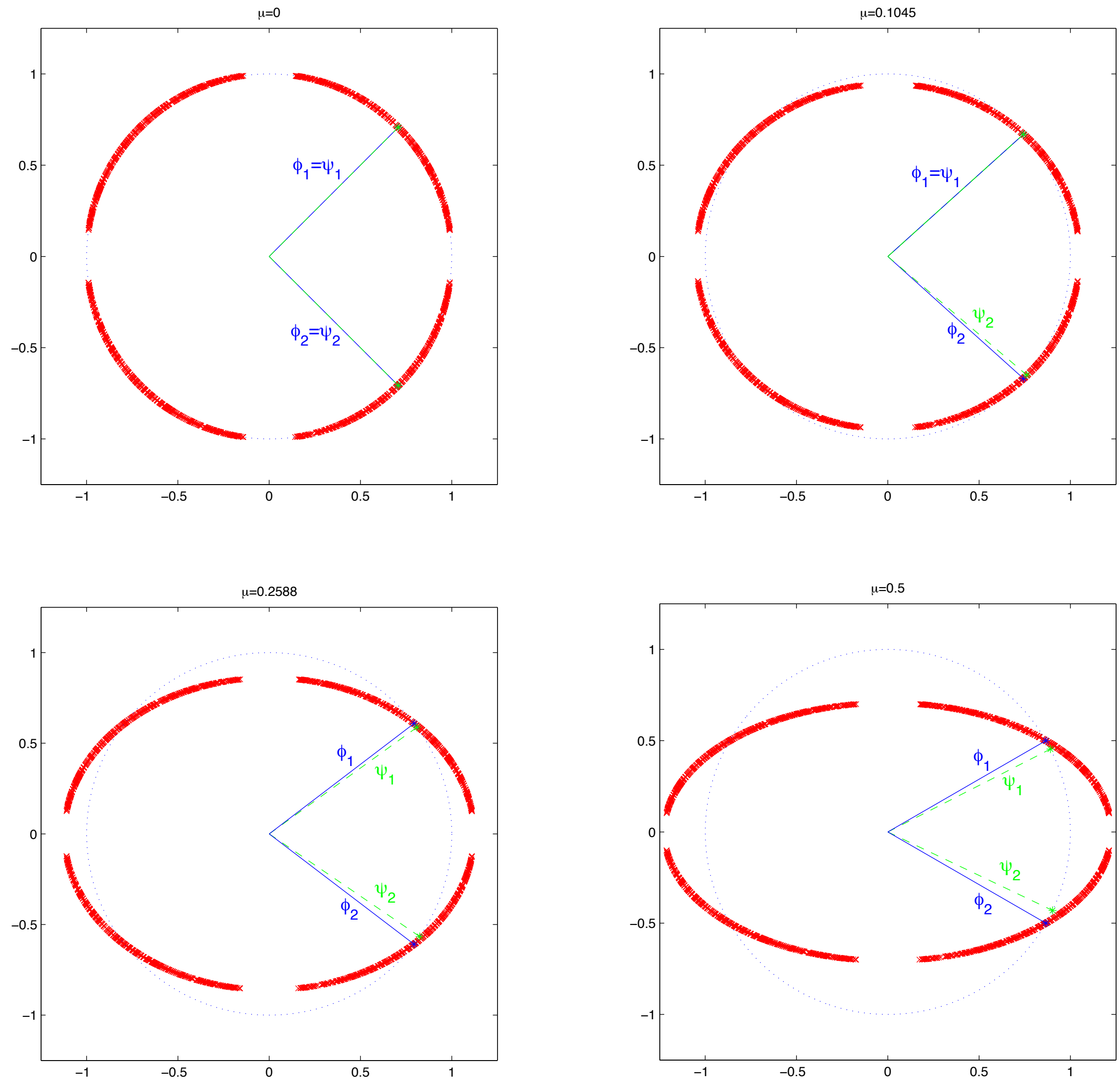
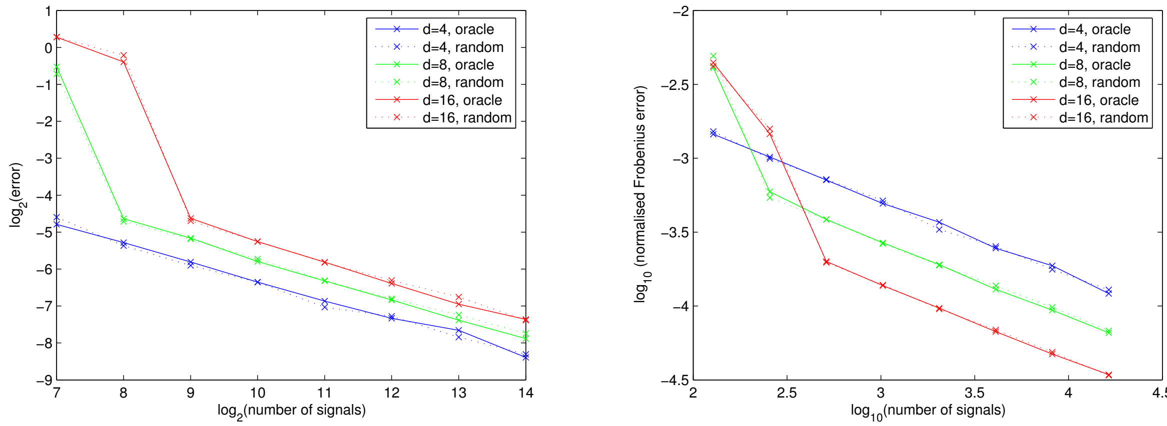


Fig. 2. Signals created from various bases $\Phi = (\phi_1, \phi_2)$ with increasing coherence μ, together with the corresponding minimiser $\Psi = (\psi_1, \psi_2)$ of the K-SVD-criterion for $S = 1$.

ℓ_1-criterion in [18] and plots the normalised Frobenius norm between the generating and the learned dictionary, $\|\Phi - \tilde{\Psi}\|_F/\sqrt{dK^3}$, averaged over 10 runs.

As expected we have a log-linear relation between the number of samples and the reconstruction error. However our predictions seem to be too pessimistic. So rather than an inclination of $-\frac{1}{4}$ we see one of $-\frac{1}{2}$ indicating that $d(\Phi, \tilde{\Psi}) \approx N^{-\frac{1}{2}}$. We also see that both the oracle and



(a) (b)

Fig. 3. Error between the generating Hadamard-Dirac dictionary Φ in $\mathbb{R}^d$ and the output $\tilde{\Psi}$ of the K-SVD algorithm with parameter $S = 1$; the error is measured as $d(\Phi, \tilde{\Psi}) = \max_i \|\phi_i - \psi_i\|_2$) in (a) and as $\|\Phi - \tilde{\Psi}\|_F / \sqrt{dK^3}$ in (b).

the random initialisation lead to the same results, raising the question of uniqueness of the equivalent local minima, compare also [18].

