Leopold-Franzens University Innsbruck

Department of Mathematics

Bachelor thesis

Theorem of Seifert and van Kampen

Student : Kenneth Ehi
Submission Date: August 7, 2023
Supervisor: Univ.-Prof. Dipl.-Math. Dr. Tim Netzer

Eidesstattliche Erklärung

Ich erkläre hiermit an Eides statt durch meine eigenhändige Unterschrift, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Alle Stellen, die wörtlich oder inhaltlich den angegebenen Quellen entnommen wurden, sind als solche kenntlich gemacht.
Ich erkläre mich mit der Archivierung der vorliegenden Bachelorarbeit einverstanden.

Contents

1 Introduction

The Theorem of Seifert and van Kampen is a powerful statement in Algebraic Topology that has many applications including the theory of covering spaces, groups, and homotopy. The theorem essentially states that if we have two open sets in a given space with a common intersection, and we know the fundamental groups of these sets and their intersection, then we can compute the fundamental group of the entire space. This allows us to determine the fundamental group of spaces where a direct approach would be much more challenging. In this thesis I will give a short introduction to Algebraic Topology. We will provide the most important definitions, prove some interesting facts about covering spaces and learn about the deep connection between group theory and topology.

2 Fundamentals

All of the definitions,theorems and propositions in this chapter were taken from [1],[2],[3],[4],[5],[6].

2.1 Category Theory

Definition 2.1. (Category) A category $\mathcal{C}$ consists of:

1. A class $Obj(\mathcal{C})$ who's elements are called objects
2. For every 2 objects $X, Y \in Obj(\mathcal{C})$ a class $\mathcal{C}(X, Y)$ who's elements are called morphisms
3. For every three objects $X, Y, Z \in Obj(\mathcal{C})$ a binary operation $\circ$ on morphisms called the *composition of morphisms*

$$comp : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$$
$$(f, g) \to g \circ f$$

such that for any morphisms $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(W, X), h \in \mathcal{C}(Y, Z)$:

$$h \circ (f \circ g) = (h \circ f) \circ g$$

4. For every object $X \in Obj(\mathcal{C})$ a morphism $id_X : X \to X$ called the identity such that for any Y and every morphism $f \in \mathcal{C}(X, Y)$

$$f \circ id_X = f$$
$$id_Y \circ f = f$$

A morphism $f : X \to Y$ is called *isomorphism* if there exists a morphism $g : Y \to X$ such that:

$$f \circ g = id_Y$$
$$g \circ f = id_X$$

Definition 2.2. (Small category) A category $\mathcal{C}$ is called small category if $Obj(\mathcal{C})$ and $\mathcal{C}(X, Y)$ are sets.

Definition 2.3. (Functor) Let $\mathcal{C}_1, \mathcal{C}_2$ be categories. A functor F consists of 2 mappings:

- $F : Obj(\mathcal{C}_1) \to Obj(\mathcal{C}_2)$
- $F : \mathcal{C}_1(X, Y) \to \mathcal{C}_2(F(X), F(Y)), \forall X, Y \in Obj(\mathcal{C})$

such that for every object X and every two morphisms f, g :

$$F(id_X) = id_{F(X)}$$
$$F(g \circ f) = F(g) \circ F(f)$$

Definition 2.4. (Diagram) Let $\mathcal{C}$ be a category and $\mathcal{D}$ a small category A diagram of type $\mathcal{D}$ in $\mathcal{C}$ is a functor $F : \mathcal{D} \to \mathcal{C}$. A morphism of diagrams is a *natural transformation*.

Definition 2.5. (Natural Transformation) Let $\mathcal{C}_1$,$\mathcal{C}_2$ be categories and $F_1, F_2 : \mathcal{C}_1 \to \mathcal{C}_2$ be functors. A natural transformation $\alpha : F_1 \to F_2$ consists of morphisms $\alpha_X : F_1(X) \to F_2(X)$ for every $X \in Obj(\mathcal{C}_1)$ such that the following diagram commutes for all $f \in \mathcal{C}_1(X, Y)$:

$$\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "\alpha_X"'] & F_1(Y) \arrow[d, "\alpha_Y"] \\
F_2(X) \arrow[r, "F_2(f)"] & F_2(Y)
\end{tikzcd}$$

Definition 2.6. (Retraction) Consider a category $\mathcal{C}$ and $X, Y \in Obj(\mathcal{C})$ and two morphisms $r : X \to Y$, $s : Y \to X$. If $rs = id_Y$ then r is called a *retraction* of s and s is called a *section* of r.

Definition 2.7. (Cocone) Let $F : \mathcal{D} \to \mathcal{C}$ be a diagram. A cocone in F is a pair (x, α) with $x \in Obj(\mathcal{C})$ and α a family of morphisms $\alpha_d : F(d) \to x$ for each object $d \in Obj(\mathcal{D})$ such that for each morphism $f \in \mathcal{D}(d_1, d_2)$,$d_1, d_2 \in Obj(\mathcal{D})$ with $F(f) : F(d_1) \to F(d_2) \in \mathcal{C}(F(d_1), F(d_2))$ the following diagram commutes:

$$\begin{tikzcd}
F(d_1) \arrow[rr, "F(f)"] \arrow[dr, "\alpha_{d_1}"'] & & F(d_2) \arrow[dl, "\alpha_{d_2}"] \\
& x &
\end{tikzcd}$$

Definition 2.8. (Colimit) Let $F : \mathcal{D} \to \mathcal{C}$ be a diagram. Let $\mathcal{K}$ be the category having all cocones of F as objects and and as morphisms from the cone (x, α) to the cocone (y, β) all morphisms $\eta : x \to y$ such that for each $d \in Obj(\mathcal{D})$ the following diagram commutes:

$$\begin{tikzcd}
x \arrow[rr, "\eta"] & & y \\
& F(d) \arrow[ul, "\alpha_d"] \arrow[ur, "\beta_d"'] &
\end{tikzcd}$$

$(u, \gamma) \in Obj(\mathcal{K})$ is called a colimit of F if for any cocone $(x, \alpha) \in Obj(\mathcal{K})$ there exists a unique morphism $\lambda_x : u \to x$ such that for every $d \in Obj(\mathcal{D})$ the following diagram commutes:

$$\begin{tikzcd}
u \arrow[rr, "\lambda_x"] & & x \\
& F(d) \arrow[ul, "\gamma_d"] \arrow[ur, "\alpha_d"'] &
\end{tikzcd}$$

I.e we can factorize over the cocone (u, γ). We obtain the following commutative diagram:

$$\begin{tikzcd}
F(d_1) \arrow[rr, "F(f)"] \arrow[dr, "\gamma_{d_1}"] \arrow[ddr, "\alpha_{d_1}"'] & & F(d_2) \arrow[dl, "\gamma_{d_2}"'] \arrow[ddl, "\alpha_{d_2}"] \\
& u \arrow[d, dotted, "\lambda"] & \\
& x &
\end{tikzcd}$$

Following are two examples of colimtits.

