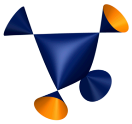
universität
innsbruck
Institut für Mathematik

Operator Algebra

Tim Netzer



∀¬6∃qʎ∀

A⌐9EbYA

Contents

Introduction

Operator algebra is a branch of mathematics at the crossroads of functional analysis and algebra, with important connections to mathematical physics, in particular to quantum theory. Rather than studying individual operators in isolation, it studies operators through the structures formed by their interactions. These structures have a strong algebraic flavor, but are enriched by additional features such as involutions, norms, and topologies. They provide a natural setting in which geometry, algebra, and analysis meet. The subject originated in the early 20th century, when mathematicians such as John von Neumann and Israel Gelfand sought to formalize the algebraic and analytic properties of bounded operators on Hilbert spaces, motivated in part by questions arising from the foundations of quantum mechanics.

The most basic objects of study are algebras of operators on Hilbert spaces that are closed under certain natural operations. Among these, C^*-algebras occupy a central position. They can be characterized axiomatically as complete normed algebras equipped with an involution satisfying the C^*-identity. In this way, they simultaneously generalize the algebra of continuous functions on a compact space and the algebra of bounded operators on a Hilbert space.

Within pure mathematics, C^*-algebras provide deep links between functional analysis, topology, representation theory, and geometry. For instance, the theorem of Gelfand–Naimark identifies every commutative C^*-algebra with the algebra of continuous functions on a compact Hausdorff space. This result unifies algebraic and topological viewpoints and opens the door to noncommutative geometry, where noncommutative C^*-algebras are interpreted as algebras of functions on “noncommutative” or “quantum” spaces.

From the standpoint of applications, C^*-algebras form one of the basic mathematical languages of quantum mechanics. Physical observables are modeled by self-adjoint elements, while states are modeled by positive linear functionals. The operator-algebraic approach therefore provides not only a rigorous formalism for quantum theory, but also a powerful framework for studying statistical mechanics, quantum information theory, and quantum field theory.

In these lecture notes, we gradually build the machinery needed to work with operator algebras and operator spaces. We begin with $*$-algebras, states, and representations; then move on to C^*-algebras and their classification; continue with operator systems; and finish with selected applications.

Among the many sources on the topic, we recommend [1, 3, 4, 5, 6, 7, 8, 9].

Chapter 1

Algebras with Involution

1.1 Preliminaries

In this section we recall some basic concepts and results from functional analysis that will be used throughout the course. We do not include proofs here, but refer the reader to standard texts on functional analysis, for example [6].

Definition 1.1.1. (i) A **Banach space** is a normed vector space over $\mathbb{C}$ that is complete with respect to the metric induced by the norm; equivalently, every Cauchy sequence converges.

(ii) A **Hilbert space** is a vector space over $\mathbb{C}$ equipped with an inner product such that the induced norm makes it into a Banach space.

(iii) If V and W are normed spaces, a **bounded linear operator from V to W** is a linear map

$$L\colon V \to W$$

for which there exists some $\lambda \geqslant 0$ such that

$$\|Lv\|_W \leqslant \lambda \|v\|_V$$

for all $v \in V$. We denote the set of all bounded linear operators from V to W by $\mathcal{B}(V, W)$, and simply write $\mathcal{B}(V)$ if $V = W$.

(iv) For a bounded linear operator $L \in \mathcal{B}(V, W)$, its **operator norm** is defined by

$$\|L\|_{\mathrm{op}} := \inf \{\lambda \geqslant 0 \mid \forall v \in V\colon \|Lv\|_W \leqslant \lambda \|v\|_V\}. \qquad \triangle$$

Example 1.1.2. (i) $\mathbb{C}^n$ with the standard inner product is a Hilbert space.

(*ii*) The space

$$\ell_2 := \left\{ (c_i)_{i\in\mathbb{N}} \mid c_i \in \mathbb{C}, \sum_{i=0}^{\infty} |c_i|^2 < \infty \right\}$$

is a Hilbert space with respect to the inner product

$$\langle (c_i)_i, (d_i)_i \rangle := \sum_{i=0}^{\infty} c_i \overline{d}_i .$$

(*iii*) Let X be a compact Hausdorff space. Then the space $\mathcal{C}(X)$ of all continuous complex-valued functions on X is a Banach space with respect to the sup-norm

$$\|f\|_\infty := \sup\left\{ |f(x)| \mid x \in X \right\}.$$

(*iv*) If V and W are Banach spaces, then $\mathcal{B}(V, W)$ is again a Banach space with respect to the operator norm. △

Remark 1.1.3. (*i*) Every finite-dimensional normed vector space over $\mathbb{C}$ is a Banach space. Every finite-dimensional inner product space is a Hilbert space.
(*ii*) If V is finite-dimensional, then every linear map $L\colon V \to W$ is bounded. Thus, after choosing bases, $\mathcal{B}(V, W)$ identifies with $\mathrm{Mat}_{m,n}(\mathbb{C})$.
(*iii*) For linear maps between normed spaces, boundedness is equivalent to continuity. The operator norm can also be described as

$$\|L\|_{\mathrm{op}} = \sup\left\{ \frac{\|Lv\|}{\|v\|} \mid v \in V, v \neq 0 \right\} = \sup\left\{ \|Lv\| \mid v \in V, \|v\| = 1 \right\}. \qquad \triangle$$

Definition 1.1.4. Let I be an arbitrary index set, and let H_i be a Hilbert space for every $i \in I$. The **direct sum of Hilbert spaces** is defined by

$$\bigoplus_{i\in I} H_i := \{(h_i)_{i\in I} \mid h_i \in H_i, \sum_{i\in I} \|h_i\|^2 < \infty\},$$

where the infinite sum is understood as the supremum of all finite partial sums. This direct sum carries the inner product

$$\langle (h_i)_i, (g_i)_i \rangle := \sum_{i\in I} \langle h_i, g_i \rangle .$$

With this inner product it is again a Hilbert space.

Now assume further that $L_i\colon H_i \to \tilde{H}_i$ is a bounded linear operator between Hilbert spaces for every $i \in I$. We would like to define the **direct sum of operators**

$$\begin{aligned}\oplus_i L_i\colon \bigoplus_i H_i &\to \bigoplus_i \tilde{H}_i\\ (h_i)_i &\mapsto (L_i h_i)_i.\end{aligned}$$

For this map to be well-defined and bounded, the operator norms of the L_i must be bounded uniformly in i. △

Definition 1.1.5. If V is a normed space, we denote by V' its topological dual space, i.e. the space of all bounded linear functionals on V. The **weak topology** on V is the coarsest vector space topology that makes all functionals from V' continuous. It is therefore coarser than the norm topology. The **weak*-topology** on V' is the coarsest vector space topology that makes all evaluation maps at points of V continuous. It is therefore coarser than the operator norm topology on V', and also coarser than the weak topology on V'. △

Remark 1.1.6. We will use two important theorems from functional analysis in these notes. First, the *Theorem of Banach–Alaoglu* states that the operator norm unit ball in V' is compact in the weak*-topology. Second, the *uniform boundedness principle*, also known as the *Theorem of Banach–Steinhaus*, says that if a sequence of bounded linear operators from a Banach space to a normed space is pointwise bounded, then it is bounded in operator norm. In particular, every weakly convergent sequence in a Banach space is bounded in norm (Exercise 1). △

1.2 $*$-Algebras

Definition 1.2.1. (i) A **$*$-algebra** is a unital, not necessarily commutative, $\mathbb{C}$-algebra $\mathcal{A}$ equipped with an **involution**, i.e. a map $a \mapsto a^*$ satisfying

$$(\lambda a + \gamma b)^* = \overline{\lambda}\, a^* + \overline{\gamma}\, b^*, \quad (a^*)^* = a, \quad (ab)^* = b^* a^*$$

for all $a, b \in \mathcal{A}$ and all $\lambda, \gamma \in \mathbb{C}$.

(ii) For a $*$-algebra $\mathcal{A}$, the real subspace

$$\mathcal{A}_{\mathrm{sa}} := \{a \in \mathcal{A} \mid a^* = a\}$$

is called the subspace of **self-adjoint** or **Hermitian elements**.

(iii) The set of **sums of squares** is defined by

$$\sum \mathcal{A}^2 := \left\{ \sum_{i=1}^{m} a_i^* a_i \mid m \in \mathbb{N},\ a_i \in \mathcal{A} \right\}.$$

It is a convex cone in $\mathcal{A}_{\mathrm{sa}}$ and satisfies

$$a^* \cdot \sum \mathcal{A}^2 \cdot a \subseteq \sum \mathcal{A}^2$$

for all $a \in \mathcal{A}$.
(iv) By $\mathcal{A}^\times$ we denote the set of invertible elements in $\mathcal{A}$, i.e.

$$\mathcal{A}^\times := \{a \in \mathcal{A} \mid \exists b \in \mathcal{A}\colon ab = ba = 1\}.$$

(v) A $*$**-algebra homomorphism** is a unital algebra homomorphism

$$\pi\colon \mathcal{A} \to \mathcal{B}$$

between $*$-algebras that is compatible with the involutions, i.e. $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. △

Example 1.2.2. If X is a compact Hausdorff space, the algebra $\mathcal{C}(X)$ of continuous complex-valued functions on X is a commutative $*$-algebra. All operations are defined pointwise; in particular,

$$f^*(x) := \overline{f(x)}$$

for $f \in \mathcal{C}(X)$ and $x \in X$. The self-adjoint elements are precisely the real-valued functions. Sums of squares coincide with single squares and are precisely the nonnegative functions. Every continuous map $\pi\colon X \to Y$ induces a $*$-algebra homomorphism

$$\pi'\colon \mathcal{C}(Y) \to \mathcal{C}(X); \quad f \mapsto f \circ \pi,$$

and every $*$-algebra homomorphism between such algebras arises in this way. △

Example 1.2.3. (i) The ring $\mathbb{C}[x_1, \ldots, x_n]$ of complex polynomials in n variables is a commutative $*$-algebra. The involution is given by complex conjugation of the coefficients, so

$$\mathbb{C}[x_1, \ldots, x_n]_{\mathrm{sa}} = \mathbb{R}[x_1, \ldots, x_n].$$

To define a $*$-algebra homomorphism $\pi\colon \mathbb{C}[x_1,\dots,x_n] \to \mathcal{B}$, one has to choose pairwise commuting self-adjoint elements $b_1,\dots,b_n \in \mathcal{B}_{\mathrm{sa}}$ and set $\pi(x_i) = b_i$.

(ii) The ring $\mathbb{C}\langle z_1,\dots,z_n\rangle$ of polynomials in noncommuting variables is defined similarly, except that the variables do not commute. Its elements are finite linear combinations of words in the variables, and the order of the letters matters. For example, $z_1z_2 - z_2z_1$ is not the zero polynomial. We define the involution by complex conjugation on the coefficients and by $z_i^* = z_i$. Since involutions reverse the order of products, we get for example

$$(z_1z_2 - iz_2z_1)^* = z_2z_1 + iz_1z_2.$$

Thus the self-adjoint polynomials are not simply the polynomials with real coefficients. To define a $*$-algebra homomorphism $\pi\colon \mathbb{C}\langle z_1,\dots,z_n\rangle \to \mathcal{B}$, one has to choose self-adjoint elements $b_1,\dots,b_n \in \mathcal{B}_{\mathrm{sa}}$ and set $\pi(z_i) = b_i$; in contrast to the commutative case, the elements b_i need not commute. △

Example 1.2.4. (i) The algebra $\mathrm{Mat}_d(\mathbb{C})$ of complex $d \times d$ matrices is a $*$-algebra, with involution given by the usual adjoint operation. It is noncommutative for $d \geqslant 2$. Its self-adjoint part $\mathrm{Mat}_d(\mathbb{C})_{\mathrm{sa}}$ consists of the Hermitian matrices and is also denoted by $\mathrm{Her}_d(\mathbb{C})$. Sums of squares coincide with single squares Q^*Q and are precisely the positive semidefinite matrices.

(ii) More generally, the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H is a $*$-algebra. Every bounded operator φ has a unique adjoint, defined by

$$\langle \varphi(h_1), h_2\rangle = \langle h_1, \varphi^*(h_2)\rangle$$

for all $h_1, h_2 \in H$. Again, sums of squares coincide with single squares and with positive semidefinite operators, i.e. self-adjoint operators φ with $\langle \varphi(h), h\rangle \geqslant 0$ for all $h \in H$. △

Example 1.2.5. Let Γ be a group with identity element e; we use multiplicative notation. The **group algebra** $\mathbb{C}\Gamma$ is the complex vector space with basis Γ,

$$\mathbb{C}\Gamma = \left\{ \sum_{g\in\Gamma} c_g \cdot g \mid c_g \in \mathbb{C},\ \text{only finitely many } c_g \neq 0 \right\}.$$

Multiplication in Γ extends bilinearly to a multiplication on $\mathbb{C}\Gamma$:

$$\left(\sum_g c_g \cdot g\right) \cdot \left(\sum_g c'_g \cdot g\right) = \sum_g \left(\sum_{f\cdot h = g} c_f c'_h\right) \cdot g.$$

In this way $\mathbb{C}\Gamma$ becomes an algebra. It is commutative if and only if Γ is commutative, and its identity element is $1 \cdot e$. We equip $\mathbb{C}\Gamma$ with the involution

$$\left(\sum_g c_g \cdot g\right)^* = \sum_g \overline{c_g} \cdot g^{-1} = \sum_g \overline{c_{g^{-1}}} \cdot g.$$

Hence an element $\sum_g c_g \cdot g$ is self-adjoint if and only if

$$\overline{c_g} = c_{g^{-1}}$$

for all $g \in \Gamma$.

For example, consider $\Gamma = \mathbb{Z}$ with addition. Then $\mathbb{C}\mathbb{Z}$ is commutative and isomorphic to the algebra

$$\mathbb{C}\mathbb{Z} \cong \mathbb{C}[t, t^{-1}]$$

of Laurent polynomials in one variable, where the basis vector $m \in \mathbb{Z}$ corresponds to t^m. The corresponding involution satisfies $(t^i)^* = t^{-i}$, so the self-adjoint elements are not the real Laurent polynomials. However, $\mathbb{C}[t, t^{-1}]$ is also generated by the self-adjoint elements

$$x := \frac{t + t^{-1}}{2} \quad \text{and} \quad y := \frac{t - t^{-1}}{2i}.$$

They satisfy $x^2 + y^2 = 1$, and we obtain a $*$-algebra isomorphism with

$$\mathbb{C}[x, y]/(x^2 + y^2 - 1) = \mathbb{C}[S^1],$$

the algebra of polynomial functions on the unit circle, now with involution $x^* = x$ and $y^* = y$. In this presentation, the self-adjoint elements are exactly the polynomials with real coefficients. More generally, $\mathbb{C}\mathbb{Z}^n$ is isomorphic to the algebra

$$\mathbb{C}[\underbrace{S^1 \times \cdots \times S^1}_{n}]$$

of polynomial functions on the n-dimensional torus, with its canonical involution.

Another important example is $\Gamma = F_n$, the **free group** on n generators $z_1, \ldots, z_n$. Its elements are words in the letters

$$z_1, \ldots, z_n \quad \text{and} \quad z_1^{-1}, \ldots, z_n^{-1}.$$

The group operation is concatenation of words, subject only to the relations

$$z_i^{-1} z_i = z_i z_i^{-1} = e$$

for all i, where e denotes the empty word. The group algebra $\mathbb{C}F_n$ contains the noncommutative polynomial algebra $\mathbb{C}\langle z_1, \dots, z_n \rangle$ as a subalgebra, but not as a $*$-subalgebra: in $\mathbb{C}\langle \underline{z} \rangle$ we have $z_i^* = z_i$, whereas in $\mathbb{C}F_n$ we have $z_i^* = z_i^{-1}$. △

1.3 Stone-Weierstraß and Burnside Theorems

In this section we prove two fundamental results that are closely related in spirit. The first is the Stone-Weierstraß theorem, which concerns $*$-subalgebras of $\mathcal{C}(X)$ for a compact Hausdorff space X. The second is Burnside's theorem, which can be viewed as a finite-dimensional noncommutative analogue for subalgebras of matrix algebras.

Lemma 1.3.1. *Let $\mathcal{A} \subseteq \mathcal{C}(X)$ be a closed subalgebra. If $f, g \in \mathcal{A}$ are real-valued, then*

$$|f|, \, \max(f, g), \, \min(f, g) \in \mathcal{A}.$$

Proof. Assume without loss of generality that $\|f\|_\infty \leqslant 1$. By the classical Weierstraß approximation theorem, there is a sequence of polynomials $p_n \in \mathbb{R}[t]$ converging uniformly on $[-1, 1]$ to the absolute value function. Hence

$$\|p_n(f) - |f|\|_\infty \overset{n\to\infty}{\to} 0.$$

Since $\mathcal{A}$ is a subalgebra, it contains all $p_n(f)$, and since $\mathcal{A}$ is closed, it follows that $|f| \in \mathcal{A}$. The remaining claims follow from

$$\max\{f, g\} = \frac{1}{2}|f - g| + \frac{1}{2}(f + g) \quad \text{and} \quad \min\{f, g\} = -\max\{-f, -g\}. \qquad \square$$

A subset $\mathcal{A} \subseteq \mathcal{C}(X)$ is said to *separate points of X* if for every pair of distinct points $x, y \in X$ there exists some $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

Theorem 1.3.2 (Stone-Weierstraß)**.** *If $\mathcal{A} \subseteq \mathcal{C}(X)$ is a closed $*$-subalgebra that separates points of X, then $\mathcal{A} = \mathcal{C}(X)$.*

Proof. By Exercise 2 (iii), it is enough to show that $\mathcal{A}$ contains all real-valued functions in $\mathcal{C}(X)$. Since $\mathcal{A}$ is closed, it suffices to show that every real-valued function $g \in \mathcal{C}(X)$ can be uniformly approximated by elements of $\mathcal{A}$.

Let such a function g and some $\varepsilon > 0$ be given. We first construct, for each $x \in X$, a real-valued function $f_x \in \mathcal{A}$ such that

$$f_x(x) = g(x) \quad \text{and} \quad g - \varepsilon \leqslant f_x \quad \text{on } X.$$

For this purpose, fix $x \in X$. For every $y \in X$, choose a real-valued function $f_{xy} \in \mathcal{A}$ with $f_{xy}(x) = g(x)$ and $f_{xy}(y) = g(y)$. If $g(x) = g(y)$, we can take a constant function. Otherwise, we choose an element of $\mathcal{A}$ separating x and y, and obtain such an f_{xy} by scaling and adding a constant. Here we use again Exercise 2 (iii), which ensures that real-valued elements of $\mathcal{A}$ already separate points.

For every $y \in X$, there is a neighborhood U_y of y such that $g - \varepsilon \leqslant f_{xy}$ on U_y. The sets U_y cover X, so compactness gives a finite subcover

$$X = U_{y_1} \cup \cdots \cup U_{y_n}.$$

Then

$$f_x := \max\{f_{xy_1}, \ldots, f_{xy_n}\}$$

has the required properties and belongs to $\mathcal{A}$ by Lemma 1.3.1.

Now choose, for each $x \in X$, a neighborhood V_x of x such that $f_x \leqslant g + \varepsilon$ on V_x. Again, compactness gives a finite subcover. Taking the minimum of the corresponding finitely many functions f_x, we obtain a function $f \in \mathcal{A}$ satisfying

$$g - \varepsilon \leqslant f \leqslant g + \varepsilon$$

on X. Hence $\|f - g\|_\infty \leqslant \varepsilon$, as required. □

We next prove Burnside's theorem, a result of similar flavor for subalgebras of the noncommutative algebra $\mathrm{Mat}_d(\mathbb{C})$. For a subalgebra $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$, a subspace $V \subseteq \mathbb{C}^d$ is called **$\mathcal{A}$-invariant** if

$$Mv \in V$$

for all $v \in V$ and all $M \in \mathcal{A}$. The subspaces $\{0\}$ and $\mathbb{C}^d$ are called the *trivial invariant subspaces*.

Theorem 1.3.3 (Burnside). *If a subalgebra $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ has only the two trivial invariant subspaces, then $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$.*

Proof. First note that $\mathcal{A}$ acts transitively on nonzero vectors in $\mathbb{C}^d$: for every $0 \neq v \in \mathbb{C}^d$, the space

$$\{Mv \mid M \in \mathcal{A}\}$$

is a nonzero $\mathcal{A}$-invariant subspace, and hence must be all of $\mathbb{C}^d$.