Finally let us point out further research directions based on a comparison of our results for the K-SVD-minimisation principle to the available identification results for the ℓ_1-minimisation principle,

$$\min_{\Phi \in \mathcal{D}, X : Y = \Phi X} \sum_{ij} |X_{ij}|. \tag{37}$$

At first glance it seems that the K-SVD-criterion requires a larger sample size than the ℓ_1-criterion, ie. $N^{1-4q}/\log N = O(K^3 d)$ as opposed to $O(d^2 \log d)$ reported in [17] for a basis and $O(K^3)$ reported in [14] for an overcomplete dictionary. Also it does not allow for exact identification with high probability but only guarantees stability. However this effect may be due to the more general signal model which assumes decay rather than exact sparsity. Indeed it is very interesting to compare our results to a recent result for a noisy version of the ℓ_1-minimisation principle, [18], which provides stability results under unbounded white noise and, omitting log factors, also derives a sampling complexity of $O(K^3 d)$.

Another difference, apparently intrinsic to the two minimisation criteria is that the K-SVD criterion can only identify tight dictionary frames exactly, while the ℓ_1-criterion allows identification of arbitrary dictionaries. Thus to support the use of K-SVD for the learning of non-tight dictionaries also theoretically, we plan to study the stability of the K-SVD criterion under non-tightness by analysing the maximal distance between an original, non tight dictionary with condition number $\sqrt{B/A} > 1$ and the closest local maximum, cp. also Figure 2.

The last research direction we want to point out is how much decay of the coefficients is actually necessary. For the one-dimensional asymptotic results we used condition $c_1 > c_2 + 2\mu\|c\|_1$ to ensure that the maximal inner product is always attained at i_p. However, typically we have $|\langle \phi_i, \Phi c_{p,\sigma}\rangle| \approx c_{p(i)} \pm \mu$. Therefore a condition such as $c_1 > c_2 + O(\mu)$, which allows for outliers, ie. signals for which the maximal inner product is not attained at i_p, might be sufficient to prove - if not exact identifiability - at least stability. Together with the inspiring techniques from [18], we expect the tools developed in the course of such an analysis to allow us also to deal with unbounded white noise.

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Appendix A

Technical details for the proof of Theorem 2.1

A.1 Expectations

We start by calculating $\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right)$ for two arbitrary unit norm frames Ψ, Φ.

$$\begin{aligned}
\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) &= \mathbb{E}_p\mathbb{E}_\sigma\left(\Big|\sum_i \sigma_i c_{p(i)}\langle \psi_{i_p}, \phi_i\rangle\Big|^2\right) \\
&= \mathbb{E}_p\left(\sum_i c_{p(i)}^2 \cdot |\langle \psi_{i_p}, \phi_i\rangle|^2\right) \\
&= \sum_i \mathbb{E}_p\left(c_{p(i)}^2 \cdot |\langle \psi_{i_p}, \phi_i\rangle|^2\right). \qquad (38)
\end{aligned}$$

For each i we now split the set of all permutations $\mathcal{P}$ into disjoint sets $\mathcal{P}^i_{jk}$, defined as

$$\mathcal{P}^i_{jk} := \{p : p(i) = k, p(j) = 1\}.$$

We then have $\mathcal{P} = \cup_{j,k} \mathcal{P}^i_{jk}$ and

$$\sharp\mathcal{P}^i_{jk} = \begin{cases} (K-1)! & \text{if } j = i \text{ and } k = 1 \\ (K-2)! & \text{if } j \neq i \text{ and } k \neq 1 \\ 0 & \text{else} \end{cases}.$$

Using these sets we can compute the expectations in (38) as follows

$$\begin{aligned}
\mathbb{E}_p\left(c^2_{p(i)} \cdot |\langle \psi_{i_p}, \phi_i\rangle|^2\right) &= \frac{1}{K!}\sum_p c^2_{p(i)} \cdot |\langle \psi_{i_p}, \phi_i\rangle|^2 \\
&= \frac{1}{K!}\sum_j \sum_k \sum_{p\in\mathcal{P}^i_{jk}} c^2_k \cdot |\langle \psi_j, \phi_i\rangle|^2 \\
&= \frac{(K-2)!}{K!}\sum_{j\neq i}\sum_{k\neq 1} c^2_k \cdot |\langle \psi_j, \phi_i\rangle|^2 + \frac{(K-1)!}{K!} c^2_1 \cdot |\langle \psi_i, \phi_i\rangle|^2 \\
&= \frac{(1-c^2_1)}{K(K-1)}\sum_{j\neq i} |\langle \psi_j, \phi_i\rangle|^2 + \frac{c^2_1}{K}\cdot |\langle \psi_i, \phi_i\rangle|^2.
\end{aligned}$$

Re-substituting the above expression into (38) finally leads to,

$$\begin{aligned}
\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) &= \frac{c^2_1}{K}\sum_i |\langle \psi_i, \phi_i\rangle|^2 + \frac{(1-c^2_1)}{K(K-1)}\sum_i\sum_{j\neq i} |\langle \psi_j, \phi_i\rangle|^2 \\
&= \frac{c^2_1}{K}\sum_i |\langle \psi_i, \phi_i\rangle|^2 + \frac{(1-c^2_1)}{K(K-1)}\left(\|\Phi^\star\Psi\|^2_F - \sum_i |\langle \psi_i, \phi_i\rangle|^2\right).
\end{aligned}$$

We can simplify the above result for three important special cases:

If Φ is a unit norm tight frame, we have,

$$\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \psi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) = \frac{c^2_1}{K}\sum_i |\langle \psi_i, \phi_i\rangle|^2 + \frac{(1-c^2_1)}{(K-1)}\left(A - \frac{1}{K}\sum_i |\langle \psi_i, \phi_i\rangle|^2\right),$$

if $\Psi = \Phi$, we have,

$$\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) = c^2_1 + \frac{(1-c^2_1)}{(K-1)}\left(\frac{\|\Phi^\star\Phi\|^2_F}{K} - 1\right),$$

and if $\Phi = \Psi$ is unit norm tight frame, we have,

$$\mathbb{E}_p\mathbb{E}_\sigma\left(|\langle \phi_{i_p}, \Phi c_{p,\sigma}\rangle|^2\right) = c^2_1 + \frac{(1-c^2_1)}{K-1}(A-1).$$