Definition 2.9. (Coproduct) Consider a category $\mathcal{C}$. Let X, Y, Z be objects of $\mathcal{C}$. A coproduct is an object P together with two morphisms $i_1 : X \to P$,$i_2 : Y \to P$ such that for every two morphisms $f : X \to Z$, $g : Y \to Z$ there exist a unique morphism $u : P \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "i_1"'] & P \arrow[u, dotted, "u"] & Y \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd}$$

Definition 2.10. (Pushout) Let $\mathcal{C}$ be a category, $f \in \mathcal{C}(X,Y), g \in \mathcal{C}(X,Z)$ two morphisms with common domain. A *pushout* of f, g consists of

- an object $P \in Obj(\mathcal{C})$
- two morphisms $i_1 : Y \to P, i_2 : Z \to P$ that satisfy:

$$i_2 \circ g = i_1 \circ f$$

such that for any $Q \in Obj(\mathcal{C})$ together with two morphisms $j_1 : Y \to Q, j_2 : z \to Q$ that satisfy $j_1 \circ f = j_2 \circ g$ there exists a unique $p : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\ Y \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"] & P \arrow[dr, dotted, "p"] & \\ & & Q \end{tikzcd}$$

Definition 2.11. (Lift) Given two morphisms $f : X \to Z$, $g : Y \to Z$ a $lift$ of f to Y is a map $h : X \to Y$ such that:

$$f = g \circ h$$

Theorem 2.12. Let C, D be commutative squares in a category $\mathcal{C}$, such that D is a pushout. If there is a retraction $r : D \to C$, then C is a pushout.

Proof. Consider the commutative squares

$$\begin{tikzcd} C_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & C_1 \arrow[d, "u_1"] \\ C_2 \arrow[r, "u_2"'] & C \end{tikzcd}$$

$$\begin{tikzcd} D_0 \arrow[r, "j_1"] \arrow[d, "j_2"'] & D_1 \arrow[d, "v_1"] \\ D_2 \arrow[r, "v_2"'] & D \end{tikzcd}$$

Let $c : C \to D, d : D \to C$ be morphisms such that $dc = 1_C$. Suppose we are given morphisms

$$w_k : C_k \to W$$

such that $w_1 i_1 = w_2 i_2$. Consider the morphisms

$$w_k d_k : D_k \to W$$

Since d is a map of squares , $d_1 j_1 = i_1 d_0$, $d_2 j_2 = i_2 d_0$ so

$$(w_1 d_1) j_1 = (w_2 d_2) j_2$$

since D is a pushout, there is a unique morphism $p : D \to W$ such that

$$pv_1 = w_1 d_1, pv_2 = w_2 d_2$$

Let $w = pc : C \to W$. Then for $k \in \{1, 2\}$

$$wu_k = pcu_k = pv_k c_k = w_k d_k c_k = w_k$$

as required. Suppose $\tilde{w} : C \to W$ also satisfied $\tilde{w}_k u_k = w_k$, $k \in \{1, 2\}$. Then

$$\tilde{w} dv_k = \tilde{w} u_k d_k = w_k d_k = pv_k$$

Hence

$$\tilde{w} d = p \text{ and so } \tilde{w} = \tilde{w} dc = pc = w$$

□

2.2 Topology

Definition 2.13. (Topology) Consider a non empty set X. A *topology* on X is a subset T of $P(X)$ that satisfies the following:

- $\varnothing, X \in T$
- $A, B \in X \Rightarrow A \cap B \in X$
- $A_i \in X \Rightarrow \bigcup_{i \in I} A_i \in X$

We call sets $A \in T$ open and sets $X \backslash A$ closed. (X, T) is called a topological space.

Definition 2.14. (Basis) Let (X, T) be a topological space then a set $\mathcal{B} \subset P(X)$ such that

$$T = \{\bigcup_{B \in \mathcal{M}} B | \mathcal{M} \subseteq \mathcal{B}\}$$

is called a *basis* of (X, T).

Definition 2.15. (Continuous map) Consider two topological spaces $(X, T), (Y, G)$. A map $f : X \to Y$ is called continuous if and only if: $\forall\, A \in G : f^{-1}(A) \in T$.

Remark 2.16. We can now define the category TOP. The *objects* of TOP are *topological spaces*, *morphisms* between topological spaces are *continuous maps*.

Definition 2.17. (Quotient topology) Consider an equivalence relation $\backsim$ on X together with the canonic projection $p : X \to X/\backsim$. We can define:

$$T/\backsim = \{A \subseteq X/\backsim | \, p^{-1}(A) \in T\}$$

Definition 2.18. (Subspace topology) Given a topological space (X, T) and a subset $A \subseteq X$. Then

$$T_A = \{A \cap B | \, B \in T\}$$

is the *subspace topology* on A.

Definition 2.19. (Adjunction space) Consider two topological spaces X, Y together with a map $f : A \subset X \to Y$ called an attaching map. Let $X \cup Y$ be the disjoint union and $j_k : X, Y \to X \cup Y$ be the respective inclusions. We define an equivalence relation $\backsim$ where $x \backsim y \Leftrightarrow y = f(x)$ for $x \in A, y \in Y$. Additionally let $i : A \to X$ be the inclusion. The following pushout is called an adjunction space:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "j_1"] \\
Y \arrow[r, "j_2"] & (X \cup Y)/\backsim
\end{tikzcd}$$

We will use the notations $(X \cup Y)/\backsim = X \cup_f Y = X \cup_\backsim Y$

Definition 2.20. (Wedge sum) Consider pointed topological spaces $(X_i, x_i), i \in I$. The wedge sum $\bigvee_i (X_i, x_i)$ is a quotient of the disjoint union $\bigcup X_i$ that is obtained by identifying the x_i with a single point. The wedged sum of circles $S^1 \subset \mathbb{R}$ is called a bouquet of circles.

Definition 2.21. (N-cell)

$$e^n = \{x \in \mathbb{R}^n | ||x|| < 1\}$$

is called an *n-cell*.

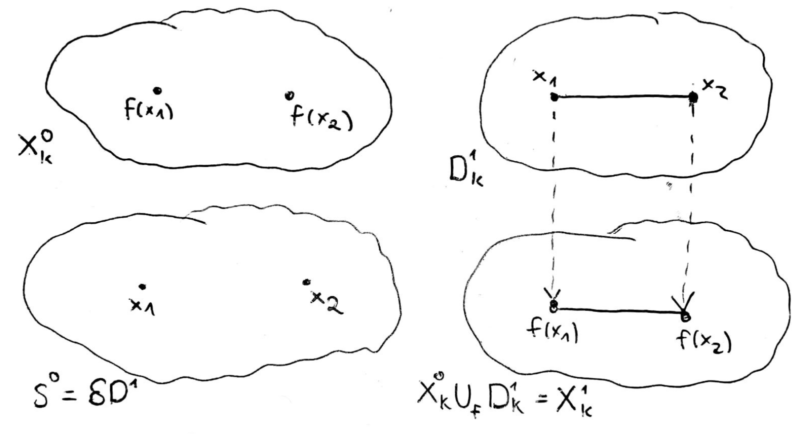
Definition 2.22. (Cell-complex) We inductively define adjunction spaces.