We now show that $\mathcal{A}$ contains a matrix of rank 1. Let $0 \neq P \in \mathcal{A}$. If $\operatorname{rank}(P) \geqslant 2$, choose $v_1, v_2 \in \mathbb{C}^d$ such that Pv_1 and Pv_2 are linearly independent. By transitivity, choose $M \in \mathcal{A}$ such that $MPv_1 = v_2$. Then $PMPv_1$ and Pv_1 are linearly independent, so $PMP - \lambda P \neq 0$ for every $\lambda \in \mathbb{C}$.

On the other hand, since $\mathbb{C}$ is algebraically closed, the linear map induced by PM on the image of P has an eigenvalue λ_0. Thus

$$(PM - \lambda_0 I_d)P$$

has rank strictly smaller than P and is nonzero. Repeating this argument, we eventually obtain a matrix $Q \in \mathcal{A}$ of rank 1.

Once $\mathcal{A}$ contains one rank-one matrix, transitivity implies that it contains all rank-one matrices; the details are left to Exercise 4. Since every matrix is a sum of rank-one matrices, it follows that $\mathcal{A} = \operatorname{Mat}_d(\mathbb{C})$. □

Remark 1.3.4. Suppose now that $\mathcal{A} \subseteq \operatorname{Mat}_d(\mathbb{C})$ is a $*$-subalgebra. If $\mathcal{A}$ has a proper invariant subspace $V \subseteq \mathbb{C}^d$, then $V^\perp$ is also invariant, because $\mathcal{A}$ is closed under the involution (see Exercise 5). After a unitary change of basis, all matrices in $\mathcal{A}$ therefore have block diagonal form, so $\mathcal{A}$ is contained in an algebra of the form

$$\operatorname{Mat}_{d_1}(\mathbb{C}) \oplus \operatorname{Mat}_{d_2}(\mathbb{C}),$$

where $1 \leqslant d_1, d_2$ and $d_1 + d_2 = d$. If no proper invariant subspace exists, then Theorem 1.3.3 gives $\mathcal{A} = \operatorname{Mat}_d(\mathbb{C})$. Iterating this decomposition, we find that $\mathcal{A}$ is contained in

$$\operatorname{Mat}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \operatorname{Mat}_{d_r}(\mathbb{C}),$$

and that, for each factor, the projection of $\mathcal{A}$ onto $\operatorname{Mat}_{d_i}(\mathbb{C})$ is surjective. △

Example 1.3.5. Let $\mathcal{A} \subseteq \operatorname{Mat}_d(\mathbb{C})$ be a commutative $*$-subalgebra. By the preceding remark, we may assume after a unitary change of basis that

$$\mathcal{A} \subseteq \operatorname{Mat}_{d_1}(\mathbb{C}) \oplus \cdots \oplus \operatorname{Mat}_{d_r}(\mathbb{C}),$$

with $d_1 + \cdots + d_r = d$, and that the projection of $\mathcal{A}$ onto each block is surjective. Since the elements of $\mathcal{A}$ commute with one another, each full matrix block must be commutative. Hence $d_i = 1$ for all i. Thus every commutative $*$-subalgebra of $\operatorname{Mat}_d(\mathbb{C})$ is unitarily conjugate to a subalgebra of the diagonal matrices. △

1.4 States and Representations

From now on, $\mathcal{A}$ will always denote a $*$-algebra.

Definition 1.4.1. (i) A **state** on $\mathcal{A}$ is a $\mathbb{C}$-linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ satisfying

$$\varphi(1) = 1 \quad \text{and} \quad \varphi(a^*a) \geqslant 0$$

for all $a \in \mathcal{A}$. One often also requires $\varphi(a^*) = \overline{\varphi(a)}$, but this already follows from positivity on squares (cf. Exercise 3). In particular, the restriction $\varphi\colon \mathcal{A}_{\mathrm{sa}} \to \mathbb{R}$ is $\mathbb{R}$-linear. A **pure state** is a state that cannot be written as a nontrivial convex combination of two other states; equivalently, it is an extreme point of the convex set of all states.

(ii) A **bounded $*$-representation** of $\mathcal{A}$ is a $*$-algebra homomorphism

$$\pi\colon \mathcal{A} \to \mathcal{B}(H)$$

for some Hilbert space H. The representation is called **finite-dimensional** if H is finite-dimensional. After choosing a basis of H, this means that

$$\pi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C}).$$

(iii) An **unbounded $*$-representation** is an algebra homomorphism

$$\pi\colon \mathcal{A} \to \mathcal{L}(D)$$

into the algebra of linear operators on an inner product space D, such that

$$\langle \pi(a)v, w\rangle = \langle v, \pi(a^*)w\rangle$$

holds for all $v, w \in D$ and all $a \in \mathcal{A}$. △

Example 1.4.2. Every Borel probability measure μ on a compact Hausdorff space X gives rise to a state

$$\begin{aligned} \varphi_\mu\colon \mathcal{C}(X) &\to \mathbb{C} \\ f &\mapsto \int_X f\,\mathrm{d}\mu. \end{aligned}$$

We will see in Theorem 1.4.6 that all states on $\mathcal{C}(X)$ arise in this way. △

Example 1.4.3. Every linear functional on $\mathrm{Mat}_d(\mathbb{C})$ is of the form

$$\begin{aligned} \varphi_Q \colon \mathrm{Mat}_d(\mathbb{C}) &\to \mathbb{C} \\ M &\mapsto \mathrm{tr}(Q^* M) \end{aligned}$$

for some $Q \in \mathrm{Mat}_d(\mathbb{C})$. This follows from finite-dimensionality and the inner product $\langle A, B \rangle = \mathrm{tr}(B^* A)$ on $\mathrm{Mat}_d(\mathbb{C})$.

The functional φ_Q is nonnegative on squares if and only if Q is positive semidefinite, using the self-duality of the positive semidefinite cone. Moreover, $\varphi_Q(I_d) = 1$ is equivalent to $\mathrm{tr}(Q) = 1$. Thus states on $\mathrm{Mat}_d(\mathbb{C})$ are in one-to-one correspondence with positive semidefinite matrices of trace one.

Every such matrix can be written as

$$Q = \sum_{i=1}^{d} \lambda_i v_i v_i^*$$

with orthonormal vectors $v_i \in \mathbb{C}^d$, coefficients $\lambda_i \geqslant 0$, and $\sum_i \lambda_i = 1$. Hence φ_Q is pure if and only if $Q = vv^*$ has rank one. Thus pure states correspond to unit vectors in $\mathbb{C}^d$, modulo multiplication by a complex number of absolute value one, called a *phase*. △

Example 1.4.4. For every Hilbert space H and every unit vector $h \in H$, we obtain a state

$$\varphi_h \colon \mathcal{B}(H) \to \mathbb{C}; \quad L \mapsto \langle Lh, h \rangle.$$ △

Example 1.4.5. On a group algebra $\mathbb{C}\Gamma$, two basic examples of states are

$$\begin{aligned} \varphi_1\left(\sum_g c_g g\right) &:= \sum_g c_g, \\ \varphi_2\left(\sum_g c_g g\right) &:= c_e. \end{aligned}$$

The verification that these are states is left to Exercise 6. △

Theorem 1.4.6 (Riesz–Markov–Kakutani). *If X is a compact Hausdorff space, then for every state φ on $C(X)$ there exists a unique regular Borel probability measure μ on X such that $\varphi = \varphi_\mu$. The pure states correspond to Dirac measures, and hence to points of X.*

Sketch of Proof. Let $\varphi\colon \mathcal{C}(X) \to \mathbb{C}$ be a state. For an open subset $U \subseteq X$, denote by $\mathbf{1}_U$ its characteristic function, which is usually not continuous. Define

$$\mu(U) := \sup\{\varphi(f) \mid f \in \mathcal{C}(X),\ 0 \leqslant f \leqslant \mathbf{1}_U \text{ on } X\}.$$

This set function extends to a probability measure on the Borel σ-algebra of X, and this measure is the desired one. □

Example 1.4.7. (i) Let $\mathcal{A}$ be a commutative $*$-algebra and let $\pi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$ be a finite-dimensional $*$-representation. By Example 1.3.5, after unitary conjugation we may assume that

$$\pi(\mathcal{A}) \subseteq \mathrm{Mat}_1(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_1(\mathbb{C}).$$

Thus π is given by a d-tuple of $*$-algebra homomorphisms $\pi_i\colon \mathcal{A} \to \mathbb{C}$.

(ii) Let $\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]$ with the involution from above. The finite-dimensional $*$-representations of $\mathbb{C}[x_1, \ldots, x_n]$ are, after a change of basis, direct sums of evaluations at points of $\mathbb{R}^n$. △

Example 1.4.8. The $*$-representations of a group algebra $\mathbb{C}\Gamma$ on a Hilbert space H correspond exactly to unitary representations of Γ, i.e. group homomorphisms $\Gamma \to \mathcal{U}(H)$. Given a $*$-representation $\pi\colon \mathbb{C}\Gamma \to \mathcal{B}(H)$, one obtains a unitary representation

$$\tilde{\pi}\colon \Gamma \to \mathcal{U}(H); \quad g \mapsto \pi(g).$$

Conversely, a unitary representation $\tilde{\pi}\colon \Gamma \to \mathcal{U}(H)$ gives a $*$-representation

$$\pi\colon \mathbb{C}\Gamma \to \mathcal{B}(H); \quad \sum_g c_g g \mapsto \sum_g c_g \tilde{\pi}(g). \qquad \triangle$$

The following result says that matrix algebras have only the expected finite-dimensional representations.

Theorem 1.4.9. *If $\pi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ is a $*$-representation, then e is a multiple of d, and after unitary conjugation in $\mathrm{Mat}_e(\mathbb{C})$, the representation π is a direct sum of copies of the identity representation.*

Proof. The image of π is a $*$-subalgebra of $\mathrm{Mat}_e(\mathbb{C})$. By Theorem 1.3.3 and the following remark, after a unitary change of basis it is contained in a direct sum of matrix algebras, and its projection onto each block is surjective. Composing π with such a projection gives a surjective $*$-algebra homomorphism

$$\pi_i\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_{e_i}(\mathbb{C}).$$

Since $\mathrm{Mat}_d(\mathbb{C})$ is simple, the kernel of π_i is either zero or all of $\mathrm{Mat}_d(\mathbb{C})$. Surjectivity excludes the latter, so π_i is injective. By dimension count, $d = e_i$ for every i. Finally, every $*$-automorphism of $\mathrm{Mat}_d(\mathbb{C})$ is of the form

$$M \mapsto U^* M U$$

for some unitary matrix U. After applying the corresponding unitary conjugation on each block, every π_i becomes the identity representation. This proves the claim. $\square$

Remark 1.4.10. (i) Let $a \in \sum \mathcal{A}^2$ and let π be a $*$-representation of $\mathcal{A}$. Then $\pi(a)$ is positive semidefinite. Indeed, if $a = \sum_i a_i^* a_i$, then

$$\langle \pi(a)v, v\rangle = \sum_i \langle \pi(a_i)^*\pi(a_i)v, v\rangle = \sum_i \langle \pi(a_i)v, \pi(a_i)v\rangle = \sum_i \|\pi(a_i)v\|^2 \geqslant 0.$$

(ii) Every $*$-representation π of $\mathcal{A}$ gives rise to many states on $\mathcal{A}$. If h is a unit vector in the representation space, define

$$\varphi(a) := \langle \pi(a)h, h\rangle.$$

The properties of a state follow directly. $\triangle$

The important construction of Gelfand, Naimark, and Segal, known as the **GNS construction**, provides a converse to the previous remark. Starting from a state $\varphi\colon \mathcal{A} \to \mathbb{C}$, it constructs a representation

$$\pi_\varphi\colon \mathcal{A} \to \mathcal{L}(D)$$

and a unit vector $h \in D$ such that

$$\varphi(a) = \langle \pi_\varphi(a)h, h\rangle$$

for all $a \in \mathcal{A}$. Before explaining the construction, we need two lemmas.

Lemma 1.4.11 (Cauchy–Schwarz Inequality). *Let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a state. Then for all $a, b \in \mathcal{A}$,*

$$|\varphi(b^*a)|^2 \leqslant \varphi(b^*b)\varphi(a^*a).$$

Proof. The Hermitian matrix

$$M := \begin{pmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{pmatrix} \in \mathrm{Her}_2(\mathbb{C})$$

is positive semidefinite. Indeed, for $v = (v_1, v_2)^t \in \mathbb{C}^2$,

$$v^*Mv = \varphi\left((v_1a + v_2b)^*(v_1a + v_2b)\right) \geqslant 0.$$

Hence M has nonnegative determinant, and

$$\det(M) = \varphi(a^*a)\varphi(b^*b) - \varphi(a^*b)\varphi(b^*a) = \varphi(a^*a)\varphi(b^*b) - |\varphi(b^*a)|^2.$$

This proves the claim. □

Lemma 1.4.12. *Let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a state. Then*

$$\mathcal{N}_\varphi := \{a \in \mathcal{A} \mid \varphi(a^*a) = 0\}$$

is a proper left ideal in $\mathcal{A}$.

Proof. See Exercise 7. □

Construction 1.4.13 (GNS Construction)**.** Let φ be a state on $\mathcal{A}$. First equip the vector space $\mathcal{A}$ with the sesquilinear form

$$\langle a, b\rangle_\varphi := \varphi(b^*a).$$

This form is positive semidefinite. To obtain a positive definite inner product, we factor out its null space $\mathcal{N}_\varphi$. Thus, on

$$D := \mathcal{A}/\mathcal{N}_\varphi,$$

the form $\langle\cdot,\cdot\rangle_\varphi$ becomes a well-defined inner product.

Since $\mathcal{N}_\varphi$ is a left ideal, left multiplication by elements of $\mathcal{A}$ is well-defined on D. Thus each $a \in \mathcal{A}$ defines a linear operator

$$m_a\colon D \to D; \quad d \mapsto ad.$$

In this way we obtain a $*$-representation

$$\begin{aligned}\pi_\varphi\colon \mathcal{A} &\to \mathcal{L}(D)\\ a &\mapsto m_a.\end{aligned}$$

If $h \in D$ is the equivalence class of 1, then $\|h\|_\varphi = 1$ and

$$\langle\pi_\varphi(a)h, h\rangle_\varphi = \varphi(1^*a1) = \varphi(a)$$

for all $a \in \mathcal{A}$. This proves the assertion. The remaining details are left to Exercise 8. △

Definition 1.4.14. We call $\mathcal{A}$ *Archimedean* if for every $a \in \mathcal{A}_{\mathrm{sa}}$ there exists $r > 0$ such that

$$r - a \in \sum \mathcal{A}^2.$$

After dividing by r, this is equivalent to saying that for every $a \in \mathcal{A}_{\mathrm{sa}}$ there exists $\varepsilon > 0$ with

$$1 - \varepsilon a \in \sum \mathcal{A}^2.$$

Equivalently, 1 is an *algebraic interior point* of the cone $\sum \mathcal{A}^2$ inside the real vector space $\mathcal{A}_{\mathrm{sa}}$. This means that one can move a small distance from 1 in any self-adjoint direction without leaving the cone of sums of squares. △

Example 1.4.15. (i) For a compact Hausdorff space X, the $*$-algebra $\mathcal{C}(X)$ is Archimedean.

(ii) For every Hilbert space H, the $*$-algebra $\mathcal{B}(H)$ is Archimedean. In particular, $\mathrm{Mat}_d(\mathbb{C})$ is Archimedean.

(iii) The polynomial algebras $\mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ are both not Archimedean. △

Proposition 1.4.16. *Every group algebra $\mathbb{C}\Gamma$ is Archimedean.*

Proof. For $a = \sum_g c_g g \in \mathbb{C}\Gamma$, set

$$\|a\|_1 := \sum_g |c_g|.$$

Omitting terms with zero coefficients, one checks the identity

$$\|a\|_1^2 - a^*a = \frac{1}{2} \sum_{g,h\in\Gamma} |c_g c_h| \left(1 - \frac{c_g\overline{c_h}}{|c_g c_h|} h^{-1} g\right)^* \left(1 - \frac{c_g\overline{c_h}}{|c_g c_h|} h^{-1} g\right).$$

Hence $\|a\|_1^2 - a^*a \in \sum \mathbb{C}\Gamma^2$. If $a \in \mathbb{C}\Gamma_{\mathrm{sa}}$ and $a \neq 0$, then for $r = \|a\|_1$ we obtain

$$r - a = \frac{1}{2r}\left((r-a)^*(r-a) + (r^2 - a^*a)\right) \in \sum \mathbb{C}\Gamma^2. \qquad \square$$

Proposition 1.4.17. *If $\mathcal{A}$ is Archimedean, then the GNS construction associated to any state φ on $\mathcal{A}$ yields a bounded $*$-representation.*

Proof. Let φ be a state on $\mathcal{A}$, and let π_φ be the corresponding GNS representation on $D = \mathcal{A}/\mathcal{N}_\varphi$. For $a \in \mathcal{A}$, Archimedeanicity applied to a^*a gives some $r > 0$ such that

$$r - a^*a \in \sum \mathcal{A}^2.$$

Every vector $v \in D$ is represented by some $b \in \mathcal{A}$, and

$$b^*(r - a^*a)b \in \sum \mathcal{A}^2.$$

Therefore

$$\|\pi_\varphi(a)v\|_\varphi^2 = \langle \pi_\varphi(a)v, \pi_\varphi(a)v\rangle_\varphi = \varphi(b^*a^*ab) \leqslant \varphi(rb^*b) = r\|v\|_\varphi^2,$$

where the inequality follows from positivity of φ on $\sum \mathcal{A}^2$. Thus $\pi_\varphi(a)$ is bounded on D, with operator norm at most $\sqrt{r}$, and extends uniquely to the Hilbert space completion H of D. Hence

$$\pi_\varphi \colon \mathcal{A} \to \mathcal{B}(H)$$

can be regarded as a bounded $*$-representation; see Exercise 9 for the remaining details. □

We can now show that Archimedean $*$-algebras have bounded $*$-representations that are as injective as possible.

Theorem 1.4.18 (Representation Theorem for Archimedean $*$-Algebras). *For every Archimedean $*$-algebra $\mathcal{A}$ there exists a bounded $*$-representation*

$$\pi \colon \mathcal{A} \to \mathcal{B}(H)$$

with

$$\operatorname{Ker}(\pi) = \{a \in \mathcal{A} \mid \varphi(a^*a) = 0 \textit{ for all states } \varphi\}.$$

If $\mathcal{A}$ is finite-dimensional, then H can be chosen finite-dimensional.

Proof. For every state φ on $\mathcal{A}$, let $\pi_\varphi \colon \mathcal{A} \to \mathcal{B}(H_\varphi)$ be the bounded $*$-representation obtained from the GNS construction. For each fixed $a \in \mathcal{A}$, the previous proposition gives a bound on $\|\pi_\varphi(a)\|_{\mathrm{op}}$ that is independent of φ. Hence, by Definition 1.1.4, we may form the direct sum representation

$$\pi \colon \mathcal{A} \to \mathcal{B}\left(\bigoplus_\varphi H_\varphi\right)$$
$$a \mapsto \oplus_\varphi \pi_\varphi(a).$$

For every state $\tilde{\varphi}$ and every $a \in \mathcal{A}$, we have

$$\|\pi(a)h_{\tilde{\varphi}}\|^2 = \langle \pi_{\tilde{\varphi}}(a^*a)h_{\tilde{\varphi}}, h_{\tilde{\varphi}}\rangle = \tilde{\varphi}(a^*a),$$

where $h_{\tilde{\varphi}} \in H_{\tilde{\varphi}} \subseteq \bigoplus_{\varphi} H_{\varphi}$ is the distinguished unit vector from the GNS construction. This gives the asserted description of the kernel.

If $\mathcal{A}$ is finite-dimensional, it is enough to take a finite set of states spanning the vector space generated by all states. For such states, each quotient $\mathcal{A}/\mathcal{N}_{\varphi}$ is finite-dimensional and therefore already complete. Thus the resulting Hilbert space is a finite direct sum of finite-dimensional spaces. $\square$

Chapter 2

C^*-Algebras

2.1 Definitions and First Properties

Definition 2.1.1. A **Banach algebra** is a complex unital algebra $\mathcal{A}$ equipped with a norm such that

(i) $(\mathcal{A}, \|\cdot\|)$ is complete,

(ii) $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$.