A.2 ε-Condition

To complete the proof of Theorem 2.1 we still need to verify that $\varepsilon \leq 1/5$ and

$$\varepsilon \leq \frac{(1-2\beta)^2}{2A\log(2AK/\lambda)} \tag{39}$$

imply

$$2AK\exp\left(-\frac{(1-\frac{\varepsilon^2}{2}-2\beta)^2}{2A\varepsilon_i^2}\right)-\lambda(\varepsilon_i^2-\varepsilon_i^4/4)<0,$$

for all $0<\varepsilon_i<\varepsilon$, where we have used the shorthand $\beta=\frac{c_2+\mu\|c\|_1}{c_2+c_1}$ and $\lambda=c_1^2-\frac{1-c_1^2}{K-1}$. Next (39) implies $\varepsilon^2/2<(1-2\beta)^4/(8\log^2 4)<(1-2\beta)/15$, so we can estimate,

$$\begin{aligned}\exp\left(-\frac{(1-\frac{\varepsilon^2}{2}-2\beta)^2}{2A\varepsilon_i^2}\right)&\le\exp\left(-\frac{(1-\frac{1-2\beta}{15}-2\beta)^2}{2A\varepsilon_i\cdot\varepsilon}\right)\\&\le\exp\left(-\frac{14^2(1-2\beta)^2\cdot 2A\log(2AK/\lambda)}{15^2\cdot 2A\varepsilon_i\cdot(1-2\beta)^2}\right)\\&\le\exp\left(-\log(2AK/\lambda)\cdot 14^2/15^2\cdot 1/\varepsilon_i\right).\end{aligned}$$

For two values $a,b>0$ we have $ab\ge a+b$ as long as $a>b/(b-1)$. Setting $a=\log(2AK/\lambda)$ and $b=14^2/15^2\cdot 1/\varepsilon_i$ we see that this condition is satisfied for $\varepsilon_i\le\varepsilon<1/5$, so we can further estimate,

$$\begin{aligned}\exp\left(-\frac{(1-\frac{\varepsilon^2}{2}-2\beta)^2}{2A\varepsilon_i^2}\right)&\le\exp\left(-\big(\log(2AK/\lambda)+14^2/15^2\cdot 1/\varepsilon_i\big)\right)\\&=\lambda/(2AK)\cdot\exp\left(-14^2/15^2\cdot 1/\varepsilon_i\right).\end{aligned}$$

As last step we will show that for $0<\varepsilon<1/5$ we have $\exp\left(-14^2/15^2\cdot 1/\varepsilon\right)\le\varepsilon^2-\varepsilon^4/4$ or equivalently that $\exp\left(14^2/15^2\cdot 1/\varepsilon\right)\ge(\varepsilon^2-\varepsilon^4/4)^{-1}$. Using a geometric series expansion we can estimate,

$$\begin{aligned}\frac{1}{\varepsilon^2-\varepsilon^4/4}=\frac{1}{\varepsilon^2}\cdot\frac{1}{1-\varepsilon^2/4}&=\frac{1}{\varepsilon^2}\cdot\sum_{i=0}^{\infty}\left(\frac{\varepsilon^2}{4}\right)^i\\&=\frac{1}{\varepsilon^2}+\frac{1}{4}+\frac{1}{\varepsilon^2}\sum_{i=2}^{\infty}\left(\frac{\varepsilon^2}{4}\right)^i\\&=\frac{1}{\varepsilon^2}+\frac{1}{4}+\frac{\varepsilon^2}{16}\sum_{i=0}^{\infty}\left(\frac{\varepsilon^2}{4}\right)^i<\frac{1}{\varepsilon^2}+\frac{25}{99}.\end{aligned}$$

At the same time we can lower bound $e^{a/\varepsilon}$, where $a=\left(\frac{14}{15}\right)^2$, as

$$\begin{aligned}e^{a/\varepsilon}&=\sum_{i=0}^{\infty}\left(\frac{a}{\varepsilon}\right)^i\cdot\frac{1}{i!}\\&>1+\frac{a}{\varepsilon}+\frac{a^2}{2\varepsilon^2}+\frac{a^3}{6\varepsilon^3}+\frac{a^4}{24\varepsilon^4}\\&>1+\frac{1}{\varepsilon^2}\left(\frac{a^2}{2}+\frac{5a^3}{6}+\frac{25a^4}{24}\right)>1+\frac{1}{\varepsilon^2},\end{aligned}$$

leading to the desired inequality.

Appendix B

Technical details for the proof of Theorem 3.1

B.1 Expectations

We calculate $\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right)$ for two arbitrary unit norm frames Ψ,Φ whose spark is larger than S, ie. any subset of S vectors is linearly independent.

$$\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right)=\sum_i \mathbb{E}_p\left(c_{p(i)}^2\|P_{I_p}(\Psi)\phi_i\|_2^2\right) \tag{40}$$

For each i we now split the set of all permutations $\mathcal{P}$ into disjoint sets $\mathcal{P}^i_{Jk}$, defined as

$$\mathcal{P}^i_{Jk}:=\{p:p(J)=\{1,\dots,S\},p(i)=k\},$$

where J is subset of $\{1,\dots,K\}$ with $|J|=S$ and $k=1\dots K$. We then have $\mathcal{P}=\cup_{J,k}\mathcal{P}^i_{jk}$ and

$$|\mathcal{P}^i_{Jk}|=\begin{cases}(K-S-1)!S! & \text{if } i\notin J \text{ and } k\geq S+1\\ (K-S)!(S-1)! & \text{if } i=j\in J \text{ and } k=p(j)\\ 0 & \text{else}\end{cases}.$$

Using these sets we can compute the expectations in (40) as follows

$$\begin{aligned}\mathbb{E}_p\left(c_{p(i)}^2\|P_{I_p}(\Psi)\phi_i\|_2^2\right)&=\frac{1}{K!}\sum_J\sum_k\sum_{p\in\mathcal{P}^i_{Jk}}c_k^2\|P_J(\Psi)\phi_i\|_2^2\\ &=\binom{K}{S}^{-1}\frac{1}{K-S}\sum_{J:i\notin J}\sum_{k\geq S+1}c_k^2\|P_J(\Psi)\phi_i\|_2^2+\binom{K}{S}^{-1}\frac{1}{S}\sum_{J:i\in J}\sum_{k\leq S}c_k^2\|P_J(\Psi)\phi_i\|_2^2\\ &=\binom{K}{S}^{-1}\left(\frac{1-c_1^2-\ldots-c_S^2}{K-S}\sum_{J:i\notin J}\|P_J(\Psi)\phi_i\|_2^2+\frac{c_1^2+\ldots+c_S^2}{S}\sum_{J:i\in J}\|P_J(\Psi)\phi_i\|_2^2\right)\end{aligned}$$