1. Start with a discrete set X_0 who's points can be regarded as 0-*cells*.
2. Inductively start attaching the closure of n-cells $e^n_k \subset D^n_k$ between the $n-1$-*cells* via a family of attaching maps f where $f_k : \delta D^n_k \to X_{n-1}$. For every attaching map we define the equivalence relation $x \backsim_k y \leftrightarrow y = f_k(x)$ and define $\backsim := (\backsim_1, ..., \backsim_k, ...)$. This gives the following adjunction space:

$$X_n = (X_{n-1} \cup \bigcup_k D^n_k)/\backsim$$

Since, $D^n_k \backslash \delta D^n_k = e^n_k$, as a set, X_n is given by:

$$X_n = X_{n-1} \cup \bigcup_k e^n_k$$



A space constructed this way is called a *cell-complex*. A set $U \subset X_n$ is open if $U \cap e^n_k$ is open for every e^n_k.

Definition 2.23. (Connected) A topological space (T, X) is called *connected* if and only if the following holds:

$$\forall A, B \in T : (A \cap B = \emptyset \wedge A \cup B = X) \Rightarrow (A = X \vee B = X)$$

Definition 2.24. (Path connected) A topological space X is called *path connected* if and only if for every two points $x, y \in X$ there exists a continuous path $\varphi : [0,1] \to X$ that satisfies

$$\varphi(0) = x,\ \varphi(1) = y$$

Definition 2.25. (Path components) We can define an equivalence relation:

$$x, y \in X : x \backsim y \Leftrightarrow \exists\, \varphi : [0,1] \to X : \varphi\, continuous,\ \varphi(0) = x, \varphi(1) = y.$$

We call the elements $[x] \in X/\backsim$ the *path components* of X.

Definition 2.26. (Locally path connected) Consider a topological space X. If for each $x \in X$ and each neighbourhood U_x there exists a path connected neighbourhood $V_x \subseteq U_x$ we call X *locally path connected.*

Definition 2.27. (Homotopy) Let $f_1, f_2 : X \to Y$ be continuous maps. A *homotopy* between f_1 and f_2 is a continuous map $H : X \times [0,1] \to Y$ such that

- $H(0,t) = f_1(t)$
- $H(1,t) = f_2(t)$

Then f_1 is *homotopic* to f_2. A map is said to be *nullhomotopic* if it is homotopic to a constant map.

Definition 2.28. (Homotopy of paths) Consider a topological space (X, T) and two paths $\varphi_1, \varphi_2 : [a,b] \to X$ where $\varphi_1(a) = \varphi_2(a)$ and $\varphi_1(b) = \varphi_2(b)$. A *path homotopy* between φ_1, φ_2 relative to $\{0,1\}$ is a continuous map $H(s,t) : [a,b] \times [0,1] \to X$ that satisfies the following:

- $\forall\, t \in [0,1] : H(a,t) = \varphi_1(a) = \varphi_2(a)$
- $\forall\, t \in [0,1] : H(b,t) = \varphi_1(b) = \varphi_2(b)$
- $H(s,0) = \varphi_1(s)$
- $H(s,1) = \varphi_2(s)$

If such a map H exists we call φ_1, φ_2 *homotopic.*

Proposition 2.29. Given a topological space (X, T), we can define an equivalence relation, where $\varphi_1 \backsim \varphi_2$ if and only if φ_1, φ_2 are homotopic. We denote the corresponding equivalence class with $[\varphi_1]$.

Definition 2.30. (Retraction on topological spaces) A subset $A \subseteq X$ is called a *retract* if the exists a continuous map $r : X \to A$ such that $r|_A = id_A$. r is then called a *retraction*

Definition 2.31. (Deformation retract) A $deformation\ retract$ of a space X onto a subspace A is a continuous map

$$H : X \times [0,1] \to X$$

such that:

- $H(\cdot, 0) = id_X$
- $H(x,1) \in A$,$\forall x \in X$
- $H(\cdot, t)|_A = id_A$, $\forall\, t \in I$

The map $r : X \to A$ defined by $H(x, 1)$ is a retraction of X onto A and H is a homotopy between id_X and $i \circ r$ where $i : A \to X$ is the inclusion.

Definition 2.32. (Contractible) A space X where id_X is homotopic to a constant map is called *contractible*. In this case there exists an $x \in X$ such that $\{x\}$ is a deformation retract of X.

Definition 2.33. (Homotopy equivalence) Two spaces X, Y are said to be *homotopy equivalent* if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

Proposition 2.34. (Lebesgue) Let X be a compact metric space. Let $\mathcal{U}$ be an open covering of X. There exists an $\epsilon > 0$ such that for each $x \in X$ the neighbourhood $U_\epsilon(x) = \{y \in X | \, d(x, y) < \epsilon\}$ is contained in some member of $\mathcal{U}$.

Proof. Since X is compact there is a finite subcover $\{A_1, A_2, ...\} \subseteq \mathcal{U}$. Let $C_i = X \backslash A_i$ (we can assume that $C_i \neq \emptyset$. If that would be the case then, since A_i is open, there exists an $\epsilon > 0$ such that $U_\epsilon(x) \subset A_i \, , \forall x \in X$). We define

$$f : X \to \mathbb{R} : x \to \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

Since f is continuous it has a minimum value ϵ on a compact set. Since $f(x) \geq \epsilon$ there must exist an i, such that $d(x, C_i) \geq \epsilon$. This means that $B_\epsilon(x) \subseteq A_i$. $\square$

2.3 Group Theory

Definition 2.35. (Groupoid) A category $\mathcal{G}$ is called a groupoid if it satisfies the following conditions:

- $Obj(\mathcal{G})$ is a set
- $\forall \, x, y \in Obj(\mathcal{G}) : \, \mathcal{G}(x, y)$ is a (possibly empty) set
- $\forall \, x, y \in Obj(\mathcal{G}) \, \exists \, inv : \, \mathcal{G}(x, y) \to \mathcal{G}(y, x) : f \to f^{-1}$

Definition 2.36. (Group as a category) A category G is called a group if it can be considered a groupoid with only one object.

Definition 2.37. (Group) A group $(G, *)$ consists of the following :

- A set G
- A total binary operation $* : G \times G \to G$

that satisfy the following conditions:

- associativity : $\forall f, g, h \in G$:
$$(f * g) * h = f * (g * h)$$
- neutral element : $\exists \, e \in G \, \forall f \in G$:
$$f * e = e * f = f$$
- inverse : $\forall f \in G \, \exists f^{-1} \in G$:
$$f * f^{-1} = f^{-1} * f = e$$

If additionally $(G, *)$ satisfies commutativity, we call it an Abelian group. A *subgroup* H of G consists of a non empty subset $H \subset G$ that is closed under $*$ and contains all inverse elements. We write $H < G$. We call H a *normal* subgroup of G, if $\forall g \in G : H * g = g * H$. This is equivalent so saying:

$$\forall h \in H \, \forall g \in G : g * h * g^{-1} \in H.$$

We will denote a group $(G, *)$ just with G for following definitions if not required otherwise.

Remark 2.38. We can easily show that the algebraic definition of a group is equivalent to the category theoretic definition. The set of morphisms $\{\mathcal{G}(x, x)\}$ is the set G of group elements. $comp_x : \mathcal{G}(x, x) \times \mathcal{G}(x, x) \to \mathcal{G}(x, x)$ is the total binary operation on $*$ G.