If $\mathcal{A}$ is also equipped with an involution $*$ satisfying

(iii) $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$,

then $\mathcal{A}$ is called a **C^*-algebra**. △

Remark 2.1.2. Condition (i) says that $\mathcal{A}$ is a Banach space, while (ii) expresses compatibility between the norm and multiplication. In particular, multiplication is continuous. Condition (iii), called the *C^*-property*, imposes a strong compatibility between the norm and the involution. For example, it implies $\|a\| = \|a^*\|$ for all $a \in \mathcal{A}$; see Exercise 10. △

Example 2.1.3. (i) If X is a compact Hausdorff space, then the $*$-algebra $\mathcal{C}(X)$, equipped with the sup-norm from Example 1.1.2 (iii), is a commutative C^*-algebra.
(ii) If H is a Hilbert space, then $\mathcal{B}(H)$, equipped with the operator norm, is a C^*-algebra. This includes the finite-dimensional case $\mathrm{Mat}_d(\mathbb{C})$.
(iii) If $\mathcal{A}$ is a C^*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ is a closed $*$-subalgebra, then $\mathcal{B}$ is again a C^*-algebra. The closedness assumption ensures that $\mathcal{B}$ is complete. △

Definition 2.1.4. Let $\mathcal{A}$ be a Banach algebra. The **spectrum** of an element $a \in \mathcal{A}$ is

$$\sigma(a) \coloneqq \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathcal{A}^\times\}.$$

Its complement

$$\rho(a) \coloneqq \mathbb{C} \setminus \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \mathcal{A}^\times\}$$

is called the **resolvent set** of a. △

Example 2.1.5. (i) An element $f \in \mathcal{C}(X)$ is invertible if and only if it has no zero on X. Hence the spectrum $\sigma(f)$ is precisely the set of values $f(X)$.
(ii) For $M \in \mathrm{Mat}_d(\mathbb{C})$, the spectrum $\sigma(M)$ coincides with the usual spectrum from linear algebra, i.e. the set of complex eigenvalues of M. △

The next proposition contains most of the analytic machinery about Banach algebras that we will need.

Proposition 2.1.6. *Let $\mathcal{A}$ be a Banach algebra and let $a, b \in \mathcal{A}$.*

- (i) *If $\|a\| < 1$, then $1 - a \in \mathcal{A}^\times$.*
- (ii) *The group of units $\mathcal{A}^\times$ is an open subset of $\mathcal{A}$.*
- (iii) *The spectrum $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$, contained in the closed disk of radius $\|a\|$.*
- (iv) *We have*
$$\max\{|\lambda| \mid \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n}.$$
- (v) $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$.

Proof. (i) Define $b_n \coloneqq \sum_{i=0}^n a^i$. Since $\|a\| < 1$, the sequence $(b_n)_n$ is Cauchy and hence converges to some $b \in \mathcal{A}$. Moreover,

$$(1-a)b_n = b_n(1-a) = 1 - a^{n+1} \overset{n\to\infty}{\longrightarrow} 1.$$

By continuity of multiplication, $(1-a)b = b(1-a) = 1$. Thus $1-a$ is invertible.
(ii) Let $e \in \mathcal{A}^\times$. If $a \in \mathcal{A}$ satisfies $\|e - a\| < \|e^{-1}\|^{-1}$, then

$$a = e\left(1 - e^{-1}(e-a)\right).$$

Since $\|e^{-1}(e-a)\| < 1$, part (*i*) shows that $1 - e^{-1}(e-a)$ is invertible, and hence so is a. Thus $\mathcal{A}^\times$ contains an open ball around each of its elements.

(*iii*) Since $\mathcal{A}^\times$ is open, the resolvent set $\rho(a)$ is open, and therefore $\sigma(a)$ is closed. If $|\lambda| > \|a\|$, then

$$\lambda - a = \lambda(1 - \lambda^{-1}a) \in \mathcal{A}^\times \tag{2.1}$$

by part (*i*). Hence $\sigma(a)$ is contained in the closed disk of radius $\|a\|$, and is compact.

It remains to prove nonemptiness. Fix $\lambda_0 \in \rho(a)$. For λ sufficiently close to λ_0, the Neumann series gives

$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 - a)^{-(n+1)}.$$

Hence, for every bounded linear functional χ on $\mathcal{A}$, the scalar-valued function

$$r_\chi(\lambda) \coloneqq \chi((\lambda - a)^{-1})$$

is holomorphic on $\rho(a)$. For $|\lambda| > \|a\|$, using (2.1), we have

$$r_\chi(\lambda) = \chi((\lambda - a)^{-1}) = \lambda^{-1} \sum_{n=0}^{\infty} \frac{\chi(a^n)}{\lambda^n}, \tag{2.2}$$

and therefore

$$|r_\chi(\lambda)| \leqslant \frac{\|\chi\|_{\mathrm{op}}}{|\lambda| - \|a\|}.$$

In particular, $r_\chi(\lambda) \to 0$ as $|\lambda| \to \infty$.

If $\sigma(a)$ were empty, then r_χ would be a bounded entire function and hence constant by Liouville's theorem. Since it tends to zero at infinity, it would be identically zero. Thus $\chi((\lambda - a)^{-1}) = 0$ for all bounded linear functionals χ, which forces $(\lambda - a)^{-1} = 0$, a contradiction. Hence $\sigma(a)$ is nonempty.

(*iv*) First, the sequence $(\|a^n\|^{1/n})_n$ converges by Exercise 11. Let

$$r \coloneqq \max\{|\lambda| \mid \lambda \in \sigma(a)\}.$$

Equation (2.2) extends to every λ with $|\lambda| > r$, because the resolvent is defined and holomorphic there. It follows that

$$\chi\left(\left(\frac{a}{\lambda}\right)^n\right) \xrightarrow{n\to\infty} 0$$

for every bounded linear functional χ. Thus $(a/\lambda)^n$ converges weakly to zero and is therefore bounded in norm by Exercise 1. Hence there is a constant c such that

$$\|a^n\|^{1/n} \leqslant c^{1/n}|\lambda|$$

for all n. Letting $n \to \infty$ and then $|\lambda| \downarrow r$ gives

$$\lim_{n\to\infty} \|a^n\|^{1/n} \leqslant r.$$

Conversely, if $|\lambda| > \lim_{n\to\infty} \|a^n\|^{1/n}$, then

$$\lim_{n\to\infty} \left\| \left(\frac{a}{\lambda}\right)^n \right\|^{1/n} < 1.$$

By the root test, the series $\sum_{n=0}^{\infty}(a/\lambda)^n$ converges in $\mathcal{A}$, and

$$\lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n = (\lambda - a)^{-1}.$$

Hence $\lambda \in \rho(a)$. Therefore every $\lambda \in \sigma(a)$ satisfies

$$|\lambda| \leqslant \lim_{n\to\infty} \|a^n\|^{1/n},$$

which proves the formula.

(v) We first show that $1 - ab$ is invertible if and only if $1 - ba$ is invertible. Assume $1 - ab$ is invertible. Then

$$\begin{aligned}
(1 - ba)\left(b(1 - ab)^{-1}a + 1\right) &= b(1 - ab)^{-1}a + 1 - bab(1 - ab)^{-1}a - ba \\
&= b\left((1 - ab)^{-1} - ab(1 - ab)^{-1}\right)a + 1 - ba \\
&= b\left((1 - ab)(1 - ab)^{-1}\right)a + 1 - ba \\
&= 1.
\end{aligned}$$

The same computation in the other order shows that $b(1 - ab)^{-1}a + 1$ is also a left inverse, so $1 - ba$ is invertible. The converse follows by symmetry. Applying this to $\lambda^{-1}a$ and b proves the statement for all $0 \neq \lambda \in \mathbb{C}$. $\square$

Corollary 2.1.7 (Gelfand–Mazur). *Let $\mathcal{A}$ be a Banach algebra with $\mathcal{A}^\times = \mathcal{A} \setminus \{0\}$. Then $\mathcal{A} = \mathbb{C}$.*

Proof. If there were an element $a \in \mathcal{A} \setminus \mathbb{C}$, then $\lambda - a \neq 0$ for every $\lambda \in \mathbb{C}$. By assumption, each $\lambda - a$ would be invertible, so $\sigma(a)$ would be empty. This contradicts Proposition 2.1.6 (iii). ☐

Corollary 2.1.8. *If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$ is normal, i.e. $a^*a = aa^*$, then*

$$\|a\| = \lim_{n\to\infty} \|a^n\|^{1/n} = \max\{|\lambda| \mid \lambda \in \sigma(a)\}.$$

If $a \in \mathcal{A}_{\mathrm{sa}}$, then $\sigma(a) \subseteq \mathbb{R}$.

Proof. The second equality is Proposition 2.1.6 (iv), so we only need to prove the first. Since a is normal,

$$\|a^2\|^2 = \|(a^2)^*a^2\| = \|(a^*a)^2\| = \|a^*a\|^2 = \|a\|^4.$$

Here the first, third, and fourth equalities use the C^*-property, while the second uses normality. Hence $\|a\| = \|a^2\|^{1/2}$, and by induction

$$\|a\| = \|a^{2^k}\|^{1/2^k}$$

for every $k \geqslant 1$. Therefore

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{k\to\infty} \|a^{2^k}\|^{1/2^k} = \|a\|.$$

Now assume $a = a^*$. For every $\gamma \in \mathbb{R}$, we have

$$\sigma(\gamma i + a) = \gamma i + \sigma(a).$$

If $\lambda \in \sigma(a)$, then

$$|\gamma i + \lambda|^2 \leqslant \|\gamma i + a\|^2 = \|(\gamma i + a)^*(\gamma i + a)\| = \|\gamma^2 + a^*a\| \leqslant \gamma^2 + \|a^*a\|.$$

Writing $\lambda = x + iy$, this gives

$$y^2 + 2\gamma y + x^2 \leqslant \|a^*a\|$$

for all $\gamma \in \mathbb{R}$. This is possible only if $y = 0$. Hence $\lambda \in \mathbb{R}$. ☐

2.2 Classification of Commutative C^*-Algebras

In the previous section we saw that $\mathcal{C}(X)$ is a commutative C^*-algebra whenever X is a compact Hausdorff space. The goal of this section is to prove the converse: every commutative C^*-algebra is of this form. Throughout the section, $\mathcal{A}$ will denote a commutative Banach algebra, and later a commutative C^*-algebra.

Definition 2.2.1. Let $\mathcal{A}$ be a commutative Banach algebra.
(i) The **Gelfand space** of $\mathcal{A}$ is

$$X_{\mathcal{A}} \coloneqq \{\xi \mid \xi \colon \mathcal{A} \to \mathbb{C} \text{ is an algebra homomorphism}\}.$$

(ii) Each element $a \in \mathcal{A}$ defines a complex-valued function $\hat{a}$ on $X_{\mathcal{A}}$ by

$$\hat{a}(\xi) \coloneqq \xi(a).$$

We equip $X_{\mathcal{A}}$ with the coarsest topology for which all functions $\hat{a}$ are continuous. △

Lemma 2.2.2. *Every algebra homomorphism* $\xi \colon \mathcal{A} \to \mathbb{C}$ *is bounded with* $\|\xi\|_{\mathrm{op}} \leqslant 1$.

Proof. Suppose there were an element $a \in \mathcal{A}$ with $\|a\| < 1$ and $|\xi(a)| = 1$. By Proposition 2.1.6 (i), the element $1 - \xi(a)^{-1}a$ is invertible. Applying ξ gives

$$0 \neq \xi(1 - \xi(a)^{-1}a) = 1 - \xi(a)^{-1}\xi(a) = 0,$$

a contradiction. Hence $|\xi(a)| < 1$ whenever $\|a\| < 1$, and therefore $\|\xi\|_{\mathrm{op}} \leqslant 1$. □

Theorem 2.2.3. *For every commutative Banach algebra* $\mathcal{A}$*, the Gelfand space* $X_{\mathcal{A}}$ *is a compact Hausdorff space. The map*

$$\begin{aligned} \Gamma \colon \mathcal{A} &\to \mathcal{C}(X_{\mathcal{A}}) \\ a &\mapsto \hat{a} \end{aligned}$$

is an algebra homomorphism with $\|\Gamma\|_{\mathrm{op}} \leqslant 1$*. Moreover, for every* $a \in \mathcal{A}$*,*

$$\sigma(a) = \{\hat{a}(\xi) \mid \xi \in X_{\mathcal{A}}\}.$$

Proof. By Lemma 2.2.2, the space $X_{\mathcal{A}}$ is contained in the unit ball of the dual space $\mathcal{A}'$. This unit ball is compact in the weak*-topology by Banach–Alaoglu. The conditions

$$\hat{1}(\xi) = 1 \quad \text{and} \quad \widehat{ab}(\xi) = \hat{a}(\xi)\hat{b}(\xi)$$

for all $a, b \in \mathcal{A}$ are closed conditions in the weak*-topology. Hence $X_{\mathcal{A}}$ is a closed subspace of a compact Hausdorff space, and is therefore compact Hausdorff. The map $a \mapsto \hat{a}$ is an algebra homomorphism because every $\xi \in X_{\mathcal{A}}$ is one.

We next prove the spectral statement. If $\lambda \notin \sigma(a)$, then $\lambda - a \in \mathcal{A}^\times$. Hence for every $\xi \in X_{\mathcal{A}}$,

$$0 \neq \xi(\lambda - a) = \lambda - \xi(a),$$

so $\lambda \neq \hat{a}(\xi)$. This shows

$$\{\hat{a}(\xi) \mid \xi \in X_{\mathcal{A}}\} \subseteq \sigma(a).$$

Conversely, let $\lambda \in \sigma(a)$. Then $\lambda - a$ is not invertible, so the ideal generated by $\lambda - a$ is proper. By Zorn's Lemma, it is contained in a maximal ideal $\mathcal{M}$ of $\mathcal{A}$. Since the closure of a proper ideal is again proper, using openness of $\mathcal{A}^\times$ from Proposition 2.1.6 (*ii*), the maximal ideal $\mathcal{M}$ is closed. Thus $\mathcal{A}/\mathcal{M}$ is a Banach algebra in which every nonzero element is invertible. By the Gelfand–Mazur Theorem, Corollary 2.1.7, we have $\mathcal{A}/\mathcal{M} \cong \mathbb{C}$. The quotient map

$$\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{M} \cong \mathbb{C}$$

is therefore an element of $X_{\mathcal{A}}$. Since $\lambda - a \in \mathcal{M}$, we get

$$0 = \pi(\lambda - a) = \lambda - \pi(a),$$

and hence $\hat{a}(\pi) = \lambda$.

It remains to show that $\|\Gamma\|_{\mathrm{op}} \leqslant 1$. If $|\lambda| > \|a\|$, then $\lambda \notin \sigma(a)$ by Proposition 2.1.6 (*iii*). Using the spectral identity just proved, this implies $|\hat{a}(\xi)| \leqslant \|a\|$ for all $\xi \in X_{\mathcal{A}}$. Hence $\|\hat{a}\|_\infty \leqslant \|a\|$. □

We now obtain the promised classification theorem for commutative C^*-algebras.

Theorem 2.2.4. *If $\mathcal{A}$ is a commutative C^*-algebra, then the Gelfand transform*

$$\Gamma\colon \mathcal{A} \to \mathcal{C}(X_{\mathcal{A}})$$

is an isomorphism of C^*-algebras, i.e. an isometric $*$-algebra isomorphism.

Proof. Since $\mathcal{A}$ is commutative, every element of $\mathcal{A}$ is normal. Combining Corollary 2.1.8 with the spectral formula from Theorem 2.2.3, we obtain

$$\|a\| = \max\{|\lambda| \mid \lambda \in \sigma(a)\} = \|\hat{a}\|_\infty$$

for every $a \in \mathcal{A}$. Thus Γ is isometric, and in particular injective.

Next let $a \in \mathcal{A}_{\mathrm{sa}}$. By Corollary 2.1.8, $\sigma(a) \subseteq \mathbb{R}$. Hence $\hat{a}$ is real-valued, so $\hat{a} \in \mathcal{C}(X_\mathcal{A})_{\mathrm{sa}}$. Since every element of $\mathcal{A}$ can be written as $a + ib$ with $a, b \in \mathcal{A}_{\mathrm{sa}}$ (see Exercise 2), it follows that Γ is compatible with the involutions.

Therefore $\Gamma(\mathcal{A})$ is a unital $*$-subalgebra of $\mathcal{C}(X_\mathcal{A})$, isometrically $*$-isomorphic to $\mathcal{A}$. It separates points of $X_\mathcal{A}$ by definition of the topology and of the functions $\hat{a}$. By the Stone-Weierstraß theorem, $\Gamma(\mathcal{A})$ is dense in $\mathcal{C}(X_\mathcal{A})$. Since $\Gamma(\mathcal{A})$ is complete, it is closed, and hence $\Gamma(\mathcal{A}) = \mathcal{C}(X_\mathcal{A})$. □

Corollary 2.2.5. *The category of commutative C^*-algebras with $*$-algebra homomorphisms is equivalent, contravariantly, to the category of compact Hausdorff spaces with continuous maps.* □

Remark 2.2.6. In view of Corollary 2.2.5, studying commutative C^*-algebras is equivalent to studying compact Hausdorff spaces. By Theorem 1.4.6, states on commutative C^*-algebras correspond to probability measures on these spaces.

The study of noncommutative C^*-algebras is therefore sometimes called *noncommutative topology*, and the study of their states *noncommutative probability theory*. Since C^*-algebras and their states also form a standard model for quantum theory, as we will explain in Section 4.1, one can view quantum theory as a form of noncommutative probability theory. △

2.3 Functional Calculus for Normal Elements

From now on, $\mathcal{A}$ denotes a C^*-algebra, not necessarily commutative.

Lemma 2.3.1. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a C^*-subalgebra, and let $b \in \mathcal{B}$ be normal. Then*

$$b \in \mathcal{A}^\times \iff b \in \mathcal{B}^\times.$$

In particular, the spectrum of b is independent of the ambient C^*-algebra in which it is computed.

Proof. The implication "$\Leftarrow$" is immediate. For the converse, assume for contradiction that $b \in \mathcal{A}^\times$ but $b \notin \mathcal{B}^\times$. Replacing $\mathcal{B}$ by the C^*-subalgebra generated by b, we may assume that $\mathcal{B}$ is commutative, because b is normal.

By Theorem 2.2.4, we may identify $\mathcal{B}$ with $\mathcal{C}(X_\mathcal{B})$ via the Gelfand transform. Since b is not invertible in $\mathcal{B}$, we have $0 \in \sigma_\mathcal{B}(b)$. Equivalently, by Theorem 2.2.3, the function $\hat{b}$ has a zero at some point $\xi \in X_\mathcal{B}$.

Choose a continuous function $f \in \mathcal{C}(X_\mathcal{B})$ such that

$$|f(\xi)| > \|b^{-1}\|_\mathcal{A} \quad \text{and} \quad |f\hat{b}| \leqslant 1 \quad \text{on } X_\mathcal{B}.$$

This is possible because $\hat{b}(\xi) = 0$. Then

$$\begin{aligned} \|b^{-1}\|_\mathcal{A} < \|f\|_\infty = \|\Gamma^{-1}(f)\|_\mathcal{B} &= \|\Gamma^{-1}(f)\|_\mathcal{A} \\ &= \|\Gamma^{-1}(f)bb^{-1}\|_\mathcal{A} \\ &\leqslant \|\Gamma^{-1}(f)b\|_\mathcal{A}\|b^{-1}\|_\mathcal{A} \\ &= \|\Gamma^{-1}(f)b\|_\mathcal{B}\|b^{-1}\|_\mathcal{A} \\ &= \|f\hat{b}\|_\infty\|b^{-1}\|_\mathcal{A} \leqslant \|b^{-1}\|_\mathcal{A}, \end{aligned}$$

a contradiction. Hence b is invertible in $\mathcal{B}$. □

Theorem 2.3.2 (Functional Calculus for Normal Elements). *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$ be normal. Then there is an isometric $*$-algebra homomorphism*

$$\Gamma_a \colon \mathcal{C}(\sigma(a)) \to \mathcal{A}$$

such that $\Gamma_a(\mathrm{id}_{\sigma(a)}) = a$.