Abbreviating $\gamma^2:=c_1^2+\ldots+c_S^2$ and re-substituting the above expression into (40) leads to,

$$\begin{aligned}\binom{K}{S}\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right)&=\frac{1-\gamma^2}{K-S}\sum_i\sum_{J:i\notin J}\|P_J(\Psi)\phi_i\|_2^2+\frac{\gamma^2}{S}\sum_i\sum_{J:i\in J}\|P_J(\Psi)\phi_i\|_2^2\\ &=\frac{1-\gamma^2}{K-S}\sum_i\sum_J\|P_J(\Psi)\phi_i\|_2^2+\left(\frac{\gamma^2}{S}-\frac{1-\gamma^2}{K-S}\right)\sum_i\sum_{J:i\in J}\|P_J(\Psi)\phi_i\|_2^2\\ &=\frac{1-\gamma^2}{K-S}\sum_J\sum_i\|P_J(\Psi)\phi_i\|_2^2+\left(\frac{\gamma^2}{S}-\frac{1-\gamma^2}{K-S}\right)\sum_J\sum_{i\in J}\|P_J(\Psi)\phi_i\|_2^2\\ &=\frac{1-\gamma^2}{K-S}\sum_J\|P_J(\Psi)\Phi\|_F^2+\left(\frac{\gamma^2}{S}-\frac{1-\gamma^2}{K-S}\right)\sum_J\|P_J(\Psi)\Phi_J\|_F^2.\end{aligned}$$

Since Φ is a tight frame we have $\|P_J(\Psi)\Phi\|_F^2 = \mathrm{tr}(\Phi^\star P_J(\Psi)^\star P_J(\Psi)\Phi) = \mathrm{tr}(P_J(\Psi)\Phi\Phi^\star) = AS$ and so we finally get

$$\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Psi)\Phi c_{p,\sigma}\|_2^2\right) = \frac{A(1-\gamma^2)S}{K-S} + \left(\frac{\gamma^2}{S} - \frac{1-\gamma^2}{K-S}\right)\binom{K}{S}^{-1}\sum_J \|P_J(\Psi)\Phi_J\|_F^2,$$

which for $\Psi = \Phi$ reduces to

$$\mathbb{E}_p\mathbb{E}_\sigma\left(\|P_{I_p}(\Phi)\Phi c_{p,\sigma}\|_2^2\right) = \gamma^2 + \frac{(A-1)(1-\gamma^2)S}{K-S},$$

B.2 Projection $P_J(\Psi)$

We want to compute the projection $P_J(\Psi) = \Psi_J(\Psi_J^\star\Psi_J)^{-1}\Psi_J^\star$ or more precisely $\|P_J(\Psi)\Phi_J\|_F^2$ and $\|P_J(\Phi) - P_J(\Psi)\|_F^2$ for $\Psi = \Phi_J A_J + Z_J W_J$ in terms of Φ_J and Z_J up to order $O(\varepsilon^3)$. Note that Condition (21) implies that any subset of S atoms of Φ is linearly independent. This means that $\Phi_J^\star\Phi_J$ is invertible and we can write $\Phi_J^\dagger = (\Phi_J^\star\Phi_J)^{-1}\Phi_J^\star$. (Ab)using the language of compressed sensing we denote the minimal eigenvalue of $\Phi_J^\star\Phi_J$ by $1-\delta_J(\Phi)$ and define $\delta_S(\Phi) := \max_{|J|\le S}\delta_J(\Phi) < 1$, which is known as lower isometry constant. In the following we will usually omit the reference to the dictionary for simplicity. We first split Ψ_J into the part contained in the span of Φ_J and the rest. Abbreviating $Q_J(\Phi) = \mathbb{I}_d - P_J(\Phi)$ and $B_J = Z_J W_J A_J^{-1}$, we have

$$\begin{aligned}\Psi_J &= P_J(\Phi)\Psi_J + Q_J(\Phi)\Psi_J \\ &= \Phi_J A_J + P_J(\Phi)Z_J W_J + Q_J(\Phi)Z_J W_J \\ &= \Big(\Phi_J(\mathbb{I}_S + \Phi_J^\dagger B_J) + Q_J(\Phi)B_J\Big)A_J. \qquad (41)\end{aligned}$$

Next we calculate $(\Psi_J^\star\Psi_J)^{-1}$. Using the expression in (41) we have

$$\Psi_J^\star\Psi_J = A_J\Big((\mathbb{I}_S + \Phi_J^\dagger B_J)^\star\Phi_J^\star\Phi_J(\mathbb{I}_S + \Phi_J^\dagger B_J) + B_J^\star Q_J(\Phi)B_J\Big)A_J.$$

Since $\|\Phi_J^\dagger B_J\|_{2,2} \le \|\Phi_J^\dagger\|_{2,2}\|B_J\|_F \le (1-\delta_J)^{-1/2}\left(\sum_{j\in J}\varepsilon_j^2/(1-\varepsilon_j^2)\right)^{1/2} < 1$ we can calculate the inverse of $(\mathbb{I}_S + \Phi_J^\dagger B_J)$ using a Neumann series, ie.

$$(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1} = \mathbb{I}_S + \sum_{i=1}^\infty(-\Phi_J^\dagger B_J)^i,$$

with $\|(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}\|_{2,2} \le (1 - \|\Phi_J^\dagger B_J\|_{2,2})^{-1}$. This allows us to rewrite $\Psi_J^\star\Psi_J$ as,

$$\Psi_J^\star\Psi_J = A_J(\mathbb{I}_S + \Phi_J^\dagger B_J)^\star\Phi_J^\star\Phi_J\left(\mathbb{I}_S + R_J\right)(\mathbb{I}_S + \Phi_J^\dagger B_J)A_J,$$
$$\text{for } R_J = (\Phi_J^\star\Phi_J)^{-1}(\mathbb{I}_S + \Phi_J^\dagger B_J)^{\star-1}B_J^\star Q_J(\Phi)B_J(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}.$$

Using the identity $\|(\Phi_J^\star \Phi_J)^{-1}\|_{2,2} = \|\Phi_J^\dagger\|_{2,2}^2$ we can estimate

$$\begin{aligned}\|R_J\|_{2,2} &\leq \|(\Phi_J^\star \Phi_J)^{-1}\|_{2,2} \|(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}\|_{2,2}^2 \|Q_J(\Phi) B_J\|_{2,2}^2 \\ &\leq \frac{\|Q_J(\Phi)B_J\|_F^2}{\left(\|\Phi_J^\dagger\|_{2,2}^{-1} - \|B_J\|_F\right)^2} \leq \frac{\sum_{j\in J} \varepsilon_j^2/(1-\varepsilon_j^2)}{1-\delta_S(\Phi) - 2\left(\sum_{j\in J}\varepsilon_j^2/(1-\varepsilon_j^2)\right)^{1/2}}.\end{aligned}$$