Definition 2.39. (Group homomorphism) Consider the groups $(G, *), (H, \circ)$. A map $\varphi : (G, *) \to (H, \circ)$ is called a group homomorphism if it satisfies the following

$$\forall g, h \in G : \varphi(g * h) = \varphi(g) \circ \varphi(h)$$

$ker(\varphi) = \{g \in G | \, \varphi(g) = e_H\}$ is called the *kernel* of φ.

Remark 2.40. We can now define the category GRP of groups. Objects of GRP are groups, morphisms are group homomorphisms.

Proposition 2.41. (Quotient group) Consider a group G and a normal subgoup N. The set G/N together with a binary operation for $g_1, g_2 \in G$

$$g_1 N \cdot g_2 N = g_1 g_2 N$$

forms a group called a quotient group of G.

Proposition 2.42. If $\varphi : G \to H$ is a group homomorphism then $ker(\varphi)$ is a normal subgroup of G, φ is injective if and only if $ker(\varphi) = \{e_G\}$.

Proof. We check conditions for a subgroup

- $\varphi(e_G) = e_H$
- $\forall g \in ker(\varphi) : \, \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1}) = e_H \varphi(g^{-1})$. So $g^{-1} \in ker(\varphi)$
- $h \in ker(\varphi)$, $g \in G : \, \varphi(g^{-1}hg) = \varphi(g^{-1})\varphi(h)\varphi(g) = \varphi(g)^{-1} e_H \varphi(g) = e_H$

So $g^{-1}hg \in H$ For $g_1, g_2 \in G$ assume $ker(\varphi) = \{e_G\}$ and $\varphi(g_1) = \varphi(g_2)$ then

$$e_H = \varphi(g_1)^{-1}\varphi(g_2) = \varphi(g_1^{-1} g_2)$$

so $g_1^{-1} g_2 \in ker(\varphi)$ and so $g_1^{-1} g_2 = e_G$. □

Theorem 2.43. Consider a group homomorphism $\varphi : G \to H$. The following homomorphism is well defined:

$$\tilde{\varphi} : G/ker(\varphi) \to H : gker(\varphi) \to \varphi(g)$$

Furthermore $G/ker(\varphi) \to im(\varphi)$ is an isomorphism.

Proof. First we show $\tilde{\varphi}$ is well defined. Consider $g_1, g_2 \in G$. If $g_1 ker(\varphi) = g_2 ker(\varphi)$ then

$$e_H = \varphi(g_1^{-1} g_2) = \varphi(g_1^{-1})\varphi(g_2)$$

so $\varphi(g_1) = \varphi(g_2)$. For injectivity note that $e = \tilde{\varphi}(gker(\varphi)) = \varphi(g)$ implies $g \in ker(\varphi)$, so $gker(\varphi) = eker(\varphi)$ in $G/ker(\varphi)$ so $ker(\tilde{\varphi}) = e$. Now because $im(\varphi) = im(\tilde{\varphi})$ its follows that $G/ker(\varphi) \to im(\phi)$ is an isomorphism. □

Definition 2.44. (Generating set) Consider a group G. A generating set of G is a set S such that:

$$\forall\, g \in G\, \exists\, s_1, s_2, .. \in S \cup S^{-1} : g = s_1 s_2 ...$$

where S^{-1} is the set of all inverses of elements in S We call the set of all possible written products of elements in $S \cup S^{-1}$ the set of *words* over S.

Definition 2.45. (Presentation of a group) Consider a group G. The presentation of G consists of a generating set S and a set R called *relations* on S. We write $G = \langle S|R \rangle$. The words in R represent the words in $S \cup S^{-1}$ that reduce to the group identity.

Definition 2.46. (Free product) A free product between groups $G = \langle S_G|R_G \rangle, H = \langle S_H|R_H \rangle$ is a coproduct in the category of groups. We define it as follows:

$$G * H = \langle S_G \cup S_H | R_G, R_H \rangle$$

For a group G will use the notation $*_i G = G * G * ... * G$, *i-times*

Definition 2.47. (Free group) A group F_S over a given set S is called $free$ if it satisfies the following:

- the set F_S consists of all *words* over S.
- the binary operation is just the written product of elements in $S \cup S^{-1}$ where aa^{-1} reduces to the *empty word*.
- the neutral element is the *empty word*.

The group operation only depends on the general group axioms. Given a set S and any $function$ from S to G, there exists a unique homomorphism $\varphi : F_S \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
& F_S \arrow[dd, "\varphi"] \\
S \arrow[ur] \arrow[dr, "f"] & \\
& G
\end{tikzcd}$$

This is called the *universal property* of free groups We say F_S is $of\ rank\ n$ if $|S| = n$.

Proposition 2.48. For every group G that is generated by a set S, $G \backsim F_S/ker(\varphi)$ so $G = \langle S|ker(\varphi) \rangle$.

Example 2.49. $\mathbb{Z}/2\mathbb{Z} = \langle a|a^2 \rangle$. Here $S = a$ and $\varphi : F_S \to \mathbb{Z}/2\mathbb{Z}$ is the homomorphisms that maps a^{2n} to the *identity* for every $n \in \mathbb{N}$ i.e $ker(\varphi) = \{a^{2n}|n \in \mathbb{Z}\}$.

Proposition 2.50. A free group of rank n is isomorphic to the free product $*_n\mathbb{Z}$. $\mathbb{Z} = \langle a|\emptyset \rangle$ is presentation of $\mathbb{Z}$. The rest follows by definition of a free group and the free product.

2.4 Graphs

Definition 2.51. (Graph) A graph is a topological space X obtained from a discrete set X_0 by attaching a collection of 1-cells e_a. Thus, X is obtained from the disjoint union of closed intervals I_a, by identifying the endpoints of I_a with points in X_0. The 1-cells are called *edges* and the points in X_0 are called *vertices*. The two endpoints of an edge can be the same vertex, so the closure of e_a can be either homeomorphic to I or S^1. A graph $Y \subset X$ is called a subgraph of X if $e_a \subseteq Y$ always implies the closure of e_a is in Y

- a *walk* in X is a sequence of edges $(e_1, ..., e_n)$ such there is a sequence of vertices $(v_1, ..., v_{n+1})$ such that the the boundary $C(e_a) = \{v_a, v_{a+1}\}$.
- a *trail* is a walk in which all edges are distinct.
- a cycle in X is a trail in which (only) the first and the last vertices are equal.

Remark 2.52. By the definition above a graph is just a 1-dimensional $cell-complex$. Vertices are 0-*cells* edges are 1-*cells*.

Definition 2.53. (Tree) A graph T is called a tree if it is connected and contains no cycles i.e every two vertices are joined by exactly one path. It follows that trees are contractible. A tree T in X is a subgraph that is a tree. A tree T in X is called maximal if it contains all vertices of X.

Proposition 2.54. A connected graph contains a maximal tree.