Proof. Let $\mathcal{B}$ be the C^*-subalgebra of $\mathcal{A}$ generated by a. Since a is normal, $\mathcal{B}$ is commutative. By Theorem 2.2.4, the Gelfand transform gives an isometric $*$-isomorphism

$$\Gamma \colon \mathcal{B} \to \mathcal{C}(X_\mathcal{B}).$$

Consider the continuous function

$$\hat{a} \colon X_\mathcal{B} \to \sigma(a); \quad \xi \mapsto \xi(a).$$

By Lemma 2.3.1, the spectrum of a is the same whether computed in $\mathcal{A}$ or in $\mathcal{B}$. By Theorem 2.2.3, the image of $\hat{a}$ is exactly $\sigma(a)$, so $\hat{a}$ is surjective.

We claim that $\hat{a}$ is also injective. Indeed, if two characters on $\mathcal{B}$ agree on a, then they also agree on a^*, since characters are $*$-homomorphisms in the commutative

C^*-algebra $\mathcal{B}$. As $\mathcal{B}$ is generated by a and a^*, the two characters agree on a dense $*$-subalgebra of $\mathcal{B}$, and by continuity they agree on all of $\mathcal{B}$. Thus $\hat{a}$ is injective.

Since $X_{\mathcal{B}}$ is compact and $\sigma(a)$ is Hausdorff, the continuous bijection $\hat{a}\colon X_{\mathcal{B}} \to \sigma(a)$ is a homeomorphism. Therefore pullback along $\hat{a}$ gives an isometric $*$-isomorphism

$$\mathcal{C}(\sigma(a)) \cong \mathcal{C}(X_{\mathcal{B}}).$$

Composing this isomorphism with $\Gamma^{-1}\colon \mathcal{C}(X_{\mathcal{B}}) \to \mathcal{B} \subseteq \mathcal{A}$ gives the desired map

$$\Gamma_a\colon \mathcal{C}(\sigma(a)) \to \mathcal{A}.$$

Under this composition, the identity function on $\sigma(a)$ is first sent to $\hat{a}$, and then to a. $\square$

Remark 2.3.3. For polynomial functions $p \in \mathcal{C}(\sigma(a))$, the functional calculus satisfies $\Gamma_a(p) = p(a)$, because Γ_a is an algebra homomorphism and sends the identity function to a. Thus Theorem 2.3.2 extends the usual polynomial calculus to all continuous functions on the spectrum. We write

$$f(a) \coloneqq \Gamma_a(f)$$

for $f \in \mathcal{C}(\sigma(a))$. This applies, in particular, to bounded normal operators on Hilbert spaces. △

2.4 Positive Elements in C^*-Algebras

In this section we define and study positive elements in C^*-algebras. Positivity is one of the central order-theoretic features of C^*-algebras, and it admits several equivalent descriptions. Throughout the section, $\mathcal{A}$ denotes a general C^*-algebra.

Proposition 2.4.1. *For $a \in \mathcal{A}_{\mathrm{sa}}$, the following are equivalent:*

(i) $\sigma(a) \subseteq [0, \infty)$.

(ii) $a = b^2$ *for some* $b \in \mathcal{A}_{\mathrm{sa}}$.

(iii) $\|t - a\| \leqslant t$ *for some, equivalently for every, real number* $t \geqslant \|a\|$.

In particular, every C^*-algebra is Archimedean as a $*$-algebra.

Proof. For $(i)\Rightarrow(ii)$, use the functional calculus $\Gamma_a\colon \mathcal{C}(\sigma(a)) \to \mathcal{A}$ from Theorem 2.3.2. Since a is self-adjoint, it is normal. If $\sigma(a) \subseteq [0,\infty)$, then the square-root function is real-valued and continuous on $\sigma(a)$. Define

$$b \coloneqq \sqrt{a} \coloneqq \Gamma_a(\sqrt{\cdot}) \in \mathcal{A}_{\mathrm{sa}}.$$

Then

$$b^2 = \Gamma_a(\sqrt{\cdot})^2 = \Gamma_a(\mathrm{id}_{\sigma(a)}) = a.$$

For $(ii)\Rightarrow(i)$, assume $a = b^2$ with $b \in \mathcal{A}_{\mathrm{sa}}$. Let $\mathcal{B}$ be the commutative C^*-subalgebra generated by b. By Lemma 2.3.1, the spectrum of a is the same whether computed in $\mathcal{A}$ or in $\mathcal{B}$. Under the Gelfand transform for $\mathcal{B}$, we have

$$\hat{a}(\xi) = \xi(a) = \xi(b^2) = \xi(b)^2 \geqslant 0$$

for all $\xi \in X_{\mathcal{B}}$, since $\xi(b) \in \mathbb{R}$ for self-adjoint b. Hence $\sigma(a) \subseteq [0,\infty)$ by Theorem 2.2.3.

For $(i)\Leftrightarrow(iii)$, Corollary 2.1.8 gives

$$\|t-a\| = \max\{|t-\lambda| \mid \lambda \in \sigma(a)\}$$

for every real t. If $t \geqslant \|a\|$, then $\sigma(a) \subseteq [-t,t]$. The inequality $\|t-a\| \leqslant t$ is therefore equivalent to $|t-\lambda| \leqslant t$ for all $\lambda \in \sigma(a)$, which is equivalent to $\lambda \geqslant 0$ for all $\lambda \in \sigma(a)$.

Finally, if $a \in \mathcal{A}_{\mathrm{sa}}$, then

$$\sigma(\|a\|-a) = \|a\| - \sigma(a) \subseteq [0,\infty).$$

Thus $\|a\|-a$ is a square by (ii). This shows $\|a\|-a \in \sum \mathcal{A}^2$, and therefore $\mathcal{A}$ is Archimedean. □

Definition 2.4.2. An element $a \in \mathcal{A}_{\mathrm{sa}}$ satisfying the equivalent conditions of Proposition 2.4.1 is called **positive**. We write $a \geqslant 0$ for this. For self-adjoint elements $a, b \in \mathcal{A}$, we write $b \geqslant a$ if $b - a \geqslant 0$. The set of all positive elements is denoted by $\mathcal{A}_+$. △

Example 2.4.3. (i) An element $f \in \mathcal{C}(X)$ is positive if and only if it is pointwise nonnegative on X.

(ii) A matrix $M \in \mathrm{Mat}_d(\mathbb{C})$ is positive if and only if it is Hermitian and all its eigenvalues are nonnegative; equivalently, if M is positive semidefinite. △

Proposition 2.4.4. *The following hold:*

(i) *$\mathcal{A}_+$ is a convex cone in $\mathcal{A}_{\mathrm{sa}}$.*

(ii) *For $a \in \mathcal{A}$, the element $-a^*a$ is positive if and only if $a = 0$.*

(iii) *Every $a \in \mathcal{A}_{\mathrm{sa}}$ can be written as*

$$a = a_+ - a_-$$

with $a_+, a_- \in \mathcal{A}_+$, $a_+a_- = a_-a_+ = 0$, and

$$\|a\| = \max\{\|a_+\|, \|a_-\|\}.$$

Proof. (*i*) Positive scalar multiples of positive elements are clearly positive. Let $a, b \in \mathcal{A}_+$. By Proposition 2.4.1 (*iii*),

$$\begin{aligned}\|(\|a\| + \|b\|) - (a+b)\| &\leqslant \|\|a\| - a\| + \|\|b\| - b\| \\ &\leqslant \|a\| + \|b\|.\end{aligned}$$

Since $\|a\| + \|b\| \geqslant \|a+b\|$, another application of Proposition 2.4.1 (*iii*) shows that $a + b$ is positive. Hence $\mathcal{A}_+$ is a convex cone.

(*ii*) Suppose that $-a^*a$ is positive. By Proposition 2.1.6 (*v*), the nonzero spectra of $-a^*a$ and $-aa^*$ agree, so $-aa^*$ is also positive. Write $a = b + ic$ with $b, c \in \mathcal{A}_{\mathrm{sa}}$. Then

$$a^*a + aa^* = 2b^2 + 2c^2,$$

which is positive by Proposition 2.4.1 and part (*i*). Since $-aa^*$ is positive, it follows that

$$a^*a = (a^*a + aa^*) - aa^*$$

is positive as well. Thus both a^*a and $-a^*a$ are positive, so

$$\sigma(a^*a) \subseteq [0, \infty) \quad \text{and} \quad \sigma(a^*a) \subseteq (-\infty, 0].$$

Since the spectrum is nonempty, $\sigma(a^*a) = \{0\}$. By Corollary 2.1.8,

$$0 = \|a^*a\| = \|a\|^2,$$

and hence $a = 0$. The converse is immediate.

(*iii*) Apply the functional calculus to the self-adjoint element a. Let $f = \mathrm{id}_{\sigma(a)} \in \mathcal{C}(\sigma(a))$, and define

$$f_+(t) := \max\{t, 0\}, \qquad f_-(t) := \max\{-t, 0\}.$$

Set $a_+ := \Gamma_a(f_+)$ and $a_- := \Gamma_a(f_-)$. Then $a_+, a_- \in \mathcal{A}_+$, $a = a_+ - a_-$, and $a_+a_- = 0$ because $f_+f_- = 0$. The norm identity follows from

$$\|a\| = \|f\|_\infty = \max\{\|f_+\|_\infty, \|f_-\|_\infty\} = \max\{\|a_+\|, \|a_-\|\}. \qquad \square$$

Theorem 2.4.5. *We have* $\mathcal{A}_+ = \sum \mathcal{A}^2$. *In particular, for every* $a \in \mathcal{A}$,

$$\|a\| = \min\{r \in [0, \infty) \mid r^2 - a^*a \in \sum \mathcal{A}^2\}.$$

Proof. By Proposition 2.4.1, every positive element is a square of a self-adjoint element, and hence belongs to $\sum \mathcal{A}^2$. Since sums of positive elements are positive by Proposition 2.4.4 (i), it remains only to show that each element of the form a^*a is positive.

Write $a^*a = b_+ - b_-$ as in Proposition 2.4.4 (iii). Since $b_+b_- = b_-b_+ = 0$, we compute

$$-(ab_-)^*(ab_-) = -b_-a^*ab_- = -b_-(b_+ - b_-)b_- = (b_-)^3.$$

The element b_- is positive, hence so is $(b_-)^3$ by functional calculus. By Proposition 2.4.4 (ii), this implies $ab_- = 0$. Therefore $(b_-)^3 = 0$, and functional calculus again gives $b_- = 0$. Thus $a^*a = b_+$ is positive.

Now $r^2 - a^*a \in \sum \mathcal{A}^2$ if and only if $r^2 - a^*a$ is positive. Since a^*a is positive, this is equivalent to $r^2 \geqslant \|a^*a\| = \|a\|^2$. Hence the smallest such $r \geqslant 0$ is $\|a\|$. $\square$

Remark 2.4.6. The preceding theorem shows in particular that the norm on a C^*-algebra is determined by the underlying $*$-algebra structure. Indeed, if a given unital $*$-algebra carries a C^*-norm, then for every element a this norm must satisfy

$$\|a\| = \min\{r \in [0, \infty) \mid r^2 - a^*a \in \sum \mathcal{A}^2\}.$$

The right-hand side only refers to the algebraic operations and the involution. Hence a $*$-algebra admits at most one norm that turns it into a C^*-algebra.

2.5 More Properties of C^*-Algebras

Theorem 2.5.1. *Every unital $*$-algebra homomorphism between C^*-algebras is bounded with operator norm* 1. *If it is injective, then it is isometric, i.e. norm-preserving.*

Proof. Let $\pi \colon \mathcal{A} \to \mathcal{B}$ be a unital $*$-algebra homomorphism. If $r^2 - a^*a \in \sum \mathcal{A}^2$, then

$$r^2 - \pi(a)^*\pi(a) \in \sum \mathcal{B}^2.$$

By Theorem 2.4.5, this implies $\|\pi(a)\| \leqslant \|a\|$ for all $a \in \mathcal{A}$. Hence π is bounded and $\|\pi\|_{\text{op}} \leqslant 1$. Since our homomorphisms are unital, equality follows from $\pi(1) = 1$.

If π is injective, then $\pi(\mathcal{A})$ carries two C^*-norms: the norm transported from $\mathcal{A}$ via π, and the norm inherited from $\mathcal{B}$. By the uniqueness of the C^*-norm on a given $*$-algebra, these two norms coincide. Thus π is isometric. $\square$

Theorem 2.5.2 (Russo–Dye). *The closed unit ball of a C^*-algebra is the closed convex hull of its unitary elements, i.e. of the elements u satisfying $u^*u = uu^* = 1$.*

Proof. Every unitary element has norm one, since $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Hence the closed convex hull of the unitary elements is contained in the closed unit ball.

For the converse, let a belong to the open unit ball. We first show that, for every unitary element u, the element

$$b := \frac{1}{2}(u + a) = \frac{1}{2}u(1 + u^*a)$$

is a convex combination of unitaries. Since $\|u^*a\| < 1$, the element $1 + u^*a$ is invertible by Proposition 2.1.6. Thus b is invertible and still belongs to the open unit ball. By the polar decomposition from Exercise 14, write $b = vp$ with v unitary and p positive invertible, with $\|p\| = \|b\| < 1$.

Since $0 \leqslant p$ and $\|p\| < 1$, we have $p \leqslant 1$. Hence the element $1 - p^2$ is positive. Thus $1 - p^2 = q^2$ for some self-adjoint q which commutes with p by functional calculus. Set $w := p + iq$. Then

$$w^*w = p^2 + q^2 = 1,$$

and also $ww^* = 1$, so w is unitary. Moreover,

$$\frac{1}{2}(w + w^*) = p \quad \text{and} \quad \frac{1}{2}(vw + vw^*) = vp = b.$$

Thus b is a convex combination of two unitaries.

Now choose any unitary a_0. If a_i is a convex combination of unitaries, say $a_i = \sum_j \lambda_j u_j$, then

$$a_{i+1} := \frac{1}{2}(a_i + a) = \sum_j \lambda_j \frac{1}{2}(u_j + a)$$

is again a convex combination of unitaries by the previous paragraph. The sequence a_i converges to a along the line segment from a_0 to a. Hence every element of the open unit ball lies in the closed convex hull of the unitary elements, and the result follows by closure. $\square$

Proposition 2.5.3. *Let $\mathcal{A}$ be a C^*-algebra and let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a linear functional. Then φ is a state if and only if*

$$\varphi(1) = 1 \quad \textit{and} \quad \|\varphi\|_{\mathrm{op}} \leqslant 1.$$

Proof. If φ is a state, then $\varphi(1) = 1$ by definition. For $a \in \mathcal{A}$, Theorem 2.4.5 gives

$$\|a\|^2 - a^*a \in \sum \mathcal{A}^2,$$

hence $\varphi(a^*a) \leqslant \|a\|^2$. The Cauchy–Schwarz inequality for states yields

$$|\varphi(a)|^2 \leqslant \varphi(a^*a) \leqslant \|a\|^2.$$

Thus $\|\varphi\|_{\mathrm{op}} \leqslant 1$.

Conversely, assume $\varphi(1) = 1$ and $\|\varphi\|_{\mathrm{op}} \leqslant 1$. For any $a \in \mathcal{A}$, the element a^*a is positive, and Proposition 2.4.1 gives

$$\|\|a^*a\| - a^*a\| \leqslant \|a^*a\|.$$

Therefore

$$|\|a^*a\| - \varphi(a^*a)| = |\varphi(\|a^*a\| - a^*a)| \leqslant \|a^*a\|.$$

Writing $\varphi(a^*a) = r + is$ with $r, s \in \mathbb{R}$, this implies $r \geqslant 0$.

It remains to show $s = 0$. For $\lambda \in \mathbb{R}$, set

$$b := a^*a - r - i\lambda s.$$

Since $a^*a - r$ is self-adjoint, we have

$$b^*b = (a^*a - r)^2 + \lambda^2 s^2.$$

Using $\|\varphi\|_{\mathrm{op}} \leqslant 1$, we obtain

$$(1-\lambda)^2 s^2 = |\varphi(b)|^2 \leqslant \|b\|^2 = \|b^*b\| \leqslant \|a^*a - r\|^2 + \lambda^2 s^2.$$

Hence

$$2\lambda s^2 + \|a^*a - r\|^2 - s^2 \geqslant 0$$

for all real λ. This is possible only if $s = 0$. Thus $\varphi(a^*a) = r \geqslant 0$ for all $a \in \mathcal{A}$, so φ is a state. □

Corollary 2.5.4. *For every normal element $a \in \mathcal{A}$,*

$$\operatorname{conv}(\sigma(a)) = \{\varphi(a) \mid \varphi \textit{ is a state on } \mathcal{A}\} \subseteq B_{\|a\|}(0).$$

Proof. It is enough to prove the equality inside the commutative C^*-algebra generated by a. Indeed, let $\mathcal{B}$ be this subalgebra. The spectrum of a is independent of the ambient algebra by Lemma 2.3.1. Every state on $\mathcal{A}$ restricts to a state on $\mathcal{B}$. Conversely, if φ is a state on $\mathcal{B}$, then Proposition 2.5.3 gives $\|\varphi\|_{\mathrm{op}} \leqslant 1$, and the Hahn–Banach theorem extends φ to a functional on $\mathcal{A}$ of the same norm. By Proposition 2.5.3 again, this extension is a state on $\mathcal{A}$.

We may therefore assume $\mathcal{A} = \mathcal{C}(X)$ for some compact Hausdorff space X. By Example 2.1.5, $\sigma(a) = a(X)$. By Theorem 1.4.6, the values of a under states are precisely

$$\left\{ \int_X a \, \mathrm{d}\mu \mid \mu \text{ is a probability measure on } X \right\}.$$

Point masses show that $a(X) \subseteq \{\varphi(a) \mid \varphi \text{ state}\}$, and the latter set is convex because the set of states is convex. Thus $\operatorname{conv}(\sigma(a))$ is contained in the set of state values.

Conversely, each integral $\int_X a \, \mathrm{d}\mu$ is a limit of integrals of step functions, hence a limit of finite convex combinations of values $a(x_i)$. Since $\sigma(a) = a(X)$ is compact, its convex hull is compact and therefore closed. Thus every such integral lies in $\operatorname{conv}(\sigma(a))$. The final inclusion in the closed ball of radius $\|a\|$ follows from Proposition 2.1.6. □

Corollary 2.5.5. *The cone $\mathcal{A}_+$ consists exactly of those self-adjoint elements that are nonnegative under every state. In particular, $\mathcal{A}_+$ is closed. If $a \in \mathcal{A}_+$, then*

$$\|a\| = \max\{\varphi(a) \mid \varphi \textit{ is a state on } \mathcal{A}\}.$$

Moreover, if $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$.

Proof. If $a \in \mathcal{A}_+$, then $\varphi(a) \geqslant 0$ for every state by the definition of states and Theorem 2.4.5. Conversely, if $a = a^*$ and all states are nonnegative on a, then Corollary 2.5.4 implies that $\sigma(a) \subseteq [0, \infty)$, hence $a \in \mathcal{A}_+$.

Since states are continuous by Proposition 2.5.3, the characterization by states shows that $\mathcal{A}_+$ is closed. If $a \in \mathcal{A}_+$, then Corollary 2.5.4 gives

$$\|a\| = \max\{\lambda \mid \lambda \in \sigma(a)\} \leqslant \max\{\varphi(a) \mid \varphi \text{ is a state}\} \leqslant \|a\|,$$

proving the norm formula. Finally, if $0 \leqslant a \leqslant b$, then $\varphi(a) \leqslant \varphi(b)$ for every state φ. Taking maxima over all states and using the norm formula yields $\|a\| \leqslant \|b\|$. □

2.6 Classification of General C^*-algebras

We have seen that closed $*$-subalgebras of $\mathcal{B}(H)$ are C^*-algebras. We can now show that these are actually all examples.

Theorem 2.6.1. *For each C^*-algebra $\mathcal{A}$, the representation $\pi\colon \mathcal{A} \to \mathcal{B}(H)$ from Theorem 1.4.18 is injective. So $\mathcal{A}$ is isomorphic to a closed $*$-subalgebra of $\mathcal{B}(H)$.*

Proof. In Proposition 2.4.1 we have seen that $\mathcal{A}$ is Archimedean, so Theorem 1.4.18 applies. For injectivity we need to show that for every $0 \neq a \in \mathcal{A}$ there exists a state φ with $\varphi(a^*a) \neq 0$. From the C^*-identity it follows that $a^*a \neq 0$. So by Corollary 2.1.8 there is some $0 \neq \lambda \in \sigma(a^*a)$, and by Corollary 2.5.4 there is a state with $\varphi(a^*a) = \lambda \neq 0$.