For ε small enough this is smaller than 1 and so we can again use a Neumann series to calculate the inverse,

$$(\Psi_J^\star \Psi_J)^{-1} = A_J^{-1}(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}\left(\mathbb{I}_S + \sum_{i=1}^\infty (-R_J)^i\right)(\Phi_J^\star \Phi_J)^{-1}(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1\star} A_J^{-1}.$$

Thus we finally get for the projection on the perturbed atoms indexed by J,

$$P_J(\Psi) = \left(\Phi_J + Q_J(\Phi)B_J(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}\right)\left(\mathbb{I}_S + \sum_{i=1}^\infty (-R_J)^i\right)(\Phi_J^\star \Phi_J)^{-1}\left(\Phi_J + Q_J(\Phi)B_J(\mathbb{I}_S + \Phi_J^\dagger B_J)^{-1}\right)^\star.$$

To calculate $\|P_J(\Phi) - P_J(\Psi)\|_F^2$ up to order $O(\varepsilon^3)$ we need to keep track of all terms involving B_J up to second order. We have,

$$\begin{aligned}\|P_J(\Phi) - P_J(\Psi)\|_F^2 &= \operatorname{tr}(P_J(\Phi)) - \operatorname{tr}(P_J(\Phi)P_J(\Psi)) + \operatorname{tr}(P_J(\Psi)) \\ &= 2S - 2\operatorname{tr}((\Phi_J^\star\Phi_J)^{-1}\Phi_J^\star \Psi_J(\Psi_J^\star\Psi_J)^{-1}\Psi_J^\star \Phi_J) \\ &= 2S - 2\operatorname{tr}\left(\mathbb{I}_S + \sum_{i=1}^\infty (-R_J)^i\right) \\ &\leq 2\sum_{i=1}^\infty \|R_J\|_F^i \\ &\leq \frac{2\|Q_J(\Phi)B_J\|_F^2}{\left(\|\Phi_J^\dagger\|_{2,2}^{-1} - \|B_J\|_F\right)^2 - \|Q_J(\Phi)B_J\|_F^2} \\ &\leq \frac{2\|Q_J(\Phi)B_J\|_F^2}{\|\Phi_J^\dagger\|_{2,2}^{-1}\left(\|\Phi_J^\dagger\|_{2,2}^{-1} - 2\|B_J\|_F\right)} = O(\|Q_J(\Phi)B_J\|_F^2).\end{aligned}$$

Similarily we get for $\|P_J(\Psi)\Phi_J\|_F^2$,

$$\begin{aligned}
\|P_J(\Psi)\Phi_J\|_F^2 &= \operatorname{tr}(\Phi_J^\star \Psi_J (\Psi_J^\star \Psi_J)^{-1} \Psi_J^\star \Phi_J) \\
&= \operatorname{tr}\left(\Phi_J^\star \Phi_J \left(\mathbb{I}_S + \sum_{i=1}^{\infty} (-R_J)^i \right)\right) \\
&= \operatorname{tr}\left(\Phi_J^\star \Phi_J\right) - \operatorname{tr}\left(\left(\mathbb{I}_S + \sum_{i=1}^{\infty} (-\Phi_J^\dagger B_J)^i \right)^\star B_J^\star Q_J(\Phi) B_J \left(\mathbb{I}_S + \sum_{i=1}^{\infty} (-\Phi_J^\dagger B_J)^i \right)\right) \\
&\qquad + \operatorname{tr}\left(\Phi_J^\star \Phi_J \sum_{i=2}^{\infty} (-R_J)^i \right) \\
&= \operatorname{tr}\left(\Phi_J^\star \Phi_J\right) - \operatorname{tr}\left(B_J^\star Q_J(\Phi) B_J\right) - 2\operatorname{tr}\left(B_J^\star Q_J(\Phi) B_J \sum_{i=1}^{\infty} (-\Phi_J^\dagger B_J)^i \right) \\
&\qquad - \operatorname{tr}\left(\left(\sum_{i=1}^{\infty} (-\Phi_J^\dagger B_J)^i \right)^\star B_J^\star Q_J(\Phi) B_J \sum_{i=1}^{\infty} (-\Phi_J^\dagger B_J)^i \right) + \operatorname{tr}\left(\Phi_J^\star \Phi_J \sum_{i=2}^{\infty} (-R_J)^i \right),
\end{aligned}$$

which leads to the upper bound,

$$\begin{aligned}
\|P_J(\Psi)\Phi_J\|_F^2 &\leq \|\Phi_J\|_F^2 - \|Q_J(\Phi)B_J\|_F^2 + 2\|Q_J(\Phi)B_J\|_F^2 \sum_{i=1}^{\infty} \|\Phi_J^\dagger B_J\|_F^i + \|\Phi_J^\star \Phi_J R_J\|_F \sum_{i=1}^{\infty} \|R_J\|^i \\
&\leq \|\Phi_J\|_F^2 - \|Q_J(\Phi)B_J\|_F^2 + \frac{2\|Q_J(\Phi)B_J\|_F^2 \|B_J\|_F}{\|\Phi_J^\dagger\|_{2,2}^{-1} - \|B_J\|_F} + \frac{\|Q_J(\Phi)B_J\|_F^4}{\|\Phi_J^\dagger\|_{2,2}^{-2} \left(\|\Phi_J^\dagger\|_{2,2}^{-1} - 2\|B_J\|_F \right)^2} \\
&= \|\Phi_J\|_F^2 - \|Q_J(\Phi)B_J\|_F^2 + O(\|Q_J(\Phi)B_J\|_F^2 \|B_J\|_F).
\end{aligned}$$

APPENDIX C

DECAY CONDITION (25)