Proof. For a connected graph X let $\{T_i | i \in I\}$ be the set of all trees in X partially ordered by the subgraph relation. Subgraph inclusion means that for every cycle in a union of trees from any chain in $\{T_i | i \in I\}$ there is a tree that contains this cycle. That is a contradiction, so the union of trees from a chain doesn't contain cycles. If the union is disconnected, then there are two vertices in the union which are not joined by a path and there is a tree which contains both vertices. So this tree is not connected. Again a contradiction. So any union of trees from a chain is connected with no cycles. Since every chain of trees totally ordered by the subgraph relation has an upper bound, it follows by Zorn's lemma that the set of all trees in X has a maximal element T. If T does not contain every vertex in X, we can find an edge from a vertex in T to a vertex in $X \backslash T$, which yields a larger tree that contains T. Therefore T must already contain every vertex in X. □

3 The Fundamental Group

All definitions,theorems and propositions in this chapter were mailny taken from [1],[3],[5]. In the following sections $[0,1] = I$.

Definition 3.1. (product of paths) Consider a topological space X and two paths $u, v : [0,1] \to X$. If $u(1) = v(0)$ the product $u * v$ exists and we define it as follows:

$$u * v = w : [0,1] \to X,\ w(t) = \begin{cases} u(2t) & t \in [0, \frac{1}{2}] \\ v(2t-1) & t \in (\frac{1}{2}, 1] \end{cases}.$$

Proposition 3.2. The product of paths satisfies the following conditions:

1. Let $\varphi : I \to I$ continuous and $\varphi(0) = 0, \varphi(1) = 1$ then $[u] = [u\varphi]$
2. $u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3$ if defined.
3. If $u_1 \simeq u_2$ and $u_3 \simeq u_4$ then $u_1 * u_3 \simeq u_2 * u_4$
4. $u * u^{-1}$ is always defined and homotopic to the constant path

Proof.
1. $H : (s,t) \to u(s(1-t) + t\varphi(s))$ is a homotopy from u to $u\varphi$
2. $\exists \varphi$ as in 1. such that $u_1 * (u_2 * u_3)\varphi = (u_1 * u_2) * u_3$ specifically

$$\varphi(t) = \begin{cases} 2t & t \in [0, \frac{1}{4}] \\ t + \frac{1}{4} & t \in (\frac{1}{4}, \frac{1}{2}] \\ \frac{t}{2} + \frac{1}{2} & t \in [\frac{1}{2}, 1] \end{cases}.$$

3. Given homotopies $F_i(s,t) : u_i \simeq u_{i+1}$ then

$$G(s,t) = \begin{cases} F_1(2s, t) & s \in [0, \frac{1}{2}] \\ F_3(2s-1, t) & s \in [\frac{1}{2}, 1] \end{cases}.$$

provides a homotopy $G(s,t) : u_1 * u_3 \simeq u_2 * u_4$

4. The map $F : I \times I \to X$ defined as

$$F(s,t) = \begin{cases} u(2s(1-t)) & s \in [0, \frac{1}{2}] \\ u(2(1-s)(1-t)) & s \in [\frac{1}{2}, 1] \end{cases}$$

is a homotopy from $u * u^{-1}$ to the constant path. (At time t we only use the path from 0 to $(1-t)$ and compose it with its inverse.)

$\square$

Remark 3.3. Whenever we use the product of paths and not the composition of maps we will use $*$. Note that the writing order differs between composition and product.

Definition 3.4. (The Fundamental Groupoid) Consider a topological space X. We define the fundamental groupoid $\Pi(X)$ of X as follows:

- $Obj(\Pi(X))$ are the points in X
- $\forall x, y \in Obj(\Pi(X)) : \Pi(X)(x,y)$ consists of all homotopy classes of paths from x to y
- $\forall x, y, z \ [f] \in \Pi(X)(x,y) \ , \ [g] \in \Pi(X)(y,z) :$

$$\begin{aligned} comp : \Pi(X)(x,y) \times \Pi(X)(y,z) &\to \Pi(X)(x,z) \\ ([f],[g]) &\to [g] \circ [f] = [f * g] \end{aligned}$$

- $\forall x, y \in Obj(\Pi(X)) \ , \ [f] \in \Pi(X)(x,y) :$

$$\begin{aligned} inv : \Pi(X)(x,y) &\to \Pi(X)(y,x) \\ [f] &\to [f^{-1}] = [f]^{-1} \end{aligned}$$

Using the properties of the product path we can see that *comp* and *inv* are well defined.

Definition 3.5. (The Fundamental Group) Consider a topological space X together with a point x_0. We denote this space with (X, x_0). By only considering paths u with $u(0) = u(1) = x_0$ we obtain a groupoid with only one object $\pi_1(X, x_0)$ called the fundamental group of (X, x_0). We denote this group $\pi_1(X, x_0)$.

Proposition 3.6. If X is path connected then

$$\forall x, y \in X : \pi_1(X, x) = \pi_1(X, y)$$

In the case that X is path connected and if not required otherwise we will denote the fundamental group of X with $\pi_1(X)$.

Proposition 3.7. Let X, Y be topological spaces. A continuous map $f : X \to Y$ induces a functor

$$\Pi(f) : \Pi(X) \to \Pi(Y) : x \to f(x), [u] \to [fu]$$

Definition 3.8. (Simply connected) A path connected topological space X is called *simply connected* if $\pi_1(X) = \{e\} = 0$.

Definition 3.9. (Semi-locally simply connected) A path connected topological space X is called *semi-locally simply connected* if for every $x \in X$ there exists a simply connected neighbourhood U_x.

Definition 3.10. (Locally simply connected) A path connected topological space X is called *locally simply connected* if the set of simply connected neighbourhoods U_x, $x \in X$ forms a basis of X.

4 Covering spaces

All of the definitions,theorems and propositions in this chapter were mainly taken from [4],[5]

Definition 4.1. (Trivialization) Let $p : E \to B$ be surjective and continuous, $U \subseteq B$ open. A homeomorphism $\varphi : p^{-1}(U) \to U \times F$ is called a trivialization of p over U if the following diagram commutes :

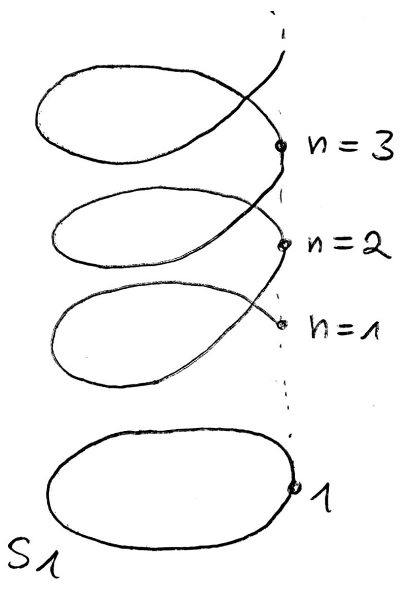
$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "p"'] & U \times F \arrow[dl, "pr_1"] \\
U &
\end{tikzcd}$$

F is uniquely determined up to homeomorphism since φ induces a homeomorphism of $p^{-1}(u)$ with $\{u\} \times F$ for all $u \in U$. F is called a *fibre*. We call the map p *locally trivial* if there exists an open covering $\mathcal{U}$ of B such that p has a trivialization over each $U \in \mathcal{U}$. A locally trivial map is also called a *fibre bundle*.

