universität innsbruck

AB Technische Mathematik

Bachelor / Masters / PhD thesis

Nonlocal Diffusion – Modelling, Analysis, Numerics

In recent years there has been much interest in diffusion processes involving "long-range" (Levy) diffusion, motivated by nonlocal phenomena in the biological and physical sciences. Discrete jump processes describe diffusion in networks [6, 12] (spreading of diseases, social networks, kernel-based learning), and jump processes in $\mathbb{R}^n$ generalise PDEs for nonlocal cell movement or elasticity [3, 4, 5, 12]. Some continuous or discrete systems are depicted in Figure 1. A typical operator is the *fractional* Laplacian:

$$(-\Delta)^\alpha u = c_{n,\alpha} \int_{\mathbf{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\, dy = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u(\xi)) \ .$$

Here, $\mathcal{F}$ the Fourier transform and $c_{n,\alpha}$ a number depending on the dimension n and the exponent $\alpha \in (0,1)$. $(-\Delta)^\alpha$ recovers the Laplacian for $\alpha = 1$.

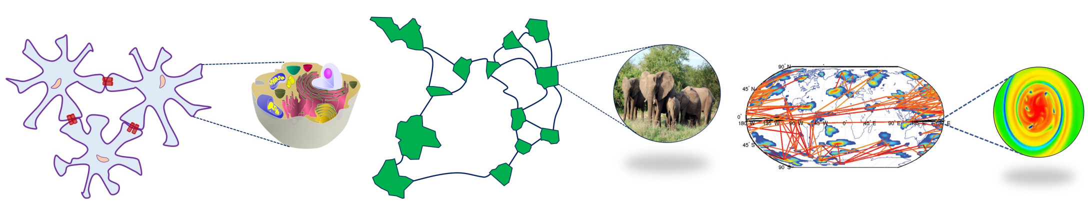
This project joins the recent work in our group in this topic. Depending on the interests of the student, possible directions include

- **Biological modelling:** macroscopic description of cell migration and self-organisation, based on microscopic movement laws [1, 4, 5, 6], effective descriptions for complex organisms [6], applications to robotic systems [4].
- **Analysis of PDEs:** calculus of variations for nonlocal problems, singularities of solutions [6, 7, 8, 9], blow-up and pattern formation in time-dependent problems [1].
- **Numerics:** fast and reliable finite elements for nonlocal heat equations and reaction-diffusion equations [10], space-time adaptive mesh refinements [10], optimal discretisations [9], *hp*-methods.

Basic questions in these areas are only starting to be addressed [3]. For the **analysis**, a basic trick popularised by [2] allows to interpret the fractional Laplacian $(-\Delta)^\alpha$ in a domain $\Omega \subset \mathbb{R}^n$ as a degenerate differential operator in the half-plane one dimension higher, $\mathbb{R}^{n+1}_+$, where more standard methods may be applied. A toolbox for the analysis of nonlocal boundary problems is developed in ongoing work [8].

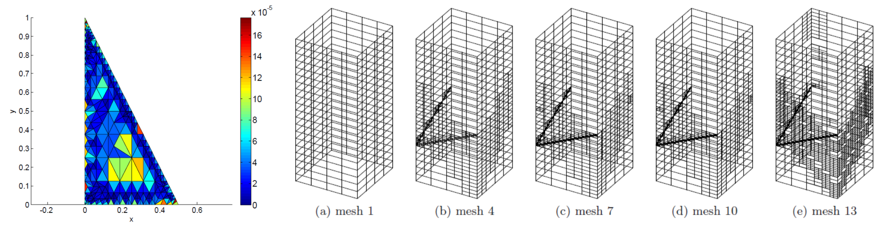
The rigorous analysis translates into fast **numerical methods** for nonlocal equations. The accuracy and convergence rate is determined by the singularities of the solution, such as spikes or boundary layers, which are a distinguishing feature of these problems. If the geometric location of the singularities is known (e.g. boundary), graded meshes which are appropriately refined towards the singularity can be proven to give optimal convergence [9]. However, typically the location of singularities is not known and theorems (sharp computable error estimates) give rise to mesh refinements [10]. Doing this in space and time, for example letting the numerical method track the collisions between two bodies [10] produces both nice geometries and provably accurate solutions.

Our work is motivated by applications to **the analysis of biological systems**: From first principles for the movement of organisms, we have found that nonlinear fractional diffusion equations describe the



(a) (b) (c)

Figure 1: (a) Illustration of a cellular system formed by biological cells connected by means of gap junctions to interchange chemicals and a zoom of the internal structure of a cell. (b) Landscape ecological system formed by patches interconnected by corridors used by some species to move from patch to patch. Zooming-in reveals the foraging movement of these species inside the patches. (c) Climate system formed by a network of climatic events correlations and the internal climatic events at local regions.



(a) (b)

Figure 2: Adaptive mesh refinements and resulting space-time meshes.

macroscopic evolution of interacting particles from swarms of robots to E. coli and cancer cells in the brain [4, 5, 6]. In its simplest form the relevant systems are given by

$$\begin{aligned} \partial_t u &= \Delta^\alpha u + f(u, \rho) \ , \\ \partial_t \rho &= \Delta \rho + g(u, \rho) \ . \end{aligned} \tag{1}$$

For organisms, u describes the density of the organism, and ρ is the density of a nutrient or chemical cue. f, g are functions determined by the biology or robotic application. For $\alpha = 1$, equation (1) reduces to the Keller-Segel model for chemotaxis. See [1] for a survey of the analysis and the rich behaviour of solutions to the Keller-Segel system, from pattern formation to blow-up and hard questions in analysis. Characteristic is the ability of self-organization, which leads to pattern formation and swarming behaviour. The formation of aggregates involves the collapse of one of the variables of the model and, consequently, the blow-up of the corresponding solution in finite time.

References

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Prerequisites

Interest in numerical methods, modelling or mathematical analysis of partial differential equations.

Contact

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