Here we sketch how to derive decay condition (25). For simplicity we write I instead of I_p. For any subset of S indices $J \neq I$ we have,

$$\|P_I(\Phi)\Phi x\|_2^2 = \|P_I(\Phi)(\Phi_{I\cap J} x_{I\cap J} + \Phi_{I/J} x_{I/J} + \Phi_{J/I} x_{J/I} + \Phi_{(I\cup J)^c} x_{(I\cup J)^c})\|_2^2,$$

and therefore,

$$
\begin{aligned}
&\|P_I(\Phi)\Phi x\|_2^2 - \|P_J(\Phi)\Phi x\|_2^2 \\
&\quad = \|\Phi_{I/J}x_{I/J}\|_2^2 - \|\Phi_{J/I}x_{J/I}\|_2^2 \\
&\qquad + \|P_I(\Phi)\Phi_{J/I}x_{J/I}\|_2^2 - \|P_J(\Phi)\Phi_{I/J}x_{I/J}\|_2^2 \\
&\qquad + \|[P_I(\Phi) - P_{I\cap J}(\Phi)]\Phi_{(I\cup J)^c}x_{(I\cup J)^c}\|_2^2 - \|[P_J(\Phi) - P_{I\cap J}(\Phi)]\Phi_{(I\cup J)^c}x_{(I\cup J)^c}\|_2^2 \\
&\qquad + 2\langle \Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle - 2\langle \Phi_{J/I}x_{J/I}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle \\
&\qquad + 2\langle P_I(\Phi)\Phi_{J/I}x_{J/I}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle - 2\langle P_J(\Phi)\Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle \\
&\quad \geq \|\Phi_{I/J}x_{I/J}\|_2^2 - \|\Phi_{J/I}x_{J/I}\|_2^2 \\
&\qquad - \|P_J(\Phi)\Phi_{I/J}x_{I/J}\|_2^2 - \|[P_J(\Phi) - P_{I\cap J}(\Phi)]\Phi_{(I\cup J)^c}x_{(I\cup J)^c}\|_2^2 \\
&\qquad - 2|\langle \Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle| - 2|\langle \Phi_{J/I}x_{J/I}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle| \\
&\qquad - 2|\langle P_I(\Phi)\Phi_{J/I}x_{J/I}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle| - 2|\langle P_J(\Phi)\Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle|. \qquad (42)
\end{aligned}
$$

We now estimate all the terms in the last sum. We have

$$
\begin{aligned}
|\langle \Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle| &\leq \|\Phi^\star_{(I\cup J)^c}\Phi_{I/J}x_{I/J}\|_\infty \|x_{(I\cup J)^c}\|_1 \\
&= \max_{j\in(I\cup J)^c} \langle \Phi^\star_{I/J}\phi_j, x_{I/J}\rangle \|x_{(I\cup J)^c}\|_1 \\
&\leq \mu\sqrt{S-n}\|x_{I/J}\|_2\|x_{(I\cup J)^c}\|_1,
\end{aligned}
$$

$$
\begin{aligned}
|\langle P_J(\Phi)\Phi_{I/J}x_{I/J}, \Phi_{(I\cup J)^c}x_{(I\cup J)^c}\rangle| &= \max_{i\in(I\cup J)^c} \langle \phi_i, \Phi_J(\Phi_J^\star\Phi_J)^{-1}\Phi_J^\star\Phi_{I/J}x_{I/J}\rangle \|x_{(I\cup J)^c}\|_1 \\
&\leq \max_{i\in(I\cup J)^c} \|\Phi_J^\star\phi_i\|_2 \|(\Phi_J^\star\Phi_J)^{-1}\Phi_J^\star\Phi_{I/J}\|_{2,2}\|x_{I/J}\|_2\|x_{(I\cup J)^c}\|_1 \\
&\leq \mu\sqrt{S-n}\frac{S\mu}{1-(S-1)\mu}\|x_{I/J}\|_2\|x_{(I\cup J)^c}\|_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\|P_J(\Phi)\Phi_{I/J}x_{I/J}\|_2^2 &\leq \|\Phi_J(\Phi_J^\star\Phi_J)^{-1}\|_{2,2}^2 \|\Phi_J^\star\Phi_{I/J}\|_{2,2}^2\|x_{I/J}\|_2^2 \\
&\leq \frac{\mu^2 S(S-n)}{1-(S-1)\mu}\|x_{I/J}\|_2^2.
\end{aligned}
$$

To estimate $\|[P_I(\Phi) - P_{I\cap J}(\Phi)]\Phi_{(I\cup J)^c}x_{(I\cup J)^c}\|_2^2$ we use the following relation between the orthogonal projections, $P_A(\Phi)$, $P_B(\Phi)$ and $P_{A\cup B}(\Phi)$, for two disjoint index sets A, B. For simplicity we leave out the reference to the dictionary Φ. We have,

$$\begin{aligned} P_{A\cup B} &= \sum_{i=0}^{\infty}(P_A P_B)^i P_A(\mathbb{I}_S - P_B) + \sum_{i=0}^{\infty}(P_B P_A)^i P_B(I - P_A) \\ &= P_A + P_B + (P_A - I)\sum_{i=1}^{\infty}(P_B P_A)^i + (P_B - I)\sum_{i=1}^{\infty}(P_A P_B)^i. \end{aligned}$$

Thus for a vector y we have

$$\|(P_{A\cup B} - P_A)y\|_2 \leq \|P_B y\|_2 + \sum_{i=1}^{\infty}\|(P_B P_A)^i y\|_2 + \sum_{i=1}^{\infty}\|(P_A P_B)^i y\|_2.$$

Setting $A = I\cap J$, $B = I/J$ and $y = \Phi_{(I\cup J)^c} x_{(I\cup J)^c}$ we can estimate the terms in the expression above as,

$$\|P_B y\|_2 = \|(\Phi_B^\dagger)^\star \Phi_B^\star y\|_2 \leq \|(\Phi_B^\dagger)^\star\|_{2,2}\|\Phi_B^\star y\|_2 \leq \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)\mu}},$$

$$\begin{aligned} \|(P_B P_A)^i y\|_2 &= \|(\Phi_B^\dagger)^\star \Phi_B^\star (P_A P_B)^{i-1} (\Phi_A^\dagger)^\star \Phi_A^\star y\|_2 \\ &\leq \|(\Phi_B^\dagger)^\star\|_{2,2}\|\Phi_B^\star \left((\Phi_A^\dagger)^\star \Phi_A^\star (\Phi_B^\dagger)^\star \Phi_B^\star\right)^{i-1} (\Phi_A^\dagger)^\star\|_{2,2}\|\Phi_A^\star y\|_2 \\ &\leq \|\Phi_B^\dagger\|_{2,2}\|\Phi_B^\star(\Phi_A^\dagger)^\star\|_{2,2}^i \|\Phi_A^\star(\Phi_B^\dagger)^\star\|_{2,2}^{i-1}\|\Phi_A^\star y\|_2 \\ &\leq \|\Phi_B^\dagger\|_{2,2}\left(\|\Phi_B^\star\Phi_A\|_{2,2}\|(\Phi_A^\star\Phi_A)^{-1}\|_{2,2}\right)^i\left(\|\Phi_A^\star\Phi_B\|_{2,2}\|(\Phi_B^\star\Phi_B)^{-1}\|_{2,2}\right)^{i-1}\|\Phi_A^\star y\|_2 \\ &\leq \frac{1}{\sqrt{1-(S-n-1)\mu}}\left(\frac{\mu\sqrt{n(S-n)}}{(1-(n-1)\mu)}\right)^i\left(\frac{\mu\sqrt{n(S-n)}}{(1-(S-n-1)\mu)}\right)^{i-1}\mu\sqrt{n}\|x_{(I\cup J)^c}\|_1 \\ &\leq \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)}}\frac{n\mu}{(1-(n-1)\mu)}\left(\frac{\mu^2 n(S-n)}{(1-(n-1)\mu)(1-(S-n-1)\mu)}\right)^{i-1}, \end{aligned}$$