Definition 4.2. (Covering space) A covering space of B is a fibre bundle $p : E \to B$ where for each trivialization of p over $U \in \mathcal{U}$, every subset in the fibre F is open and closed so $U \times F$ is homeomorphic to the coproduct of topological spaces : $\bigcup_{x \in F} U \times \{x\}$. The summands $U \times \{x\}$ are canonically homeomorphic to U. Hence, $p|_{\varphi^{-1}(U\times\{x\})} : \varphi^{-1}(U \times \{x\}) \to U$ is a homeomorphism and consequently, p is a local homeomorphism. We call the summands $\varphi^{-1}(U \times \{x\})$ the *sheets* of p. If $|F| = n \in \mathbb{N}$ we talk about an n-fold covering.

Example 4.3. (Covering of S^1) The exponential function $p : \mathbb{R} \to S^1 : t \to exp(2\pi i t)$ is a covering with fibre $\mathbb{Z}$. For each $t \in \mathbb{R}$ and $p(t) = z$ we have a homeomorphism

$$p^{-1}(S^1 \backslash z) = \bigcup_{n \in \mathbb{Z}}]t + n, t + n + 1[\simeq]t, t + 1[\times \mathbb{Z}$$



Proposition 4.4. (Uniqueness of liftings) Let X be a connected topological space, $p : E \to B$ be a covering and $F_1, F_2 : X \to E$ be liftings of $f : X \to B$, such that $S = \{x \in X | \, F_1(x) = F_2(x)\} \neq \emptyset$. Then $F_1 = F_2$

Proof. We show that S is open and closed in X. By connectedness of X, it then follows that $S = X$. Let $U_{f(x)}$ be an evenly covered neighbourhood of $f(x) \in B$, so $p^{-1}(U_{f(x)})$ is composed of disjoint sheets mapped homeomorphically onto $U_{f(x)}$ by p. Let U_1, U_2 be the sheets containing $F_1(x), F_2(x)$. Because $F_1(x), F_2(x)$ are continuous, there is a neighbourhood V_x mapped into U_1, U_2 by $F_1(x), F_2(x)$

- For $x \notin S$, $F_1(x) \neq F_2(x)$ so $U_1 \neq U_2$, hence U_1, U_2 are disjoint and $F_1|_{V_x} \neq F_2|_{V_x}$. Thus we have found an open neighbourhood V_x such that $V_x \cap S = \emptyset$, so S^c is open and S is closed.

- For $x \in S$, $F_1(x) = F_2(x)$ then $U_1 = U_2$, so $F_1|_{V_x} = F_2|_{V_x}$. Since $pF_1 = pF_2$ and p is injective on $U_1 = U_2$, we have found an open neighbourhood $V_x \subseteq S$, so S is open.

□

Definition 4.5. (Homotopy lifting property) A map $p : E \to B$ is said to have a the hlp for the space B if the following holds: For each homotopy $h : X \times I \to B$ and each map $f : X \to E$ such that $pf(x) = hi(x), i(x) = (x, 0)$ there exists a homotopy $H : X \times I \to E$ with $pH = h$ and $Hi = f$. We call H a lifting of h with initial condition f. The following diagram commutes

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "f"] \arrow[d, "i"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "h"] \arrow[ur, "H"] & B
\end{tikzcd}$$

The map p is called a fibration.

Proposition 4.6. A projection $p : B \times F \to B$ is a fibration

Proof. Let $f(x) = (f_1(x), f_2(x))$. The condition $pf = hi$ says $f_1(x) = h(x, 0)$. If we set $H(x, t) = (h(x, t), f_2(x))$ then H is a lifting of h with initial condition f. □

Theorem 4.7. A covering $p : E \to B$ is a fibration. The obtained lifting H is unique.

Proof. Let the homotopy $h : X \times I \to B$ and an initial condition f be given. I is connected. We already showed that a lifting with initial condition is uniquely determined. Therefore we just need to show that

$$\forall\, x \in X \,\exists\, V_x \text{ open: For } h|_{V_x \times I} \text{ there's a lifting } H_x \text{ with initial condition } f|_{V_x}.$$

such that the following diagram commutes:

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
V_x \times I \arrow[r, "h"] \arrow[ur, "H_x"] & B
\end{tikzcd}$$

Since h is continuous, every point $(x, t) \in X \times I$ has a neighbourhood $V_x \times (a_t, b_t)$ such that $h(V_x \times (a_t, b_t)) \subseteq U_i$ where p is trivial over U_i. By 2.34 we can choose a Lebesgue number for the open cover $(a_t, b_t)\,, t \in I$. Then there exists an $n \in \mathbb{N}$ such that $[i/n, (i+1)/n] \subset (a_t, b_t)$ for some $i \in (0, ..., n)$. If we set $i/n = t_i$ we get a partition

$$0 = t_0 < t_1 < t_2 < ... < t_n = 1$$

such that $h(V_x \times [t_i, t_{i+1}]) \subseteq U_i$ where p is trivial over U_i. Now since $p : p^{-1}(U_i) \to U_i$ is by 4.6 a fibration, $h|_{V_x} \times [t_i, t_j]$ has a lifting for each initial condition. Thus we can use induction:

- For $l = 1$ the lifting H_x on $V_x \times [0, t_1]$ to one of the sheets $p^{-1}(U_i)$ is determined by the initial condition $f|_{V_x \times \{0\}}$.

$$
\begin{tikzcd}
 & p^{-1}(U_i) \arrow[d, "p"] \\
V_x \times I \arrow[ur, "H_x"] \arrow[r, "h"] & U_i
\end{tikzcd}
$$

- Suppose we have constructed a lifting H_x on $V_x \times [t_i, t_j]$. The lifting of h on $V_x \times [t_j, t_{j+1}]$ to one of the sheets $p^{-1}(U_i)$ is now uniquely determined by $f|_{V_x} \times \{t_j\}$, which can be glued to the lifting on $V_x \times [t_i, t_j]$ to give a lifting on $V_x \times [t_j, t_{j+1}]$.

Because every two such liftings H_x, H'_x agree on every initial condition $f|_{V_x}$ they are the same by uniqueness of liftings. Likewise, if for two sets $(V_x \times I) \cap (V_y \times I) \neq \emptyset$ the respected liftings H_x, H_y must agree on $(V_x \times I) \cap (V_y \times I)$. So the constructed liftings $H_x, x \in X$ glue to give the desired unique lifting H. □

Proposition 4.8. For each path $w : I \to B$ starting at point $p(e) = w(0) \in B$ there is a unique lifting of w starting at e. Furthermore two paths in E which start at the same points are homotopic if and only if their images in B are homotopic.

Proof. We already showed the uniqueness and existence in 4.7. Now let $h : I \times I \to B$ be a homotopy of paths and $H : I \times I \to E$ be a lifting of h. Since $t \to H(s,t), s \in \{0,1\}$ are continuous maps into a discrete fibre, they are constant. Hence H is indeed a homotopy of paths. Let $u_1, u_2 : I \to E$ be paths which start at x and suppose pu_1, pu_2 are homotopic. If we lift a homotopy $g : I \times I \to E$ between them with constant initial condition $pu_0(x) = gi(x)$ the result is a homotopy between u_1, u_2. □

Proposition 4.9. For each homotopy $h : I \times I \to B$ starting at point $p(e) = h(0,0) \in B$ there is a unique lifting of h starting at e.