Now $\mathcal{A}$ is isomorphic to $\pi(\mathcal{A}) \subseteq \mathcal{B}(H)$, and from completeness we obtain that $\pi(\mathcal{A})$ is closed in $\mathcal{B}(H)$. □

Remark 2.6.2. We now know that every C^*-algebra is, up to isomorphism, a closed subalgebra of some $\mathcal{B}(H)$. So one might ask why we did not just define them like that, and thus get rid of all the hassle with the above proofs. One reason is that we can often construct C^*-algebras easier in an abstract way, without having a concrete embedding into $\mathcal{B}(H)$. For example, given any Archimedean $*$-algebra $\mathcal{A}$, we define a seminorm by

$$\|a\| := \inf\{r \geqslant 0 \mid r^2 - a^*a \in \sum \mathcal{A}^2\}.$$

After modding out the space of elements of seminorm zero and completing, we obtain a C^*-algebra. See Exercise 15 for an interesting example. △

2.7 Positive and Completely Positive Maps

Throughout this section, let $\mathcal{A}$ and $\mathcal{B}$ always be C^*-algebras. So far, the maps between them have been (isometric) $*$-algebra homomorphisms, which preserve all of the available structure. We now weaken this notion in a way that is still rich enough for many applications, especially in quantum theory. The resulting maps are called *positive* and *completely positive*.

Definition 2.7.1. A **positive map** is a $*$-linear map $\psi\colon \mathcal{A} \to \mathcal{B}$ that maps positive elements to positive elements, i.e. satisfies

$$\psi\left(\mathcal{A}_+\right) \subseteq \mathcal{B}_+.$$

Positive maps form a convex cone. △

Remark 2.7.2. The definition of positive elements in a C^*-algebra still uses much of the algebraic structure. Thus positive maps need not preserve multiplication, but they preserve substantially more structure than merely the Banach space structure.

Example 2.7.3. (i) States are positive maps $\varphi\colon \mathcal{A} \to \mathbb{C}$.
(ii) Every $*$-algebra homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}$ is a positive map. This follows immediately from $\pi(a^*a) = \pi(a)^*\pi(a)$.
(iii) For every $V \in \mathrm{Mat}_{d,e}(\mathbb{C})$ we obtain a positive map

$$\begin{aligned} \psi\colon \mathrm{Mat}_d(\mathbb{C}) &\to \mathrm{Mat}_e(\mathbb{C}) \\ M &\mapsto V^*MV. \end{aligned}$$

(iv) Transposition $\tau\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_d(\mathbb{C}); M \mapsto M^t$ is a positive map. △

Proposition 2.7.4. *Positive maps between C^*-algebras are bounded.*

Proof. Every element $a \in \mathcal{A}$ can be written as

$$a = a_1 - a_2 + ia_3 - ia_4$$

with positive elements a_i and $\|a_i\| \leqslant \|a\|$, see Exercise 19. It is therefore enough to show boundedness on positive elements. If $a \in \mathcal{A}_+$, then both a and $\|a\| - a$ are positive, so

$$0 \leqslant \psi(a) \leqslant \|a\|\psi(1).$$

By Corollary 2.5.5, this implies $\|\psi(a)\| \leqslant \|\psi(1)\|\|a\|$. □

Before we can define an important strengthening of positivity for maps, we need the following observation. The algebraic tensor product fulfills

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H) \cong \mathrm{Mat}_s(\mathcal{B}(H)) \cong \mathcal{B}(H^s).$$

The first isomorphism identifies $(a_{ij})_{i,j} \otimes L$ with $(a_{ij}L)_{i,j}$, and the second lets a matrix of operators act on H^s by matrix multiplication and then applying the operators. Since $\mathcal{B}(H^s)$ is a C^*-algebra, this equips $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H)$ with a C^*-algebra structure. Furthermore, if $\mathcal{A}$ is an arbitrary C^*-algebra, we can realize $\mathcal{A} \subseteq \mathcal{B}(H)$ for some Hilbert space H and obtain

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} \subseteq \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}(H),$$

which also makes it a C^*-algebra. Since the norm in a C^*-algebra is unique, it does not matter which embedding $\mathcal{A} \subseteq \mathcal{B}(H)$ we choose, i.e. we can use one of the intrinsic definitions of the norm.

Example 2.7.5. Let X be a compact Hausdorff space and

$$F = (f_{ij})_{i,j} \in \mathrm{Mat}_s(\mathcal{C}(X)) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{C}(X).$$

Then F is invertible in $\mathrm{Mat}_s(\mathcal{C}(X))$ if and only if each $F(x) = (f_{ij}(x))_{i,j}$ is invertible in $\mathrm{Mat}_s(\mathbb{C})$, for $x \in X$. This is true since the inverse of a complex matrix depends continuously on the matrix entries. Thus

$$\sigma(F) = \bigcup_{x \in X} \sigma(F(x)),$$

and in particular, F is positive if and only if all $F(x)$ are positive semidefinite. This also gives the C^*-norm on $\mathrm{Mat}_s(\mathcal{C}(X))$. We have

$$\|F\| = \max\{\|F(x)\| \mid x \in X\},$$

where we use the norm in the C^*-algebra $\mathrm{Mat}_s(\mathbb{C})$. △

Lemma 2.7.6. *Every positive element in* $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{A})$ *is a sum of at most* s *elements of the form*

$$(a_i^* a_j)_{i,j}$$

with $a_1, \ldots, a_s \in \mathcal{A}$.

Proof. Every positive element is of the form A^*A for some $A \in \mathrm{Mat}_s(\mathcal{A})$. Let $a_{k1}, \ldots, a_{ks}$ be the elements in the k-th row of A and observe that

$$A^*A = \sum_{k=1}^{s} (a_{ki}^* a_{kj})_{i,j}. \qquad \square$$

Definition 2.7.7. A $*$-linear map ψ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is called s-**positive** if

$$\mathrm{id}_s \otimes \psi \colon \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} \to \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{B}$$

is positive. It is called **completely positive** if it is s-positive for all $s \geqslant 1$. The set of completely positive maps is a convex subcone of the cone of all positive maps. △

Remark 2.7.8. (i) Under the identification $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{A})$, the map $\mathrm{id}_s \otimes \psi$ is the map that applies ψ to each matrix entry.
(ii) Since smaller matrices can be embedded into larger matrices in a positivity-preserving way, s-positivity implies $(s-1)$-positivity.

Remark 2.7.9. There are several reasons why completely positive maps are important. First, positive maps are difficult to describe or classify in general, while completely positive maps are often much more tractable; see for example Corollary 2.7.15 below. Second, tensor products of C^*-algebras are not functorial for positive maps: the tensor product of two positive maps need not be positive. Although we have not introduced tensor products of C^*-algebras in full generality, this phenomenon already appears for matrix algebras; see Example 2.7.11 below. Third, this obstruction has a natural physical interpretation. Positivity-preserving maps describe operations on quantum systems (see Section 4.1), and a composite system is described by a tensor product. If an operation ψ is applied only to one part of such a system, while the other part is left unchanged, it should again be positive. This requirement leads naturally to complete positivity. △

Example 2.7.10. (i) Every $*$-homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}$ is completely positive. This is because $\mathrm{id}_s \otimes \pi$ is again an $*$-homomorphism, and thus positive.

(ii) Maps of the form $\psi\colon M \mapsto V^*MV$ as in Example 2.7.3(iii) are completely positive. This is because

$$\mathrm{id}_s \otimes \psi\colon \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C})$$

is again of the same type, $X \mapsto (I_s \otimes V)^* X (I_s \otimes V)$, and thus positive. △

Example 2.7.11. The transposition map $\tau\colon \mathrm{Mat}_s(\mathbb{C}) \to \mathrm{Mat}_s(\mathbb{C})$ is positive but *not* completely positive if $s \geqslant 2$. To see this, consider the positive semidefinite matrix

$$E := \sum_{i,j=1}^{s} E_{ij} \otimes E_{ij} = \left(\sum_{i=1}^{s} e_i \otimes e_i\right)\left(\sum_{i=1}^{s} e_i \otimes e_i\right)^*$$

in $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_s(\mathbb{C}) = \mathrm{Mat}_{s^2}(\mathbb{C})$. For $s = 2$, for example, we have

$$E = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

We compute

$$(\mathrm{id}_s \otimes \tau)(E) = \sum_{i,j} E_{ij} \otimes E_{ji},$$

and this matrix is not positive semidefinite, for example since the $(2,2)$-entry is zero, but the $(2,s+1)$-entry is 1. For $s=2$, this gives

$$(\mathrm{id}_s \otimes \tau)(E) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map $\mathrm{id}_s \otimes \tau$ is called the **partial transposition** and is denoted by Γ. △

Theorem 2.7.12. *If $\mathcal{A}$ or $\mathcal{B}$ is commutative, then every positive map $\psi\colon \mathcal{A} \to \mathcal{B}$ is completely positive.*

Proof. First assume that $\mathcal{A}$ is commutative. In view of Theorem 2.2.4 we can assume $\mathcal{A} = \mathcal{C}(X)$ for some compact Hausdorff space X. Let $F \in \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{A} = \mathrm{Mat}_s(\mathcal{C}(X))$ be positive, which means that for each $x \in X$ the matrix $F(x) \in \mathrm{Mat}_s(\mathbb{C})$ is positive semidefinite, see Example 2.7.5. We have to show that $(\mathrm{id}_s \otimes \psi)(F) \in \mathrm{Mat}_s(\mathcal{B})$ is positive. Fix some $\varepsilon > 0$. We can find a finite covering of X by open sets U_i, and positive semidefinite matrices $P_i \in \mathrm{Mat}_s(\mathbb{C})$, such that

$$\|F(x) - P_i\| \leqslant \varepsilon$$

for all $x \in U_i$. We choose a continuous partition of unity subordinate to that cover, i.e. a collection of continuous maps $u_i\colon X \to [0,1]$ with $\sum_i u_i = 1$ and $u_i(x) = 0$ for $x \notin U_i$. For every $x \in X$ we then have

$$\|F(x) - \sum_i u_i(x) P_i\| \leqslant \varepsilon,$$

which means that $\|F - \sum_i u_i P_i\| \leqslant \varepsilon$ in $\mathrm{Mat}_s(\mathcal{C}(X))$. From

$$(\mathrm{id}_s \otimes \psi)\left(\sum_i u_i P_i\right) = \sum_i \psi(u_i) P_i$$

and positivity of ψ we see that this is a positive element in $\mathrm{Mat}_s(\mathcal{B})$. Since ψ is continuous by Proposition 2.7.4, so is $\mathrm{id}_s \otimes \psi$. Thus $(\mathrm{id}_s \otimes \psi)(F)$ can be approximated arbitrarily well by positive elements, and is thus itself positive, by Corollary 2.5.5.

Now assume that $\mathcal{B}$ is commutative, say $\mathcal{B} = \mathcal{C}(X)$. Since positivity of $G \in \mathrm{Mat}_s(\mathcal{C}(X))$ means positivity of each $G(x)$, it is enough to show that $\varphi \circ \psi$ is completely positive for each positive functional φ on $\mathcal{B}$, so we have reduced to the case $\mathcal{B} = \mathbb{C}$. So let $\varphi\colon \mathcal{A} \to \mathbb{C}$ be a positive linear functional and let $A \in \mathrm{Mat}_s(\mathcal{A})$ be positive. In view of Lemma 2.7.6 it is enough to assume $A = (a_i^* a_j)_{i,j}$. For $v \in \mathbb{C}^s$ we now have

$$\begin{aligned} v^*(\mathrm{id}_s \otimes \varphi)(A)v = v^*(\varphi(a_i^*a_j))_{i,j}v &= \sum_{i,j} \overline{v}_i v_j \varphi(a_i^* a_j) \\ &= \varphi\left(\sum_{i,j} \overline{v}_i v_j a_i^* a_j\right) \\ &= \varphi\left(\left(\sum_i v_i a_i\right)^* \left(\sum_i v_i a_i\right)\right). \end{aligned}$$

This is a nonnegative real number, by positivity of φ. So $(\mathrm{id}_s \otimes \varphi)(A)$ is a positive semidefinite matrix, $\mathrm{id}_s \otimes \varphi$ thus a positive map, and therefore φ is completely positive. ☐

In the following result we use the matrix

$$E = \sum_{i,j=1}^{d} E_{ij} \otimes E_{ij} \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) = \mathrm{Mat}_{d^2}(\mathbb{C})$$

that we have already encountered in Example 2.7.11. We have seen that it is positive semidefinite and of rank 1.

Theorem 2.7.13. *Let $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathcal{B}$ be a $*$-linear map between C^*-algebras. Then the following are equivalent:*

- *(i) ψ is completely positive.*
- *(ii) ψ is d-positive.*
- *(iii) $(\mathrm{id}_d \otimes \psi)(E) = \sum_{i,j=1}^d E_{ij} \otimes \psi(E_{ij}) = \left(\psi(E_{ij})\right)_{i,j}$ is positive in $\mathrm{Mat}_d(\mathcal{B})$.*

Proof. The only nontrivial direction is "(*iii*)⇒(*i*)". So let $s \geqslant 1$ be arbitrary. In view of Theorem 2.6.1 we can assume $\mathcal{B} = \mathcal{B}(H)$, and in view of Lemma 2.7.6 we must show that

$$(\psi(M_i^* M_j))_{i,j} \in \mathrm{Mat}_s(\mathcal{B}) = \mathcal{B}(H^s)$$

is positive, for all $M_1, \ldots, M_s \in \mathrm{Mat}_d(\mathbb{C})$. Write $M_i = \sum_{r,t=1}^d m_{irt} E_{rt}$ and compute

$$M_i^* M_j = \sum_{k=1}^{d} \sum_{r,t=1}^{d} \overline{m}_{ikr} m_{jkt} E_{rt}.$$

Now let $h = (h_1, \dots, h_s)^t \in H^s$ be arbitrary and compute

$$\begin{aligned}\left\langle (\psi(M_i^* M_j))_{i,j}\, h, h \right\rangle &= \sum_{i,j,k,r,t} \overline{m}_{ikr} m_{jkt} \langle \psi(E_{rt}) h_j, h_i \rangle \\ &= \sum_{k,r,t} \left\langle \psi(E_{rt}) \underbrace{\sum_j m_{jkt} h_j}_{=: g_{kt}}, \underbrace{\sum_i m_{ikr} h_i}_{g_{kr}} \right\rangle \\ &= \sum_k \left\langle (\psi(E_{rt}))_{r,t}\, g_k, g_k \right\rangle,\end{aligned}$$

where $g_k = (g_{k1}, \dots, g_{kd})^t \in H^d$. By assumption (*iii*), each of the terms in the last sum is nonnegative, proving the claim. □

Definition 2.7.14. For a linear map $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathcal{B}$, the matrix

$$(\mathrm{id}_d \otimes \psi)(E) = \sum_{i,j=1}^{d} E_{ij} \otimes \psi(E_{ij}) = (\psi(E_{ij}))_{i,j} \in \mathrm{Mat}_d(\mathcal{B})$$

is called the **Choi matrix** of ψ. This is very similar to the representing matrix of a linear map from linear algebra, but slightly differently arranged. So Theorem 2.7.13 states that a map is completely positive if and only if its Choi matrix is positive. △

Corollary 2.7.15 (Choi–Kraus Decomposition). *For every completely positive map* $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ *there exist* $V_1, \dots, V_r \in \mathrm{Mat}_{d,e}(\mathbb{C})$ *such that*

$$\psi(M) = \sum_{i=1}^{r} V_i^* M V_i$$

for all $M \in \mathrm{Mat}_d(\mathbb{C})$.

Proof. We know that the Choi matrix

$$C = \sum_{i,j} E_{ij} \otimes \psi(E_{ij}) = (\psi(E_{ij}))_{i,j} \in \mathrm{Mat}_{de}(\mathbb{C})$$

is positive semidefinite, so it can be written as a finite sum of elements vv^* with $v \in \mathbb{C}^{de} = \mathbb{C}^d \otimes \mathbb{C}^e$. Since the sum just comes out of all of the following computations, we ignore it for better readability and assume $C = vv^*$. Write $v =$

$\sum_{i=1}^d e_i \otimes y_i \in \mathbb{C}^d \otimes \mathbb{C}^e$, which implies $\psi(E_{ij}) = y_i y_j^*$ for all i, j. Now write $M = \sum_{i,j} m_{ij} E_{ij} \in \mathrm{Mat}_d(\mathbb{C})$ and compute

$$\begin{aligned} \psi(M) &= \sum_{i,j} m_{ij} \psi(E_{ij}) \\ &= \sum_{i,j} m_{ij} y_i y_j^* \\ &= V^* M V \end{aligned}$$

where $V \in \mathrm{Mat}_{d,e}(\mathbb{C})$ is the matrix with rows $y_1^*, \dots, y_d^*$. □

Remark 2.7.16. The number of matrices V_i can be chosen to be the number of rank-one summands vv^* needed to represent the Choi matrix of ψ, namely its rank. In particular, it is bounded by de. △

Remark 2.7.17. Before we study maps into $\mathrm{Mat}_d(\mathbb{C})$, we recall a fact from linear algebra. If X, Y are vector spaces and Y is finite-dimensional, there is an isomorphism

$$\mathrm{Lin}(X, Y) \cong \mathrm{Lin}(Y' \otimes X, \mathbb{C}).$$

Explicitly, from left to right a linear map $\psi\colon X \to Y$ is identified with the functional

$$\begin{aligned} \varphi_\psi \colon Y' \otimes X &\to \mathbb{C} \\ \sum_i f_i \otimes x_i &\mapsto \sum_i f_i(\psi(x_i)). \end{aligned}$$

For the other direction we fix a basis $y_1, \dots, y_d$ of Y and denote the dual basis of Y' by $y_1', \dots, y_d'$. Then a functional φ on $Y' \otimes X$ is identified with the linear map

$$\begin{aligned} \psi_\varphi \colon X &\to Y \\ x &\mapsto \sum_i \varphi(y_i' \otimes x) y_i. \end{aligned}$$

These two constructions are inverse to each other and provide the desired isomorphism. △

Example 2.7.18. We will use this in the case that $X = \mathcal{A}$ is a C^*-algebra and $Y = \mathrm{Mat}_d(\mathbb{C})$. We identify $\mathrm{Mat}_d(\mathbb{C})$ with its dual space via the inner product

$\langle M, N\rangle := \operatorname{tr}(N^*M)$. Then, given a linear map $\psi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$, we obtain the functional

$$\begin{aligned} \varphi_\psi\colon \mathrm{Mat}_d(\mathbb{C}) \otimes \mathcal{A} &= \mathrm{Mat}_d(\mathcal{A}) \to \mathbb{C} \\ \sum_{i,j} E_{ij} \otimes a_{ij} = (a_{ij})_{i,j} &\mapsto \sum_{i,j} \operatorname{tr}(E_{ij}^*\psi(a_{ij})) = \sum_{i,j} \psi(a_{ij})_{ij}. \end{aligned}$$

We will use this in the following. △

Theorem 2.7.19. *Let $\psi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$ be a $*$-linear map between C^*-algebras. Then the following are equivalent:*

- *(i)* *ψ is completely positive.*
- *(ii)* *ψ is d-positive.*
- *(iii)* *φ_ψ is a positive functional on $\mathrm{Mat}_d(\mathcal{A})$.*

Proof. "(*i*)⇒(*ii*)" is trivial. For "(*ii*)⇒(*iii*)" observe that

$$\begin{aligned} \varphi_\psi\left((a_{ij})_{i,j}\right) &= \sum_{i,j} \psi(a_{ij})_{ij} = \sum_{i,j} e_i^*\psi(a_{ij})e_j \\ &= \sum_i e_i^* \left(\sum_j \psi(a_{ij})e_j\right) \\ &= \left\langle \left(\psi(a_{ij})\right)_{i,j} e, e\right\rangle \end{aligned}$$

where $e = \sum_{i=1}^d e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$. Thus d-positivity of ψ implies positivity of φ_ψ.