$$\begin{aligned} \|(P_A P_B)^i y\|_2 &\leq \|\Phi_A^\dagger\|_{2,2}\left(\|\Phi_A^\star\Phi_B\|_{2,2}\|(\Phi_B^\star\Phi_B)^{-1}\|_{2,2}\right)^i\left(\|\Phi_B^\star\Phi_A\|_{2,2}\|(\Phi_A^\star\Phi_A)^{-1}\|_{2,2}\right)^{i-1}\|\Phi_B^\star y\|_2 \\ &\leq \frac{1}{\sqrt{1-(n-1)\mu}}\left(\frac{\mu\sqrt{n(S-n)}}{(1-(S-n-1)\mu)}\right)^i\left(\frac{\mu\sqrt{n(S-n)}}{(1-(n-1)\mu)}\right)^{i-1}\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1 \\ &\leq \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)}}\left(\frac{\mu^2 n(S-n)}{(1-(n-1)\mu)(1-(S-n-1)\mu)}\right)^{i-1/2} \end{aligned}$$

This leads to the following bound for the difference of the projections.

$$\|(P_{A\cup B}-P_A)y\|_2 \le \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)\mu}}\left(1+\frac{\frac{n\mu}{(1-(n-1)\mu)}+\left(\frac{\mu^2 n(S-n)}{(1-(n-1)\mu)(1-(S-n-1)\mu)}\right)^{1/2}}{1-\frac{\mu^2 n(S-n)}{(1-(n-1)\mu)(1-(S-n-1)\mu)}}\right)$$

$$\le \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)\mu}}\left(1+\frac{2(S-1)\mu}{(1-(S-1)\mu)}\frac{(1-(n-1)\mu)(1-(S-n-1)\mu)}{1-(S-2)\mu-(S-1)\mu^2}\right)$$

$$\le \frac{\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1}{\sqrt{1-(S-n-1)\mu}}\left(1+\frac{2(S-1)\mu}{(1-(S-1)\mu)^2}\right)$$

Substituting all the estimates into (**??**) we get,

$$\begin{aligned}
&\|P_I(\Phi)\Phi x\|_2^2-\|P_J(\Phi)\Phi x\|_2^2\\
&\quad\ge (1-(S-n-1)\mu)\|x_{I/J}\|_2^2-(1+(S-n-1)\mu)\|x_{J/I}\|_2^2\\
&\qquad-\ \frac{\mu^2 S(S-n)}{1-(S-1)\mu}\|x_{I/J}\|_2^2-\frac{\mu^2(S-n)\|x_{(I\cup J)^c}\|_1^2}{1-(S-n-1)\mu}\left(1+\frac{2(S-1)\mu}{(1-(S-1)\mu)^2}\right)^2\\
&\qquad-2\mu\sqrt{S-n}\|x_{I/J}\|_2\|x_{(I\cup J)^c}\|_1-2\mu\sqrt{S-n}\|x_{J/I}\|_2\|x_{(I\cup J)^c}\|_1\\
&\qquad-2\mu\sqrt{S-n}\frac{S\mu}{1-(S-1)\mu}\|x_{I/J}\|_2\|x_{(I\cup J)^c}\|_1-2\mu\sqrt{S-n}\frac{S\mu}{1-(S-1)\mu}\|x_{J/I}\|_2\|x_{(I\cup J)^c}\|_1\\
&\quad= a\|x_{I/J}\|_2^2-2b\|x_{I/J}\|_2-c\|x_{J/I}\|_2^2-2b\|x_{J/I}\|_2-d\\
&\quad= (\sqrt{a}\|x_{I/J}\|_2-b/\sqrt{a})^2-(\sqrt{c}\|x_{J/I}\|_2+b/\sqrt{c})^2-b^2/a+b^2/c-d
\end{aligned}$$

where

$$\begin{aligned}
a&=1-(S-n-1)\mu-\frac{\mu^2 S(S-n)}{1-(S-1)\mu}\\
b&=\mu\sqrt{S-n}\|x_{(I\cup J)^c}\|_1\left(1+\frac{S\mu}{1-(S-1)\mu}\right)=\tilde{b}\sqrt{S-n}\\
c&=1+(S-n-1)\mu\\
d&=\frac{\mu^2(S-n)\|x_{(I\cup J)^c}\|_1^2}{1-(S-n-1)\mu}\left(1+\frac{2(S-1)\mu}{(1-(S-1)\mu)^2}\right)^2=\tilde{d}(S-n)
\end{aligned}$$

Thus to have $\|P_I(\Phi)\Phi x\|_2^2-\|P_J(\Phi)\Phi x\|_2^2>0$ it is sufficient to have,

$$(\sqrt{a}\|x_{I/J}\|_2-b/\sqrt{a})^2-(\sqrt{c}\|x_{J/I}\|_2+b/\sqrt{c})^2-b^2/a+b^2/c-d>0,$$

which is in turn implied by

$$\|x_{I/J}\|_2>\|x_{J/I}\|_2\sqrt{c/a}+b/\sqrt{ca}+b/a+\sqrt{b^2/a^2-b^2/(ca)+d/a}>0,$$

Using the bounds $\|x_{I/J}\|_2 \geq \sqrt{S-n}c_S$ and $\|x_{J/I}\|_2 \leq \sqrt{S-n}c_{S+1}$ we can further simplify to

$$c_S > c_{S+1}\sqrt{c/a} + \tilde{b}/\sqrt{ca} + \tilde{b}/a + \sqrt{\tilde{b}^2/a^2 - \tilde{b}^2/(ca) + \tilde{d}/a} > 0,$$

For $S\mu < 1/2$ we have the bounds,

$$\sqrt{c/a} \leq \frac{1-S\mu}{1-2S\mu}, \qquad \tilde{b}/\sqrt{ca} \leq \tilde{b}/a \leq \frac{\mu\|x_{(I\cup J)^c}\|_1}{1-2S\mu},$$
$$\text{and } \sqrt{\tilde{b}^2/a^2 - \tilde{b}^2/(ca) + \tilde{d}/a} \leq 2\frac{\mu\|x_{(I\cup J)^c}\|_1}{1-2S\mu},$$

leading to the final condition,

$$c_S > \frac{1-S\mu}{1-2S\mu}c_{S+1} + \frac{4\mu}{1-2S\mu}\sum_{i>S+1}|c_i|.$$

References

[1] M. Aharon, M. Elad, and A.M. Bruckstein. K-SVD: An algorithm for designing overcomplete dictionaries for sparse representation. *IEEE Trans. on Signal Processing.*, 54(11):4311–4322, November 2006.