Proof. Follows directly from theorem 4.7 □

Proposition 4.10. (Fundamental group of a circle) $\pi_1(S^1, 1)$ is an infinite cyclic group generated by the homotopy class of the loop $u(s) = exp(2\pi i s)$ based at $1 \in \mathbb{C}$.

Proof. Let $u : I \to S^1$ be a loop at basepoint $x_0 = 1$ representing a given element of $\pi_1(S^1, x_0)$. By 4.8 there is a unique lift $\tilde{u} : I \to \mathbb{R}$ starting at 0. The path $\tilde{u}$ ends at some integer n since $p\tilde{u}(1) = u(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z}$. Consider the paths $\tilde{w_n} : I \to \mathbb{R}$ on $\mathbb{R}$ from 0 to n, then $\tilde{u} \simeq \tilde{w_n}$ via the homotopy $\tilde{H} : I \times I \to \mathbb{R} : (s,t) \to (1-t)\tilde{u} + t\tilde{w_n}$. Then $p\tilde{H} = H$ is a homotopy between u, w_n so $[u] = [w_n]$. We now have to show that n is uniquely determined by $[u]$. Suppose that $u \simeq w_n$ and $u \simeq w_m$. Let H be a homotopy from $w_m(s) = H(s,0)$ to $w_n(s) = H(s,1)$. By 4.9 this homotopy lifts to a homotopy $\tilde{H}$ of paths starting at 0. By 4.8 $\tilde{w_m}(s) = \tilde{H}(s,0)$ and $\tilde{w_n}(s) = \tilde{H}(s,1)$. Since $\tilde{H}$ is a homotopy of paths the endpoint $\tilde{H}(1,t)$ is independent of t . For $t = 0$ this endpoint is m, for $t = 1$ this endpoint is n, so $m = n$. □

Proposition 4.11. If a space B is simply connected, then there is a unique homotopy class of paths connecting any two points in B .

Proposition 4.12. The map $\pi_1(p) : \pi_1(E,e) \to \pi_1(B,b)$ induced by a covering space $p : (E,e) \to (B,b)$ is injective. The image subgroup $\pi_1(p)(\pi_1(E,e))$ in $\pi_1(B,b)$ consists of homotopy classes of loops in B based at b who's lifts in E starting at e are loops.

Proof. An element in $ker(\pi_1(p))$ is represented by a loop $u : I \to E$ with a homotopy $H : I \times I \to B$ of $H(s,0) = pu$ to the trivial loop $H(s,1)$. By 4.9 there is a lifted homotopy of loops $\tilde{H}$ starting with $u = \tilde{H}(s,0)$ and ending with a constant loop. Hence $[u] = 0$ in $\pi_1(E,e)$ and $\pi_1(p)$ is injective. For the second statement of the proposition, loops at b lifting to loops at e represent elements of the image of $\pi_1(p)$. By the hlp there must be such a lift. □

Note that by this proposition a subgroup of a free group on two generators can be isomorphic to a group on any number of generators.

Theorem 4.13. Let (B, b_0) be locally path connected, path connected, semi-locally simply-connected. Then B has a simply connected covering space. This covering space is called a *universal covering.*

Proof. We first define a covering space $p : (E, e_0) \to (B, b_0)$ who's points are homotopy classes of paths in B starting at b relative to $\{0,1\}$. Let

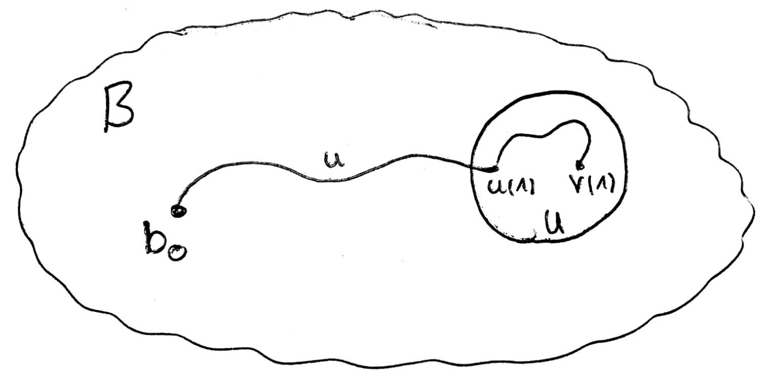
$$E = \{[u] | u \text{ is a path in } B \text{ starting at } b_0 = p(e_0)\}$$

and

$$p : E \to B : [u] \to u(1)$$

Since all paths in $[u]$ have the same endpoints this is well defined. Since B is path connected, p is surjective. Using 4.8, E consists of paths that start at e_0 lifting paths in B that start at b_0 (here e_0 can be considered the homotopy class of the constant path in B starting at b_0). We will now define a topology on E such that p is a covering map. Let $\mathcal{U}$ be the set of all open path connected subsets U of B. Suppose for two sets $U_1, U_2 \in \mathcal{U}$, $b \in U_1 \cap U_2$, then $U_1 \cap U_2$ is open neighbourhood of b. Since B is locally path connected there is a path connected set $V \subseteq U_1 \cap U_2$ containig b, thus $V \in \mathcal{U}$. Since B is locally path connected and semi-locally simply connected any open set in B can be written a s the union of set in $\mathcal{U}$. It follows that $\mathcal{U}$ is a basis for the topology on B. For a set $U \in \mathcal{U}$ and a path u in B from b_0 to a point in U define

$$U_{[u]} = \{[u * v] | v \text{ is a path in } U \text{ with } v(0) = u(1)\}$$



where $[w] \in U_{[u]}$ are homotopy classes relative to $\{0,1\}$. Now because E consists of homotopy classes of paths starting at b_0, $U_{[u]}$ is a subset of E. We show that the collection of such $U_{[u]}$ is a basis for the topology on E.

- First we show that

$$[u'] \in U_{[u]} \Rightarrow U_{[u]} = U_{[u']}$$

  This fact is rather trivial. If $[u'] \in U_{[u]}$ then $[u'] = [u * v]$ for some path in U with $v(0) = u(1)$. Then $[w] \in U_{[u']}$ is of the from $[u*v*v']$ for some $[v'] \in U$ with $v'(0) = u*v(1)$. But then $v * v'$ is a path in U with $v * v'(0) = u(1)$ so $[u * v * v'] \in U_{[u]}$. Thus $U_{[u']} \subseteq U_{[u]}$. The proof of the reverse inclusion works the same. Now, since X is path connected and locally path connected, every element of E is contained in some $U_{[u]}$.

- Suppose $[w]$ is contained in some intersection $U_{[u]} \cap V_{[v]}$ for some $U, V \in \mathcal{U}$. Then $U_{[w]} = U_{[u]}$ and $V_{[w]} = V_{[v]}$. Since $\mathcal{U}$ is a basis for the topology on B, we can choose $W \in \mathcal{U}$ with $W \subseteq U \cap V$. Then $W_{[w]} \subseteq U_{[w]} \cap V_{[w]}$ thus $W_{[w]} \subseteq U_{[u]} \cap V_{[v]}$. Since $[w] \in W_{[w]}$, it follows that the set of all $U_{[u]}$ is a basis for the topology on E, namely the collection of all possible unions of the basis elements of the form $U_{[u]}$ where $U \in \mathcal{U}$.