For "(*iii*)⇒(*i*)" fix $s \geqslant 1$ and $A = \left(a_i^*a_j\right)_{i,j} \in \mathrm{Mat}_s(\mathcal{A})$ positive. Take $v = \sum_{i=1}^s e_i \otimes v_i \in \mathbb{C}^{sd} = \mathbb{C}^s \otimes \mathbb{C}^d$ and write $v_i = \sum_{k=1}^d \lambda_{ik}e_k \in \mathbb{C}^d$. We now have

$$\begin{aligned} v^*\left(\psi(a_i^*a_j)\right)_{i,j} v &= \sum_{i,j} v_i^*\psi(a_i^*a_j)v_j \\ &= \sum_{i,j,k,\ell} \overline{\lambda}_{ik}\lambda_{j\ell} e_k^*\psi(a_i^*a_j)e_\ell \\ &= \sum_{k,\ell} e_k^*\psi\left(\left(\sum_i \lambda_{ik}a_i\right)^* \sum_i \lambda_{i\ell}a_i\right) e_\ell \\ &= \varphi_\psi\left((b_k^*b_\ell)_{k,\ell}\right) \geqslant 0, \end{aligned}$$

where $b_k = \sum_i \lambda_{ik}a_i$. □

Remark 2.7.20. The fact that d-positivity implies positivity also holds for maps into matrices over arbitrary commutative C^*-algebras. Indeed, if $\mathcal{B} = \mathcal{C}(X)$ and

$$\psi\colon \mathcal{A} \to \mathrm{Mat}_d(\mathcal{B}) = \mathrm{Mat}_d(\mathcal{C}(X))$$

is $*$-linear, then positivity in $\mathrm{Mat}_s(\mathrm{Mat}_d(\mathcal{C}(X)))$ can be checked pointwise on X. Thus ψ is completely positive if and only if each map

$$\psi_x\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C}), \qquad a \mapsto \psi(a)(x)$$

is completely positive. By the preceding theorem, this is already equivalent to d-positivity of all ψ_x, and hence to d-positivity of ψ. △

Completely positive maps also have a useful characterization in terms of representations. The following result generalizes both the Choi–Kraus decomposition and the GNS construction.

Theorem 2.7.21 (Stinespring's Dilation Theorem)**.** *Let $\psi\colon \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map. Then there is a Hilbert space K, a $*$-representation $\pi\colon \mathcal{A} \to \mathcal{B}(K)$, and a bounded linear map $V\colon H \to K$, such that*

$$\psi(a) = V^*\pi(a)V$$

for all $a \in \mathcal{A}$.

Proof. On the vector space tensor product $\mathcal{A} \otimes H$ we define a Hermitian bilinear form by

$$\langle a \otimes g, b \otimes h\rangle := \langle \psi(b^*a)g, h\rangle,$$

extended by bilinearity. From complete positivity of ψ we obtain that this is positive semidefinite. Indeed,

$$\begin{aligned}\left\langle \sum_{i=1}^{s} a_i \otimes h_i, \sum_{i=1}^{s} a_i \otimes h_i \right\rangle &= \sum_{i,j=1}^{s} \langle \psi(a_i^* a_j)h_j, h_i\rangle \\ &= \sum_i \left\langle \sum_j \psi(a_i^* a_j)h_j, h_i \right\rangle \\ &= \langle (\mathrm{id}_s \otimes \psi)\left((a_i^* a_j)_{i,j}\right) h, h\rangle,\end{aligned}$$

where $h = (h_1, \ldots, h_s)^t \in H^s$. Since $(a_i^* a_j)_{i,j} \in \mathrm{Mat}_s(\mathcal{A})$ is a square and thus positive, we see that this is a nonnegative number.

For every positive semidefinite Hermitian bilinear form, the set of elements of norm 0 is a subspace $\mathcal{N} \subseteq \mathcal{A} \otimes H$, and if we pass to $(\mathcal{A} \otimes H)/\mathcal{N}$, the bilinear form becomes a well-defined inner product. We denote by K the completion of $(\mathcal{A} \otimes H)/\mathcal{N}$ with respect to this inner product.

For $a \in \mathcal{A}$, we define a linear map m_a on $\mathcal{A} \otimes H$ by

$$m_a\left(\sum_i a_i \otimes h_i\right) := \sum_i aa_i \otimes h_i.$$

In $\mathcal{A}$ we have $\|a^*a\| - a^*a \geqslant 0$, and thus $\|a^*a\|I_s - a^*aI_s \geqslant 0$ in $\mathrm{Mat}_s(\mathcal{A})$. This implies that

$$\|a^*a\|(a_i^*a_j)_{i,j} - (a_i^*a^*aa_j)_{i,j} \geqslant 0$$

in $\mathrm{Mat}_s(\mathcal{A})$ and thus

$$\|a^*a\|(\psi(a_i^*a_j))_{i,j} - (\psi(a_i^*a^*aa_j))_{i,j} \geqslant 0$$

in $\mathrm{Mat}_s(\mathcal{B}(H))$. It follows that

$$\left\|m_a\left(\sum_i a_i \otimes h_i\right)\right\|^2 \leqslant \|a^*a\| \left\|\sum_i a_i \otimes h_i\right\|^2.$$

This implies that m_a is well-defined and bounded on $(\mathcal{A} \otimes H)/\mathcal{N}$, actually with $\|m_a\|_{\mathrm{op}} \leqslant \sqrt{\|a^*a\|} = \|a\|$. Thus m_a extends uniquely to a bounded linear operator $\pi(a)$ on K with $\|\pi(a)\|_{\mathrm{op}} \leqslant \|a\|$. We have therefore constructed a bounded $*$-representation

$$\pi\colon \mathcal{A} \to \mathcal{B}(K).$$

We now define

$$\begin{aligned} V\colon H &\to K \\ h &\mapsto 1 \otimes h \bmod \mathcal{N} \end{aligned}$$

and compute

$$\|Vh\|^2 = \langle 1 \otimes h, 1 \otimes h\rangle = \langle \psi(1^*1)h, h\rangle \leqslant \|\psi(1)\|\|h\|^2.$$

So V is bounded. Finally, we have

$$\langle V^*\pi(a)Vh_1, h_2\rangle = \langle a \otimes h_1, 1 \otimes h_2\rangle = \langle \psi(a)h_1, h_2\rangle$$

for all $h_1, h_2 \in H$. This implies $V^*\pi(a)V = \psi(a)$ in $\mathcal{B}(H)$, and the proof is complete. $\square$

Example 2.7.22. A **projective measurement** in quantum information consists of orthogonal projections $P_1, \ldots, P_m \in \mathcal{B}(H)$ with $P_1 + \cdots + P_m = \mathrm{id}_H$. A **positive operator valued measurement** (POVM) is a generalization, consisting of positive semidefinite operators $Q_1, \ldots, Q_m \in \mathcal{B}(H)$ with $Q_1 + \cdots + Q_m = \mathrm{id}_H$. For a given POVM we consider the $*$-linear map

$$\begin{aligned} Q\colon \mathbb{C}^m = \mathcal{C}(\{1, \ldots, m\}) &\to \mathcal{B}(H) \\ e_i &\mapsto Q_i \end{aligned}$$

between C^*-algebras, which is positive and thus completely positive. By Theorem 2.7.21 we get a $*$-representation $\pi\colon \mathbb{C}^m \to \mathcal{B}(K)$ and $V\colon H \to K$ with

$$Q_i = V^* \pi(e_i) V$$

for $i = 1, \ldots, m$. Since $e_i^* = e_i^2 = e_i$ and $e_1 + \cdots + e_m = 1$ holds in $\mathbb{C}^m = \mathcal{C}(\{1, \ldots, m\})$ we see that the $P_i := \pi(e_i)$ define a projective measurement. This is known as *Naimark's Dilation Theorem*: every POVM dilates to a projective measurement. Note that if H is finite-dimensional, i.e. the Q_i are matrices, then K is finite-dimensional as well (this follows from the proof of Theorem 2.7.21), and hence the P_i are matrices too. △

Example 2.7.23. We demonstrate how to recover the Choi–Kraus representation from Corollary 2.7.15 from Stinespring's Dilation Theorem.

Let $\psi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ be completely positive. The proof of Theorem 2.7.21 shows that the space K is finite-dimensional if both $\mathcal{A}$ and H are. Thus we obtain a $*$-representation $\pi\colon \mathrm{Mat}_d(\mathbb{C}) \to \mathrm{Mat}_t(\mathbb{C})$, for which we can assume $t = ds$ and $\pi = \mathrm{id}_d \oplus \cdots \oplus \mathrm{id}_d$, by Theorem 1.4.9. The linear map $V\colon \mathbb{C}^e \to \mathbb{C}^{ds} = \mathbb{C}^d \oplus \cdots \oplus \mathbb{C}^d$ is thus of the form $V = (V_1, \ldots, V_s)$ with $V_i\colon \mathbb{C}^e \to \mathbb{C}^d$ and

$$\psi(M) = V^* \pi(M) V = V^*(M \oplus \cdots \oplus M) V = \sum_i V_i^* M V_i. \qquad △$$

Chapter 3

Operator Systems

We now pass from algebras of operators to linear spaces of operators. This is a genuine weakening: multiplication will usually be lost, but the involution, the unit, and the order structure coming from positivity remain. These remaining structures are still rich enough to support a useful theory of positive and completely positive maps.

A basic source of such spaces is compression. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^*-algebra and let $K \subseteq H$ be a closed subspace. If $P\colon H \to K$ is the orthogonal projection and $P^*\colon K \to H$ the inclusion, then an operator $T \in \mathcal{A}$ restricts to K as $PTP^* \in \mathcal{B}(K)$. Hence

$$P\mathcal{A}P^* := \{PTP^* \mid T \in \mathcal{A}\} \subseteq \mathcal{B}(K).$$

In general this space is no longer closed under multiplication, and therefore is not a C^*-algebra. It is, however, closed under adjoints and contains the identity on K, so it is an example of what we will call an operator system. Another motivation will appear later in Section 4.2.

We first define operator systems concretely as subspaces of C^*-algebras. Later we give an intrinsic axiomatic description and show that the two points of view agree.

3.1 Concrete Operator Systems

Definition 3.1.1. A **(concrete) operator system** is a unital $*$-subspace of a C^*-algebra, i.e. a subspace

$$\mathcal{S} \subseteq \mathcal{A}$$

that is closed under the involution and contains 1. By Theorem 2.6.1, we may equivalently view a concrete operator system as a unital $*$-subspace of some $\mathcal{B}(H)$. We set

$$\mathcal{S}_{\mathrm{sa}} := \{s \in \mathcal{S} \mid s^* = s\}. \qquad \triangle$$

Throughout this section, $\mathcal{S}$ and $\mathcal{T}$ denote operator systems, while $\mathcal{A}$ and $\mathcal{B}$ denote C^*-algebras.

Example 3.1.2. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C^*-algebra and let $P\colon H \to K$ be the orthogonal projection onto a closed subspace. Then $P\mathcal{A}P^* \subseteq \mathcal{B}(K)$ is an operator system. $\triangle$

What makes such subspaces special, compared to arbitrary vector spaces with involution, is the positivity inherited from the ambient C^*-algebra.

Definition 3.1.3. (i) For an operator system $\mathcal{S} \subseteq \mathcal{A}$ we define

$$\mathcal{S}_+ := \mathcal{S} \cap \mathcal{A}_+$$

and call its elements the **positive elements** of $\mathcal{S}$. This is a closed convex cone in $\mathcal{S}_{\mathrm{sa}}$, with respect to the norm inherited from $\mathcal{A}$, and it has algebraic interior point 1.
(ii) For every $s \geqslant 1$ we have

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S} = \mathrm{Mat}_s(\mathcal{S}) \subseteq \mathrm{Mat}_s(\mathcal{A})$$

and we set

$$\mathcal{S}_+^{(s)} := \mathrm{Mat}_s(\mathcal{S}) \cap \mathrm{Mat}_s(\mathcal{A})_+. \qquad \triangle$$

Definition 3.1.4. Let $\mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{T} \subseteq \mathcal{B}$ be operator systems. A $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{T}$ is called s**-positive** if

$$(\mathrm{id}_s \otimes \psi)\left(\mathcal{S}_+^{(s)}\right) \subseteq \mathcal{T}_+^{(s)}$$

holds. It is called **completely positive** if it is s-positive for all $s \geqslant 1$. $\triangle$

Proposition 3.1.5. *Positive maps between operator systems are bounded. If $\varphi\colon \mathcal{S} \to \mathbb{C}$ is a positive functional, then $\|\varphi\|_{\mathrm{op}} = \varphi(1)$.*

Proof. The proof of boundedness is the same as for Proposition 2.7.4, using the decomposition from Exercise 24 instead of Exercise 19. Now assume $\varphi\colon \mathcal{S} \to \mathbb{C}$ is a positive functional. For $s \in \mathcal{S}_{\mathrm{sa}}$ we have $\|s\| - s \in \mathcal{S}_+$ and thus $\varphi(s) \leqslant \|s\|\varphi(1)$. For general $s \in \mathcal{S}$ choose $\theta \in [0, 2\pi)$ with $|\varphi(s)| = e^{i\theta}\varphi(s)$ and consider

$$\tilde{s} := e^{i\theta}s = s_1 + is_2 \in \mathcal{S}$$

with $s_1, s_2 \in \mathcal{S}_{\mathrm{sa}}$ and $\|s_i\| \leqslant \|\tilde{s}\| = \|s\|$. Then

$$\mathbb{R} \ni |\varphi(s)| = \varphi(\tilde{s}) = \underbrace{\varphi(s_1)}_{\in\mathbb{R}} + i\underbrace{\varphi(s_2)}_{\in\mathbb{R}} = \varphi(s_1) \leqslant \|s_1\|\varphi(1) \leqslant \|s\|\varphi(1),$$

by what we have just shown. This proves the claim. □

Proposition 3.1.6. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and let $\varphi\colon \mathcal{S} \to \mathbb{C}$ be a positive functional. Then φ extends to a positive functional $\widetilde{\varphi}\colon \mathcal{A} \to \mathbb{C}$.*

Proof. If $\varphi = 0$, there is nothing to prove. Otherwise, after positive scaling, we may assume $\varphi(1) = 1$, and hence $\|\varphi\|_{\mathrm{op}} = 1$ by Proposition 3.1.5. By the Hahn–Banach theorem there is an extension to a functional $\widetilde{\varphi}$ on $\mathcal{A}$ with $\|\widetilde{\varphi}\| = \|\varphi\| = 1$. Since also $\widetilde{\varphi}(1) = 1$, Proposition 2.5.3 implies that $\widetilde{\varphi}$ is positive. □

The following important result allows us to transfer many facts about completely positive maps from C^*-algebras to operator systems.

Theorem 3.1.7 (Arveson's Extension Theorem). *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system and $\psi\colon \mathcal{S} \to \mathcal{B}(H)$ a completely positive map. Then ψ extends to a completely positive map $\widetilde{\psi}\colon \mathcal{A} \to \mathcal{B}(H)$.*

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "\widetilde{\psi}"] & & . \\
\mathcal{S} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\psi"] & \mathcal{B}(H) &
\end{tikzcd}$$

If $\dim(H) = d < \infty$, then d-positivity of ψ already implies complete positivity.

Proof. First assume $\dim(H) = d < \infty$, so $\mathcal{B}(H) = \mathrm{Mat}_d(\mathbb{C})$. The map $\psi\colon \mathcal{S} \to \mathrm{Mat}_d(\mathbb{C})$ corresponds to a functional $\varphi_\psi\colon \mathrm{Mat}_d(\mathcal{S}) \to \mathbb{C}$ as described in Remark 2.7.17. The same proof as in Theorem 2.7.19 shows that d-positivity of ψ implies positivity of φ_ψ. Since $\mathrm{Mat}_d(\mathcal{S}) \subseteq \mathrm{Mat}_d(\mathcal{A})$ is an operator system, Proposition 3.1.6 guarantees the existence of a positive extension $\widetilde{\varphi}_\psi\colon \mathrm{Mat}_d(\mathcal{A}) \to \mathbb{C}$, which, by Theorem 2.7.19, gives rise to a completely positive map $\widetilde{\psi}\colon \mathcal{A} \to \mathrm{Mat}_d(\mathbb{C})$. Since $\widetilde{\varphi}_\psi$ extends φ_ψ, the corresponding map $\widetilde{\psi}$ extends ψ.

Now in the general case consider the orthogonal projection $P\colon H \to K$ onto a finite-dimensional subspace $K \subseteq H$, and the compressed map

$$\begin{aligned} \psi_K\colon \mathcal{S} &\to \mathcal{B}(K) \\ s &\mapsto P\psi(s)P^*, \end{aligned}$$

which is still completely positive. By the finite-dimensional case, ψ_K has a completely positive extension to $\mathcal{A}$. We then view this extension as a map into $\mathcal{B}(H)$ by letting its values act as zero on $K^\perp$. We denote this extension by

$$\widetilde{\psi}_K\colon \mathcal{A} \to \mathcal{B}(H).$$

The proof of Proposition 2.7.4 shows that the operator norms of the maps $\widetilde{\psi}_K$ are bounded only in terms of

$$\|\widetilde{\psi}_K(1)\| = \|\psi_K(1)\| = \|P\psi(1)P^*\| \leqslant \|\psi(1)\|.$$

Thus, as K ranges over all finite-dimensional subspaces of H, the maps $\widetilde{\psi}_K$ remain bounded in operator norm.

Now consider the coarsest vector space topology on $\mathcal{B}(\mathcal{A}, \mathcal{B}(H))$ that makes the functionals

$$\psi \mapsto \langle \psi(a)h_1, h_2\rangle$$

continuous, for all $a \in \mathcal{A}, h_1, h_2 \in H$. It can be shown that this is actually a weak*-topology[1], for which closed balls in operator norm are compact by Banach-Alaoglu. Thus the net of all $\widetilde{\psi}_K$, where K ranges through all finite-dimensional subspaces, has a subnet that converges to some $\widetilde{\psi} \in \mathcal{B}(\mathcal{A}, \mathcal{B}(H))$. By the definition of the topology, the limit map $\widetilde{\psi}$ is completely positive and extends ψ. □

Corollary 3.1.8. *Let $\mathcal{S} \subseteq \mathrm{Mat}_d(\mathbb{C})$ be an operator system, and let $\psi\colon \mathcal{S} \to \mathrm{Mat}_e(\mathbb{C})$ be a completely positive map. Then there exist $V_1, \ldots, V_r \in \mathrm{Mat}_{d,e}$ with*

$$\psi(M) = \sum_j V_j^* M V_j$$

for all $M \in \mathcal{S}$.

Proof. By Theorem 3.1.7, ψ extends to a completely positive map on $\mathrm{Mat}_d(\mathbb{C})$, to which Corollary 2.7.15 applies. □

---

[1] The pair (h_1, h_2) corresponds to the rank one operator $h \mapsto \langle h, h_1\rangle h_2$ on H, the closed linear span of those are called trace class operators on H, and for the correct notion of tensor product, $\mathcal{B}(\mathcal{A}, \mathcal{B}(H))$ is the dual of $\mathcal{A} \otimes \mathrm{TC}(H)$. The weak*-topology is then exactly the one we consider here.

3.2 Abstract Operator Systems

We now define operator systems intrinsically, without first choosing an embedding into a C^*-algebra. The main result of this section is that this abstract notion is equivalent to the concrete one from the previous section.

Remark 3.2.1. Let $\mathcal{S}$ be a $\mathbb{C}$-vector space with involution $*$. As usual, we denote by $\mathcal{S}_{\mathrm{sa}} = \{s \in \mathcal{S} \mid s^* = s\}$ the real subspace of self-adjoint elements. The space $\mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S} = \mathrm{Mat}_s(\mathcal{S})$ carries the canonical involution, given entrywise by

$$\left(s_{ij}\right)_{i,j}^* \coloneqq \left(s_{ji}^*\right)_{i,j}.$$

We write $\mathrm{Her}_s(\mathcal{S})$ for the self-adjoint elements in $\mathrm{Mat}_s(\mathcal{S})$; equivalently, $\mathrm{Her}_s(\mathcal{S}) = \mathrm{Her}_s(\mathbb{C}) \otimes \mathcal{S}_{\mathrm{sa}}$. △

Definition 3.2.2. Let $C \subseteq X$ be a convex cone in the real vector space X. A point $u \in C$ is an **algebraic interior point** or **order unit** of C if, for every $x \in X$, there exists $\varepsilon > 0$ such that $u - \varepsilon x \in C$. Equivalently, $\lambda u - x \in C$ for all sufficiently large $\lambda > 0$. △

Remark 3.2.3. Algebraic interior points coincide with interior points with respect to the finest locally convex topology on X. If $u \in C$ is such an interior point, then the closure of C consists of all points $x \in X$ such that $x + \varepsilon u \in C$ holds for all $\varepsilon > 0$. △

Definition 3.2.4. Let $\mathcal{S}$ be a $\mathbb{C}$-vector space with involution. An **abstract operator system on** $\mathcal{S}$ consists of a closed convex cone

$$\mathcal{S}_+^{(s)} \subseteq \mathrm{Her}_s(\mathcal{S})$$

for each $s \geqslant 1$, such that:

(i) $\mathcal{S}_+^{(1)} \subseteq \mathcal{S}_{\mathrm{sa}}$ has an algebraic interior point and satisfies $\mathcal{S}_+^{(1)} \cap -\mathcal{S}_+^{(1)} = \{0\}$.