[2] M. Aharon, M. Elad, and A.M. Bruckstein. On the uniqueness of overcomplete dictionaries, and a practical way to retrieve them. *Journal of Linear Algebra and Applications*, 416:48–67, July 2006.

[3] T. Blumensath and M.E. Davies. Iterative thresholding for sparse approximations. *Journal of Fourier Analysis and Applications*, 14(5-6):629–654, 2008.

[4] E. Candès, L. Demanet, D.L. Donoho, and L. Ying. Fast discrete curvelet transforms. *Multiscale Modeling & Simulation*, 5(3):861–899, 2006.

[5] E. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inf. Theory*, 52(2):489–509, 2006.

[6] S.S. Chen, D.L. Donoho, and M.A. Saunders. Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing*, 1998.

[7] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2003.

[8] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Lecture Notes. SIAM, 1992.

[9] I. Daubechies, R.A. DeVore, M. Fornasier, and S. Güntürk. Iteratively reweighted least squares minimization for sparse recovery. *Comm. Pure Appl. Math.*, 63(1):1–38, January 2010.

[10] G. Davis, S. Mallat, and M. Avellaneda. Adaptive greedy approximations. *Constructive Approximation*, 13:57–98, 1997. Springer-Verlag New York Inc.

[11] D.L. Donoho. Compressed sensing. *IEEE Trans. Inform. Theory*, 52(4):1289–1306, 2006.

[12] D.L. Donoho, M. Elad, and V.N. Temlyakov. Stable recovery of sparse overcomplete representations in the presence of noise. *IEEE Transactions on Information Theory*, 52(1):6–18, January 2006.

[13] D.J. Field and B.A. Olshausen. Emergence of simple-cell receptive field properties by learning a sparse code for natural images. *Nature*, 381:607–609, 1996.

[14] Q. Geng, H. Wang, and J. Wright. On the local correctness of ℓ^1-minimization for dictionary learning. *arXiv:1101.5672v1 [cs.IT]*, 2011.

[15] P. Georgiev, F.J. Theis, and A. Cichocki. Sparse component analysis and blind source separation of underdetermined mixtures. *IEEE Transactions on Neural Networks*, 16(4):992–996, 2005.

[16] R. Gribonval, H. Rauhut, K. Schnass, and P. Vandergheynst. Atoms of all channels, unite! Average case analysis of multi-channel sparse recovery using greedy algorithms. *Journal of Fourier Analysis and Applications*, 14(5):655–687, 2008.

[17] R. Gribonval and K. Schnass. Dictionary identifiability - sparse matrix-factorisation via l_1-minimisation. *IEEE Transactions on Information Theory*, 56(7):3523–3539, July 2010.

[18] R. Jenatton, F. Bach, and R. Gribonval. Local stability and robustness of sparse dictionary learning in the presence of noise. *preprint*, 2012.

[19] K. Kreutz-Delgado, J.F. Murray, B.D. Rao, K. Engan, T. Lee, and T.J. Sejnowski. Dictionary learning algorithms for sparse representation. *Neural Computations*, 15(2):349–396, 2003.

[20] K. Kreutz-Delgado and B.D. Rao. FOCUSS-based dictionary learning algorithms. In *Proc. SPIE 4119*, 2000.

[21] M. Ledoux and M. Talagrand. *Probability in Banach spaces. Isoperimetry and processes.* Springer-Verlag, Berlin, Heidelberg, NewYork, 1991.

[22] J. Mairal, F. Bach, and J. Ponce. Task-driven dictionary learning. Technical Report 7500, INRIA, 2010.

[23] D. Needell and J.A. Tropp. CoSaMP: Iterative signal recovery from incomplete and inaccurate samples. *Applied Computational Harmonic Analysis*, 26(3):301–321, 2009.

[24] M.D. Plumbley. Dictionary learning for ℓ_1-exact sparse coding. In M.E. Davies, C.J. James, and S.A. Abdallah, editors, *International Conference on Independent Component Analysis and Signal Separation*, volume 4666, pages 406–413. Springer, 2007.

[25] DSP Rice University. Compressive sensing resources. *http://www.compressedsensing.com/*.

[26] R. Rubinstein, A. Bruckstein, and M. Elad. Dictionaries for sparse representation modeling. *Proceedings of the IEEE*, 98(6):1045–1057, 2010.

[27] K. Schnass. On the identifiability of overcomplete dictionaries via the minimisation principle underlying K-SVD. *arXiv:1301.3375*, 2013.

[28] K. Schnass and P. Vandergheynst. Average performance analysis for thresholding. *IEEE Signal Processing Letters*, 14(11):828–831, 2007.

[29] K. Skretting and K. Engan. Recursive least squares dictionary learning algorithm. *IEEE Transactions on Signal Processing*, 58(4):2121–2130, April 2010.

[30] J.A. Tropp. Greed is good: Algorithmic results for sparse approximation. *IEEE Transactions on Information Theory*, 50(10):2231–2242, October 2004.

[31] J.A. Tropp. On the conditioning of random subdictionaries. *Applied Computational Harmonic Analysis*, 25(1-24), 2008.

[32] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. In Y. Eldar and G. Kutyniok, editors, *Compressed Sensing, Theory and Applications*, chapter 5. Cambridge University Press, 2012.

[33] J. Vybíral. Average best m-term approximation. *Constructive Approximation*, 36(1):83, 115 2012.

[34] M. Yaghoobi, T. Blumensath, and M.E. Davies. Dictionary learning for sparse approximations with the majorization method. *IEEE Transactions on Signal Processing*, 57(6):2178–2191, June 2009.

[35] M. Zibulevsky and B.A. Pearlmutter. Blind source separation by sparse decomposition in a signal dictionary. *Neural Computations*, 13(4):863–882, 2001.