(ii) For all $s, t \geqslant 1, Q \in \mathrm{Mat}_{s,t}(\mathbb{C})$ and $A \in \mathcal{S}_+^{(s)}$ we have $Q^*AQ \in \mathcal{S}_+^{(t)}$.

If $(\mathcal{S}_+^{(s)})_{s\geqslant 1}$ and $(\mathcal{T}_+^{(s)})_{s\geqslant 1}$ are abstract operator systems on $\mathcal{S}$ and $\mathcal{T}$, respectively, then a $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{T}$ is called **completely positive** if $\mathrm{id}_s \otimes \psi$ maps $\mathcal{S}_+^{(s)}$ to $\mathcal{T}_+^{(s)}$, for all $s \geqslant 1$. △

Remark 3.2.5. Under the identification $\mathrm{Mat}_s(\mathcal{S}) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathcal{S}$, condition (ii) says that the compressions $A \mapsto Q^*AQ$ are positive with respect to the family of cones. This is the matrix-level analogue of taking conic combinations, so abstract operator systems are closely related to *noncommutative* or *matrix convex cones*.

In particular, since every positive semidefinite matrix $P \in \mathrm{Mat}_s(\mathbb{C})_+$ is a sum $P = \sum_i v_i v_i^*$ with $v_i \in \mathbb{C}^s$, condition (ii) implies that for every $a \in \mathcal{S}_+^{(1)}$ we have

$$P \otimes a = \sum_i v_i a v_i^* \in \mathcal{S}_+^{(s)},$$

where $v_i a v_i^*$ denotes the matrix with (k, ℓ)-entry $(v_i)_k a \overline{(v_i)_\ell}$. △

Lemma 3.2.6. *Given an abstract operator system and an algebraic interior point* $u \in \mathcal{S}_+^{(1)}$*, the point*

$$u^{(s)} := I_s \otimes u = \mathrm{diag}(u, \dots, u) \in \mathrm{Her}_s(\mathcal{S})$$

is an algebraic interior point of $\mathcal{S}_+^{(s)}$ *for all* $s \geqslant 1$. *Furthermore,* $\mathcal{S}_+^{(s)} \cap -\mathcal{S}_+^{(s)} = \{0\}$ *holds for all* $s \geqslant 1$.

Proof. First note that, by Remark 3.2.5, $u^{(s)}$ indeed belongs to $\mathcal{S}_+^{(s)}$. Now let $A \in \mathrm{Her}_s(\mathcal{S})$ be arbitrary, and write it as $A = \sum_{i=1}^n M_i \otimes s_i$ with $M_i \in \mathrm{Her}_s(\mathbb{C})$ and $s_i \in \mathcal{S}_{\mathrm{sa}}$. Since $u \in \mathcal{S}_+^{(1)}$ is an algebraic interior point, choose $\lambda > 0$ such that $\lambda u \pm s_i \in \mathcal{S}_+^{(1)}$ for all i, and write $M_i = P_i - Q_i$ as a difference of two positive semidefinite matrices. Then

$$\sum_i (P_i + Q_i) \otimes \lambda u - \sum_i M_i \otimes s_i = \sum_i P_i \otimes (\lambda u - s_i) + Q_i \otimes (\lambda u + s_i) \in \mathcal{S}_+^{(s)},$$

again by Remark 3.2.5. Thus if $\gamma \geqslant 0$ is large enough to ensure

$$\gamma I_s - \sum_i (P_i + Q_i) \geqslant 0,$$

then

$$\gamma\lambda\,(I_s \otimes u) - A = \sum_i (P_i + Q_i) \otimes \lambda u - A + \left(\gamma I_s - \sum_i (P_i + Q_i)\right) \otimes \lambda u \in \mathcal{S}_+^{(s)}.$$

This shows that $I_s \otimes u$ is indeed an algebraic interior point of $\mathcal{S}_+^{(s)}$.

Now assume $A, -A \in \mathcal{S}_+^{(s)}$, and write $A = \sum_i M_i \otimes s_i$ with $M_i \in \operatorname{Her}_s(\mathbb{C})$ and $s_i \in \mathcal{S}_{\mathrm{sa}}$ linearly independent. Then for all $v \in \mathbb{C}^s$ we have

$$\pm v^* A v = \pm \sum_i v^* M_i v \cdot s_i \in \mathcal{S}_+^{(1)},$$

thus $\sum_i v^* M_i v \cdot s_i = 0$, and linear independence implies $v^* M_i v = 0$ for all i. This is only possible if all M_i and thus A are zero. □

Remark 3.2.7. In view of Lemma 3.2.6, closedness of the cones $\mathcal{S}_+^{(s)}$ can be understood either with respect to the finest locally convex topology, or algebraically as explained in Remark 3.2.3. △

Example 3.2.8. Every concrete operator system $\mathcal{S} \subseteq \mathcal{A}$ can be viewed as an abstract operator system. Indeed, it comes equipped with the cones $\mathcal{S}_+^{(s)}$ from Definition 3.1.3 (*ii*), and these satisfy the axioms above. In particular, each C^*-algebra can be understood as an abstract operator system. △

Example 3.2.9. Let $C \subseteq \mathcal{S}_{\mathrm{sa}}$ be a closed convex cone with nonempty interior. Then there are several ways to construct an abstract operator system structure with $\mathcal{S}_+^{(1)} = C$.

The *smallest* one is obtained by applying all operations allowed by condition (*ii*) to elements of C, cf. Remark 3.2.5. This yields

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i P_i \otimes c_i \mid P_i \in \operatorname{Mat}_s(\mathbb{C})_+, c_i \in C \right\}.$$

This is called the *smallest abstract operator system over* C, or the operator system *generated by* C, or the *non-commutative conic hull of* C.

The *largest abstract operator system over* C consists of those elements that are mapped to C under all maps from condition (*ii*), i.e.

$$\mathcal{S}_+^{(s)} = \{ A \in \operatorname{Her}_s(\mathcal{S}) \mid \forall v \in \mathbb{C}^s \colon v^* A v \in C \}.$$

The axioms imply that every other abstract operator system with first-level cone C lies, level by level, between these smallest and largest choices, i.e. for all $s \geqslant 1$. △

Theorem 3.2.10 (Effros-Winkler Separation Theorem). *Let* $(\mathcal{S}_+^{(s)})_{s\geqslant 1}$ *be an abstract operator system on* $\mathcal{S}$, *and* $A \in \mathrm{Her}_t(\mathcal{S}) \setminus \mathcal{S}_+^{(t)}$, *for some* $t \geqslant 1$. *Then there is some* $t' \leqslant t$ *and a completely positive map* $\psi\colon \mathcal{S} \to \mathrm{Mat}_{t'}(\mathbb{C})$ *with*

$$(\mathrm{id}_t \otimes \psi)(A) \notin \mathrm{Mat}_{tt'}(\mathbb{C})_+$$

and $\psi(u) = I_{t'}$ *for any chosen interior point* u *of* $\mathcal{S}_+^{(1)}$.

Proof. Since $A \notin \mathcal{S}_+^{(t)}$ and this convex cone is closed, there exists a $*$-linear functional

$$\varphi\colon \mathrm{Mat}_t(\mathcal{S}) = \mathrm{Mat}_t(\mathbb{C}) \otimes \mathcal{S} \to \mathbb{C}$$

which is nonnegative on $\mathcal{S}_+^{(t)}$ and $\varphi(A) < 0$. With the construction from Remark 2.7.17, φ corresponds to a $*$-linear map $\psi\colon \mathcal{S} \to \mathrm{Mat}_t(\mathbb{C})$, and we now proceed as in Theorem 2.7.19. For

$$e := \sum_{i=1}^{t} e_i \otimes e_i \in \mathbb{C}^t \otimes \mathbb{C}^t$$

it is easy to compute that

$$e^*(\mathrm{id}_t \otimes \psi)(A)e = \varphi(A)$$

holds, and $(\mathrm{id}_t \otimes \psi)(A)$ is thus not positive semidefinite. To prove complete positivity of ψ, let $B \in \mathrm{Mat}_s(\mathcal{S})$, $v = \sum_{i=1}^{s} e_i \otimes v_i \in \mathbb{C}^s \otimes \mathbb{C}^t$, and compute

$$v^*(\mathrm{id}_s \otimes \psi)(B)v = \varphi(V^*BV)$$

where $V = (v_1, \ldots, v_s)^* \in \mathrm{Mat}_{s,t}(\mathbb{C})$. Thus if $B \in \mathcal{S}_+^{(s)}$, then $V^*BV \in \mathcal{S}_+^{(t)}$, and since φ is nonnegative on $\mathcal{S}_+^{(t)}$, the expression is nonnegative. This shows that $(\mathrm{id}_s \otimes \psi)(B)$ is positive semidefinite, and hence ψ is completely positive.

If the image of ψ does not meet the interior of the cone $\mathrm{Mat}_t(\mathbb{C})_+$, then its intersection with the positive cone is a face, and each face is isomorphic to a potentially smaller full cone $\mathrm{Mat}_{t'}(\mathbb{C})$, see Exercise 25. After replacing $\mathrm{Mat}_t(\mathbb{C})$ by $\mathrm{Mat}_{t'}(\mathbb{C})$ we can thus assume that for a fixed interior point u of $\mathcal{S}_+^{(1)}$, the matrix $\psi(u) > 0$ is positive definite. Upon conjugation of ψ with a suitable invertible matrix we obtain $\psi(u) = I_{t'}$. □

Theorem 3.2.11 (Choi-Effros Realization Theorem). *Every abstract operator system is isomorphic to a concrete one, i.e. there is an (injective) $*$-linear map $\psi\colon \mathcal{S} \to \mathcal{B}(H)$ for some Hilbert space H, such that*

$$A \in \mathcal{S}_+^{(s)} \Leftrightarrow (\mathrm{id}_s \otimes \psi)(A) \in \mathrm{Mat}_s(\mathcal{B}(H))_+.$$

In addition, we can ensure that ψ maps any fixed interior point of $\mathcal{S}_+^{(1)}$ to id_H.

Proof. We proceed similarly to the proof of Theorem 1.4.18, and take the direct sum of all unital completely positive maps ψ into all matrix algebras. For $s \in \mathcal{S}_{\mathrm{sa}}$ we have $\lambda u \pm s \in \mathcal{S}_+^{(1)}$ for some $\lambda > 0$, and thus $\psi(s)$ is bounded in operator norm by λ for all ψ. Splitting everything in real and imaginary part shows that the same is true for all $s \in \mathcal{S}$. Thus the direct sum of all $\psi(s)$ is an operator on a direct-sum Hilbert space. Complete positivity of the ψ proves "$\Rightarrow$", and "$\Leftarrow$" follows from Theorem 3.2.10. $\square$

Example 3.2.12. Since $\mathrm{Mat}_d(\mathbb{C})$ is a C^*-algebra, it is in particular a concrete operator system. The convex cone at base level is $\mathrm{Mat}_d(\mathbb{C})_+$, the cone at level s is

$$\mathrm{Mat}_{sd}(\mathbb{C})_+ \subseteq \mathrm{Mat}_{sd}(\mathbb{C}) = \mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}).$$

But if we consider $\mathcal{S} = \mathrm{Mat}_d(\mathbb{C})$ with cone $C = \mathrm{Mat}_d(\mathbb{C})_+$, there are other ways to equip the higher levels with convex cones to make it an abstract operator system. For example, the smallest one, as explained in Example 3.2.9, consists of

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i P_i \otimes Q_i \mid P_i \in \mathrm{Mat}_s(\mathbb{C})_+, Q_i \in \mathrm{Mat}_d(\mathbb{C})_+ \right\},$$

whose elements are also called *separable matrices*. The largest one is

$$\mathcal{S}_+^{(s)} = \left\{ \sum_i A_i \otimes B_i \mid \forall v \in \mathbb{C}^s\colon\ \sum_i v^* A_i v \cdot B_i \geqslant 0 \right\},$$

whose elements are called *block positive matrices*. In view of Theorem 3.2.11, each of these systems is concrete, but except for the first one, which is concrete by the very definition, it is not obvious how a concrete realization on a Hilbert space looks like. △

Chapter 4

Some Applications

4.1 Mathematical Quantum Theory

In this section we explain a mathematical formalism for quantum theory, going back to Heisenberg, and describe how it naturally leads to C^*-algebras and to the Dirac–von Neumann axioms.

We cannot give a full account of the experimental and theoretical developments that led to quantum physics. Very roughly, phenomena such as the double-slit experiment with photons or electrons suggest that a physical system need not occupy one definite classical state before measurement. Instead, it may behave as if several classical alternatives coexist until the system is measured, or more generally interacts with its environment. One says that the object is in a *superposition* of classical states.

Assume, for simplicity, that there are two classical states, called 0 and 1. Here, 0 vs. 1 could stand for *charged* vs. *uncharged*, *excited* vs. *non-excited*, or *spin up* vs. *spin down* [1]. Now one could try to model the uncertainty with classical probabilities, by saying that the system is in state 0 with probability p and in state 1 with probability $1 - p$, where $p \in [0, 1]$. This model cannot predict all experimentally observed phenomena. It describes only incomplete knowledge: the system is in one of the two states, but we do not know which one. Quantum theory uses a different model, which is initially surprising but much more successful.

One describes the superposition of classical states by a unit vector in $\mathbb{C}^2$. Mathematicians usually write vectors in $\mathbb{C}^2$ as $(\alpha, \beta)^t$ with $\alpha, \beta \in \mathbb{C}$, and denote the standard basis vectors by e_1 and e_2. In the *bra/ket-notation,* introduced by Dirac,

---

[1] Reddit says: *Imagine a ball that is spinning, except it is not a ball and it is not spinning.*

the two basic vectors are denoted $|0\rangle$ and $|1\rangle$. This resembles the name of classical states as 0 and 1, and the funny symbols $|\ \rangle$ can make certain computations more intuitive. For example, if by $\langle i| := |i\rangle^*$ we denote the conjugate transposed vector, the product $\langle i||j\rangle$ of a row and a column vector is really the standard inner product of the two. To summarize, we can write an element $\varphi \in \mathbb{C}^2$ as

$$\varphi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha e_1 + \beta e_2 = \alpha|0\rangle + \beta|1\rangle.$$

The fact that φ has norm one means

$$|\alpha|^2 + |\beta|^2 = 1,$$

so the two numbers $|\alpha|^2, |\beta|^2$ can indeed be interpreted as classical probabilities for being in the states $|0\rangle$ and $|1\rangle$, respectively. But again, the probabilities alone do not determine the quantum state; the complex amplitudes α and β are essential.

When the object is measured[2], it passes into one of the two classical states $|0\rangle, |1\rangle$, with probabilities $|\alpha|^2, |\beta|^2$. Geometrically, it is projected onto one of the coordinate axes and re-normalized to length one again. The closer it is to an axis, the more likely it is to decide for this classical state.

Now imagine your colleague performs a measurement on the object, but does not tell you the outcome. If you still want to describe the system after the measurement, you must also use classical probabilities. For you, the system is in state $|0\rangle$ with probability $|\alpha|^2$ and in state $|1\rangle$ with probability $|\beta|^2$. But note that this is very different from the initial state $\alpha|0\rangle + \beta|1\rangle$. You *know* that the object is not in a superposition anymore, you just do not know which of the two classical states it is in, and this is just a lack of information on your side. So how to describe the new state? Something like $|\alpha||0\rangle + |\beta||1\rangle$ would for example still indicate a superposition.

The solution is to pass from vectors to matrices. Identify the unit vector $\varphi \in \mathbb{C}^2$ with the matrix

$$\varphi\varphi^* \in \mathrm{Her}_2(\mathbb{C}).$$

This is a positive semidefinite matrix of rank 1 with trace 1:

$$\mathrm{tr}(\varphi\varphi^*) = \mathrm{tr}(\varphi^*\varphi) = \varphi^*\varphi = \|\varphi\|^2 = 1.$$

---

[2] No one knows what counts as a measurement in general.

Indeed, every positive semidefinite rank 1 matrix with trace 1 is of the form $\varphi\varphi^*$, so up to $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ (a so-called *phase*) we can reconstruct the unit vector φ from this matrix. The two classical states correspond to

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we now take a classical probabilistic mixture of these matrices we obtain

$$|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}.$$

This matrix is not of rank 1, and thus does not correspond to one single state as above. From it one can recover the classical probabilities $|\alpha|^2$ for the state $|0\rangle$ and $|\beta|^2$ for $|1\rangle$, indeed as its eigenvalues and eigenvectors. This is exactly what we wanted to describe the classical uncertainty arising from incomplete knowledge. To summarize:

Definition 4.1.1. The **state** of a quantum object which can attain d classical states is described by a positive semidefinite matrix of trace 1 in $\mathrm{Mat}_d(\mathbb{C})$. The state is called **pure** if the matrix is of rank 1. Up to a phase, this corresponds to a unit vector in $\mathbb{C}^d$. Such a pure state contains intrinsic quantum-theoretic uncertainty, via superposition. A state of rank $\geqslant 2$ is called a **mixed state**. By the spectral theorem, each mixed state σ can be written as

$$\sigma = \sum_{i=1}^{d} \lambda_i \varphi_i \varphi_i^*,$$

where $\lambda_i \geqslant 0$ with $\sum_{i=1}^{d} \lambda_i = 1$ are the eigenvalues, and the φ_i form an orthonormal basis of eigenvectors of σ. Thus every state is a classical probabilistic mixture of pure states; the mixture expresses classical uncertainty due to incomplete knowledge. △

When a measurement is performed, a pure state $\varphi \in \mathbb{C}^2$ passes to one of the classical states, with probabilities determined by the entries of φ. We now formulate this more generally in vector and matrix language.

Definition 4.1.2. A **measurement** consists of a decomposition of $\mathbb{C}^d$ into pairwise orthogonal subspaces:

$$\mathbb{C}^d = U_1 \oplus \cdots \oplus U_m.$$

Equivalently, it consists of orthogonal projections $P_1, \ldots, P_m \in \mathrm{Mat}_d(\mathbb{C})$ with $\sum_i P_i = I_d$. If this measurement is performed on an object in state σ, there are m possible outcomes, with classical probabilities

$$\mathrm{tr}(\sigma P_1), \ldots, \mathrm{tr}(\sigma P_m).$$

If the i-th outcome appears, the state passes to the **post-measurement state**

$$\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \sigma P_i^* \in \mathrm{Her}_d(\mathbb{C}). \qquad \triangle$$

Let us check that these definitions make sense and agree with the preceding discussion. First, all numbers $\mathrm{tr}(\sigma P_i)$ are nonnegative, since states and orthogonal projections are positive semidefinite matrices, and the cone of positive semidefinite matrices is self-dual with respect to the trace inner-product. Second, these nonnegative numbers sum to one, and can thus indeed be interpreted as classical probabilities:

$$\sum_i \mathrm{tr}(\sigma P_i) = \mathrm{tr}\left(\sum_i \sigma P_i\right) = \mathrm{tr}\left(\sigma \sum_i P_i\right) = \mathrm{tr}(\sigma I_d) = \mathrm{tr}(\sigma) = 1.$$

Third, the post-measurement states are indeed states. Positive semidefiniteness is obvious, and

$$\mathrm{tr}\left(\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \sigma P_i^*\right) = \frac{\mathrm{tr}(P_i \sigma P_i^*)}{\mathrm{tr}(\sigma P_i)} = \frac{\mathrm{tr}(\sigma P_i^* P_i)}{\mathrm{tr}(\sigma P_i)} = \frac{\mathrm{tr}(\sigma P_i)}{\mathrm{tr}(\sigma P_i)} = 1.$$

To compare this with the preceding explanation, assume that $\sigma = \varphi\varphi^*$ with $\varphi \in \mathbb{C}^d$ is a pure state. Then

$$\mathrm{tr}(\sigma P_i) = \mathrm{tr}(\varphi\varphi^* P_i) = \varphi^* P_i \varphi = \langle P_i \varphi, \varphi \rangle$$

and this is the squared length of the projection of φ to U_i. This is exactly the probability rule described above. The post-measurement state upon observation of outcome i is

$$\frac{1}{\mathrm{tr}(\sigma P_i)} P_i \varphi\varphi^* P_i^*,$$

which is the pure state corresponding to the normalized vector $P_i\varphi$. So just as in the beginning, the post-measurement state is the normalized projection to the subspace. Thus the geometric interpretation remains the same: the closer the vector is to the subspace U_i, the more likely it is to collapse to it during measurement. Also note that the probabilities $\operatorname{tr}(\sigma P_i)$ are linear in σ, so a classical mixture of pure states leads to the same mixture of probabilities in the measurement. Finally, if the measurement is performed without observing the outcome, the post-measurement state has to be considered as the classical probabilistic mixture

$$\sum_{i=1}^{m} \operatorname{tr}(\sigma P_i) \frac{1}{\operatorname{tr}(\sigma P_i)} P_i \sigma P_i^* = \sum_{i=1}^{m} P_i \sigma P_i^*.$$

We have now defined states as positive semidefinite matrices of trace 1, and hence as elements of $\operatorname{Mat}_d(\mathbb{C})$. Since this matrix algebra is naturally identified with its dual space via the trace pairing, we can equivalently view states as linear functionals on a C^*-algebra. This brings us closer to the notion of a state on a C^*-algebra that we have already defined. It is also motivated by the following consideration: Measurements are defined by orthogonal projections, which are elements satisfying $P_i = P_i^* = P_i^2$. Stating this requires the elements to live in a $*$-algebra. States, however, need not themselves be algebra elements. Considered as linear functionals, the probabilities $\operatorname{tr}(\sigma P_i)$ are just the value that the functional takes on the element P_i. The trace one condition means that the functional defined by σ maps I_d to 1. This leads to a formulation close to what is usually called the Dirac–von Neumann axioms.

Definition 4.1.3. Let $\mathcal{A}$ be a C^*-algebra.

(*i*) A **state** is a positive linear functional on $\mathcal{A}$ that maps 1 to 1. A state is **pure** if it cannot be written nontrivially as a convex combination of other states.

(*ii*) A **measurement** consists of projections $p_1, \dots, p_m \in \mathcal{A}$ (i.e. $p_i = p_i^* = p_i^2$) with $p_1 + \cdots + p_m = 1$.

(*iii*) If a measurement $p_1, \dots, p_m$ is performed in the state φ, the *probability of outcome i* is given by $\varphi(p_i)$. If this outcome occurs, the **post-measurement state** is

$$\varphi_i := \frac{1}{\varphi(p_i)} \varphi(p_i^* \cdot p_i),$$

where $\varphi(p_i^* \cdot p_i)$ denotes the functional $a \mapsto \varphi(p_i^* a p_i)$. If the outcome is not observed, the post-measurement state is modeled as

$$\widetilde{\varphi} = \sum_i \varphi(p_i)\varphi_i = \sum_i \varphi(p_i^* \cdot p_i). \qquad \triangle$$

We now explain the notion of *entanglement*, which can arise when two or more quantum systems are combined. For simplicity, we will go back to the setup from the start of the section, i.e. consider states as positive semidefinite matrices. Of course, by self-duality, this makes no difference to the concept of states as positive functionals on a matrix algebra.

Now assume we have two quantum systems whose states are described by positive matrices from $\mathrm{Mat}_d(\mathbb{C})$ and $\mathrm{Mat}_e(\mathbb{C})$, respectively. The states of the combined system are then described by positive matrices from

$$\mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C}) \cong \mathrm{Mat}_{de}(\mathbb{C}),$$

and also called *bipartite states*. Note that positivity is defined in the matrix algebra on the right in the usual sense. However, for the definition of entanglement, the decomposition as a tensor product is crucial.

Definition 4.1.4. Let $\sigma \in \mathrm{Mat}_{de}(\mathbb{C})_+$ be a state. Then σ is called **separable** (with respect to the above tensor decomposition), if it can be written as

$$\sigma = \sum_i \sigma_{1i} \otimes \sigma_{2i}$$

with $\sigma_{1i} \in \mathrm{Mat}_d(\mathbb{C})_+$ and $\sigma_{2i} \in \mathrm{Mat}_e(\mathbb{C})_+$. A state which is not separable is called **entangled**. See Exercise 26 for what this means for pure states in unit-vector notation.

Lemma 4.1.5. *Let* $\psi \colon \mathrm{Mat}_e(\mathbb{C}) \to \mathrm{Mat}_e(\mathbb{C})$ *be a positive map. If* $\sigma \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C})$ *is positive, but* $(\mathrm{id}_d \otimes \psi)(\sigma)$ *is not positive, then* σ *is entangled.*

Proof. Assume that σ is separable, and write $\sigma = \sum_i \sigma_{1i} \otimes \sigma_{2i}$ with positive σ_{ji}. Then

$$(\mathrm{id}_d \otimes \psi)(\sigma) = \sum_i \sigma_{1i} \otimes \psi(\sigma_{2i})$$

is again positive (even separable), since ψ is positive. □

Example 4.1.6. Consider the matrix

$$E = \sum_{i,j=1}^{2} E_{ij} \otimes E_{ij} \in \mathrm{Mat}_2(\mathbb{C}) \otimes \mathrm{Mat}_2(\mathbb{C}) = \mathrm{Mat}_4(\mathbb{C})$$

from Example 2.7.11. In quantum physics notation, where $|00\rangle$ is short for $|0\rangle \otimes |0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$, we have

$$E = (|00\rangle + |11\rangle)(\langle 00| + \langle 11|),$$

so E is actually a pure state (we omit the normalization for ease of notation). The transposition τ is a positive map, and we have already seen in Example 2.7.11 that $(\mathrm{id}_2 \otimes \tau)(E)$ is not positive. This shows that E is entangled. $\triangle$

Remark 4.1.7. A positive map that is not completely positive is also called an *entanglement witness.* It can prove for at least some positive matrices that they are not separable, and thus entangled. In this sense, the existence of entanglement is the dual statement to the existence of positive maps that are not completely positive. $\triangle$

Finally, let us indicate what entanglement can be used for. Let $\sigma \in \mathrm{Mat}_d(\mathbb{C}) \otimes \mathrm{Mat}_e(\mathbb{C}) = \mathrm{Mat}_{de}(\mathbb{C})$ be a bipartite state. It describes a combined system with two parts (for example two electrons). Now assume that a measurement is performed only on the first part of the system. If this measurement is described by the projection operators $P_1, \ldots, P_m \in \mathrm{Mat}_d(\mathbb{C})$, then the measurement for the full system is described by $P_1 \otimes I_e, \ldots, P_m \otimes I_e$. If this measurement yields the i-th outcome, then the post-measurement state, up to normalization, is

$$(P_i \otimes I_e)\sigma(P_i \otimes I_e)^*.$$

For entangled states, this can have a non-trivial effect on the state of the second system, which has not been measured.

Example 4.1.8. Consider the state E from Example 4.1.6 again, and assume that a measurement with respect to the standard basis of $\mathbb{C}^2$ is performed on the first part of the system. If the first outcome appears, the state becomes

$$(P_1 \otimes I_2)E(P_1 \otimes I_2)^* = \sum_{i,j} P_1 E_{ij} P_1^* \otimes E_{ij} = E_{11} \otimes E_{11} = |00\rangle\langle 00|.$$

Thus not just the state of the first system has collapsed to $|0\rangle$, but also the state of the second system. In this way, strong correlations between measurements of the two subsystems arise. $\triangle$

4.2 Free Convexity and Optimization

We finish by indicating how the language of operator systems clarifies a natural question in convex optimization. The point is that ordinary convexity sees only positivity at one matrix level, while complete positivity records positivity at all matrix levels at once.

Definition 4.2.1. A *spectrahedron* is a convex cone of the form

$$S = \{a \in \mathbb{R}^m \mid a_1 M_1 + \cdots + a_m M_m \geqslant 0\},$$

where $M_1, \ldots, M_m \in \mathrm{Her}_d(\mathbb{C})$. △

Thus S is the inverse image of the positive semidefinite cone $\mathrm{Mat}_d(\mathbb{C})_+$ under the linear map

$$\begin{aligned} L_M \colon \mathbb{R}^m &\to \mathrm{Her}_d(\mathbb{C}) \\ a &\mapsto a_1 M_1 + \cdots + a_m M_m. \end{aligned}$$

If L_M is injective, we can identify S with the intersection of $\mathrm{Mat}_d(\mathbb{C})_+$ with an m-dimensional real subspace.

Spectrahedra are central objects in semidefinite optimization, where one optimizes a linear function over an affine slice of such a cone. Many semidefinite programs can be solved efficiently by interior point methods.

From our point of view, a spectrahedron is the first-level positive cone of an operator system. Assuming L_M is injective, set

$$\mathcal{S} = \mathrm{span}_{\mathbb{C}}\{M_1, \ldots, M_m\} \subseteq \mathrm{Mat}_d(\mathbb{C}).$$

Then $\mathcal{S}_{\mathrm{sa}} \cong \mathbb{R}^m$ and $\mathcal{S}_+ = S$. The higher cones $\mathcal{S}_+^{(s)}$ are therefore matrix-level refinements of the same spectrahedron.

A basic containment problem, considered in [2], is the following. Given

$$M_1, \ldots, M_m \in \mathrm{Her}_d(\mathbb{C}), \qquad N_1, \ldots, N_m \in \mathrm{Her}_e(\mathbb{C}),$$

decide whether the spectrahedron defined by the N_i contains the one defined by the M_i, i.e. whether

$$\sum_i a_i M_i \geqslant 0 \;\Rightarrow\; \sum_i a_i N_i \geqslant 0 \tag{4.1}$$

holds for all $a \in \mathbb{R}^m$. Since this problem is difficult, one considers the stronger condition that there exist matrices $V_1, \ldots, V_r \in \mathrm{Mat}_{d,e}(\mathbb{C})$ with

$$\sum_j V_j^* M_i V_j = N_i \tag{4.2}$$

for all $i = 1, \ldots, m$. This condition certainly implies (4.1): if $\sum_i a_i M_i \geqslant 0$, then

$$\sum_i a_i N_i = \sum_i a_i \sum_j V_j^* M_i V_j = \sum_j V_j^* \left(\sum_i a_i M_i \right) V_j \geqslant 0.$$

However, (4.2) is not necessary for (4.1) in general. Operator systems explain exactly what extra information (4.2) captures.

Let $\mathcal{S} = \mathrm{span}_{\mathbb{C}}\{M_1, \ldots, M_m\}$ and consider the $*$-linear map

$$\psi \colon \mathcal{S} \to \mathrm{Mat}_e(\mathbb{C}), \qquad M_i \mapsto N_i.$$

Then (4.1) says precisely that ψ is positive. On the other hand, by Corollary 3.1.8, the representation (4.2) is equivalent to complete positivity of ψ. Thus the relaxation does not merely ask for positivity of ψ on $\mathcal{S}_+$; it asks for positivity of $\mathrm{id}_s \otimes \psi$ on every matrix cone $\mathcal{S}_+^{(s)}$.

Concretely, $\mathcal{S}_+^{(s)}$ consists of those elements $\sum_i A_i \otimes M_i$ that are positive in

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) = \mathrm{Mat}_{sd}(\mathbb{C}).$$

Since

$$(\mathrm{id}_s \otimes \psi) \left(\sum_i A_i \otimes M_i \right) = \sum_i A_i \otimes N_i,$$

complete positivity of ψ is equivalent to the implication

$$\sum_i A_i \otimes M_i \geqslant 0 \ \Rightarrow \ \sum_i A_i \otimes N_i \geqslant 0$$

for all $s \geqslant 1$ and all $A_1, \ldots, A_m \in \mathrm{Her}_s(\mathbb{C})$.

This is the free, or matrix-valued, version of spectrahedral containment. The *free spectrahedron* associated to $M_1, \ldots, M_m$ is the sequence of cones

$$S_s := \left\{ (A_1, \ldots, A_m) \in \mathrm{Her}_s(\mathbb{C})^m \mid \sum_i A_i \otimes M_i \geqslant 0 \right\}, \qquad s \geqslant 1.$$

Thus (4.2) is equivalent to inclusion of the corresponding free spectrahedra at all matrix levels. This is stronger than the ordinary inclusion (4.1), which only tests scalar coefficients, i.e. level 1. Operator system theory therefore gives a conceptual explanation for the gap between ordinary spectrahedral containment and the relaxation proposed in [2].

Exercises

Exercise 1. Show that if a sequence in a Banach space converges weakly, then it is bounded in norm.

Exercise 2. Let $\mathcal{A}$ be a unital $*$-algebra. Show the following:

(i) $1^* = 1$.

(ii) $a \in \mathcal{A}^\times \Leftrightarrow a^* \in \mathcal{A}^\times$.

(iii) $\mathcal{A}_{\rm sa} + i\mathcal{A}_{\rm sa} = \mathcal{A}$.

(iv) $\sum \mathcal{A}^2 - \sum \mathcal{A}^2 = \mathcal{A}_{\rm sa}$.

Exercise 3. (i) Show that states on $*$-algebras fulfill $\varphi(a^*) = \overline{\varphi(a)}$ (i.e. they are $*$-linear).
(ii) Show that if $\varphi\colon \mathcal{A} \to \mathbb{C}$ is $*$-linear, nonnegative on $\sum \mathcal{A}^2$, and fulfills $\varphi(1) = 0$, then $\varphi = 0$.

Exercise 4. Complete the proof of Burnside's Theorem 1.3.3: If $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ is a subalgebra that acts transitively on $\mathbb{C}^d \setminus \{0\}$ and contains a matrix of rank 1, then $\mathcal{A} = \mathrm{Mat}_d(\mathbb{C})$.

Exercise 5. Let $\mathcal{A} \subseteq \mathrm{Mat}_d(\mathbb{C})$ be a $*$-subalgebra. Show that if $V \subseteq \mathbb{C}^d$ is an $\mathcal{A}$-invariant subspace, then $V^\perp$ is also $\mathcal{A}$-invariant.

Exercise 6. Prove that φ_1 and φ_2 from Example 1.4.5 are states on the group algebra $\mathbb{C}\Gamma$.

Exercise 7. Prove Lemma 1.4.12.

Exercise 8. Fill in all details in the GNS construction.

Exercise 9. Let $\mathcal{D}$ be an inner product space, and assume $\pi\colon \mathcal{A} \to \mathcal{L}(\mathcal{D})$ is a $*$-representation for which $\pi(a)$ is a bounded operator on $\mathcal{D}$ for all $a \in \mathcal{A}$. Show that π extends to a bounded $*$-representation

$$\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

on the completion $\mathcal{H}$ of $\mathcal{D}$.

Exercise 10. Show that in a C^*-algebra we have

(i) $1^* = 1$,

(ii) $\|1\| = 1$,

(iii) $\|a\| = \|a^*\|$ for all elements a.

Exercise 11. Let a_n be nonnegative real numbers with $a_{n+m} \leqslant a_n a_m$ for all n, m. Prove that the sequence $(a_n^{1/n})_{n\in\mathbb{N}}$ converges to $\inf_{n\in\mathbb{N}} a_n^{1/n}$.

Exercise 12. Let A_0 and A_1 be the algebras of all complex 2×2-matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix},$$

respectively. Prove that every two-dimensional complex unital algebra A is isomorphic to one of these, and that A_0, A_1 are not isomorphic. Hint: Show that A has a basis $\{1, a\}$ in which $a^2 = \lambda 1$ for some $\lambda \in \mathbb{C}$. Distinguish between the cases $\lambda = 0, \lambda \neq 0$.

Exercise 13. Show that there exists a three-dimensional noncommutative Banach algebra.

Exercise 14. Show that every invertible element a in a C^*-algebra $\mathcal{A}$ can be written as $a = up$ with a unitary element u and a positive invertible element p satisfying $\|p\| = \|a\|$.

Exercise 15. Let Γ be a group and consider the group $*$-algebra $\mathcal{A} = \mathbb{C}\Gamma$.
(i) Show that

$$\|a\| := \inf\{r \geqslant 0 \mid r^2 - a^*a \in \sum \mathcal{A}^2\}$$

is a norm on $\mathcal{A}$, fulfilling the C^*-identity.
(ii) Show that the GNS construction, applied to the state φ_2 from Example 1.4.5, gives rise to an injective $*$-representation of $\mathcal{A}$.
(iii) Show that the operator norm induced on $\mathcal{A}$ by the representation from (ii) is smaller than or equal to the norm from (i).

Exercise 16. Compute and compare the two (potentially different) C^*-algebras arising from Exercise 15 (i) and (ii) for the group $\Gamma = \mathbb{Z}^n$.

Exercise 17. Let X be a compact Hausdorff space. Show that there is a bijection between closed subsets of X and closed ideals in $\mathcal{C}(X)$.

Exercise 18. Let $\mathcal{A}$ be a Banach algebra with an involution which satisfies

$$\|a\|^2 \leqslant \|a^*a\|$$

for all $a \in \mathcal{A}$. Show that $\mathcal{A}$ is a C^*-algebra.

Exercise 19. Show that every element in a C^*-algebra can be written as

$$a = a_1 - a_2 + ia_3 - ia_4$$

with positive elements a_i and $\|a_i\| \leqslant \|a\|$.

Exercise 20. Let $L_{ij} \in \mathcal{B}(H)$ be bounded operators on a Hilbert space, for $i, j = 1, \ldots, s$. Show that the operator L on H^s defined by the matrix $(L_{ij})_{i,j} \in \mathrm{Mat}_s(\mathcal{B}(H))$ is bounded with

$$\|L\|_{\mathrm{op}}^2 \leqslant \sum_{i,j} \|L_{ij}\|_{\mathrm{op}}^2.$$

Exercise 21 (Spectral Mapping Theorem)**.** Let a be a normal element of a C^*-algebra and let $f \in \mathcal{C}(\sigma(a))$. Show that the spectrum of $f(a)$ is $f(\sigma(a))$.

Exercise 22. Prove $\mathrm{Mat}_s(\mathcal{B}(H)) \cong \mathcal{B}(H^s)$.

Exercise 23. Let $\mathcal{A}$ be a C^*-algebra. Prove that the map

$$\begin{aligned} \mathrm{tr}\colon \mathrm{Mat}_s(\mathcal{A}) &\to \mathcal{A} \\ (a_{ij})_{i,j} &\mapsto \sum_i a_{ii} \end{aligned}$$

is completely positive.

Exercise 24. Show that every element in an operator system can be written as

$$s = s_1 - s_2 + is_3 - is_4$$

with positive elements s_i and $\|s_i\| \leqslant \|s\|$.

Exercise 25. Show that every face of $\mathrm{Mat}_t(\mathbb{C})_+$ is isomorphic to $\mathrm{Mat}_{t'}(\mathbb{C})_+$ for some $t' \leqslant t$. More explicitly, after conjugation with a suitable invertible matrix, the face contains all matrices of the form

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

with $P \in \mathrm{Mat}_{t'}(\mathbb{C})_+$.

Exercise 26. Let $\sigma \in \mathbb{C}^d \otimes \mathbb{C}^e$ be a pure state (in vector notation). Show that σ is separable (which we have only defined in matrix notation) if and only if the *tensor rank* of σ is one, i.e. if $\sigma = x \otimes y \in \mathbb{C}^d \otimes \mathbb{C}^e$ is an elementary tensor.

Bibliography

[1] W. Arveson. *An invitation to C^*-algebras*. Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976.

[2] A. Ben-Tal and A. Nemirovski. On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM J. Optim.*, **12 (3)**, 811–833, 2002.

[3] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*, vol. 15 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[4] G. J. Murphy. *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.

[5] V. Paulsen. *Completely bounded maps and operator algebras*, vol. 78 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002.

[6] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edn., 1991.

[7] M. Takesaki. *Theory of operator algebras. I*, vol. 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[8] ———. *Theory of operator algebras. II*, vol. 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.

[9] ———. *Theory of operator algebras. III*, vol. 127 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.