Thus we have defined a topology on E. We now show that p is a local homeomorphism. Since $U \in \mathcal{U}$ is path connected, for any $b \in U$ we can choose a path v in U from $u(1)$ to b so that $u * v = b$. Then $p([u * v]) = b$, so $p|_{U_{[u]}}$ is surjective. Suppose we have paths v, v' from $u(1)$ to some point $b \in U$, with $p([u * v']) = p([u * v])$. Since the inclusion $\pi_1(p) : \pi_1(U) \to \pi_1(B)$ is trivial by definition of the set $\mathcal{U}$, all paths that connect $u(1)$ to the fixed point $b \in U$ are homotopic in B. Thus $[v] = [v']$ and so $[u * v] = [u * v']$. It follows that $p|_{U_{[u]}}$ is bijective. The image of any basis element $U_{[u]}$,$[u] \in E$ for the topology on E is a set $U \in \mathcal{U}$ and the inverse image $p^{-1}(U)$ is the union of the sets $U_{[u]}$ where u is a path from b_0 to a point in U. Thus inverse images of open sets in B are open in E, so p is continuous. Hence p is a local homeomorphism. Note that $p^{-1}(U) = \bigcup_u U_{[u]}$ where u is a path from b_0 to a point in U. By earlier reasoning this union is disjoint. Now for every $b \in B$, each of these $U_{[u]}$ intersects the set $p^{-1}(b)$ in a single point, therefore these points are open and closed in $p^{-1}(b)$ (where the topology on $p^{-1}(b)$ is induced by all intersections of open sets in $p^{-1}(U)$ with $p^{-1}(b)$). We can choose any point $b \in U$ as a representative to obtain the discrete fibre $p^{-1}(b)$ and construct the relevant trivializations of p over any set $U \in \mathcal{U}$ with the morphism $\varphi : p^{-1}(U) \to U \times p^{-1}(b)$. Since $\mathcal{U}$ is a basis, E is a covering space of B. It remains to show that E is simply connected. To show that it is path connected, let u be any path in B starting at b_0 and define u_t

$$u_t(s) = \begin{cases} u(s) & s \in [0, t] \\ u(t) & s \in (t, 1] \end{cases}$$

That is u_t traces out u on the intervall $[0, t]$ and is constant on $(t, 1]$. The function $f_u : t \to [u_t]$ is a path in E lifting u that starts at e_0 and ends at $[u]$. We can do this for any $[u] \in E$. So E is path connected. This leaves to show that $\pi_1(E, e_0) = 0$. Since $\pi_1(p)$ is injective we just need to show that $\pi_1(p)(\pi_1(E, e_0)) = 0$. Elements in $im(\pi_1(p))$ are represented by loops u starting at b_0 lifting to loops in E starting at e_0 where $e_0 = [u_0]$ the homotopy class of the constant paths starting at b_0 . We have observed that the path $f_u : t \to [u_t]$ lifts u, for this lifted path to be a loop means that $[u_1] = e_0$. Since $u_1 = u$ this means that $[u] = e_0$, so u is nullhomotopic. □

Proposition 4.14. Suppose B is path connected, locally path connected, and semi-locally simply connected. For every subgroup of $H < \pi_1(B, b)$ there is a covering space $p : E_H \to B$ such that $\pi_1(p)(\pi_1(E_H, e)) = H$ for a suitable basepoint $e \in E_H$.

Proof. We construct a universal cover E. For points $[u], [v] \in E$ we define an equivalence relation $[u] \backsim [v]$ if $u(1) = v(1)$ and $[u * v^{-1}] \in H$. So $u * v^{-1}$ is a loop and this loop is contained in an element of the subgroup H. Since H is a subgroup this indeed satisfies the conditions for an equivalence relation.

- Since H contains the identity element it is reflexive
- Since H is closed under inverses its symmetric
- Since H is closed under multiplication its transitive

Let E_H be the quotient space $E/\backsim$. Note that $[u] \backsim [v]$ only if $[v * w] \backsim [u * w]$. So if any two points in the neighbourhoods $U_{[u]}$ and $U_{[v]}$ are equal under $\backsim$ then $U_{[u]}, U_{[v]}$ are equal under $\backsim$. In this way the quotient map sending $[u]$ to its equivalence class under $\backsim$ gives a quotient topology on E_H. The covering map $p_H : E_H \to B$ is just the extension of the universal cover $p : E \to B$ defined by $p([u]) = u(1)$ to equivalence classes under $\backsim$. To see that p_H is well defined note that $[u] \backsim [v]$ only if $u(1) = v(1)$, so it sends members of the same equivalence class under $\backsim$ to the same point in B. With the properties of the quotient topology, following from the previous theorem, we conclude that $p_H^{-1}(U)$ is a disjoint union of open sets in E_H . Thus E_H is a covering space of B. If we choose a basepoint $e_0 \in E_H$ as the equivalence class of the constant path starting at b then the image of $\pi_1(p)(\pi_1(E_H, e_0))$ is exactly H. Because for a loop u based at b its lift to E starting at e_0 ends at $[u]$, so the image of this lifted path in E_H is a loop if and only if $[u] \simeq e_0$ or equivalently $[u] \in H$. $\square$

Proposition 4.15. Every covering space of a connected graph is a connected graph.

Proof. Let B be a graph and $p : E \to B$ be a covering space of B. By definition of a graph, we can write B as $B_0 \cup \bigcup_{\backsim} I_k$ where I_k are copy's of the unit interval joining vertices in the discrete set B_0. If we take $p^{-1}(B_0) = E_0$ to be a set of vertices in E and consider each I_k to be the image of a path $u : I \to B$ then by 4.8, there is a unique lift $\tilde{u} : I \to E$ passing through every point $p^{-1}(b)$ for $b \in I_k$. These lifts define the edges in E. Since p is a local homeomorphism we can use the uniqueness of liftings on neighbourhoods U_e, where U_e is homeomorphic to $p(U_e)$ via p, to show that every point $e \in E \backslash E_0$ must be an interior point of exactly one of these edges: For paths in $p(U_e)$ a lifting is unique. If e lies in the interior of two paths in U_e then both paths in U_e get mapped onto separate paths in $p(U_e)$. But since $p(e) \in B \backslash B_0$ only one of those paths can exist in $p(U_e)$. $\square$

5 The Theorem of Seifert and van Kampen

All of the definitions,theorems and propositions in this chapter were mainly taken from [5],[7],[8].

Theorem 5.1. (Brown's Theorem) Let $X = X_1 \cup X_2$ be a topological space where $X_1 \cap X_2 = X_{12} \neq \emptyset$ and X_1, X_2, X_{12} are open. Let $i_k : X_{12} \to X_k, j_k : X_k \to X$ be inclusions. The following diagram is a pushout in the category of groupoids:

$$\begin{tikzcd} \Pi(X_{12}) \arrow[r, "\Pi(i_1)"] \arrow[d, "\Pi(i_2)"'] & \Pi(X_1) \arrow[d, "\Pi(j_1)"] \\ \Pi(X_2) \arrow[r, "\Pi(j_2)"'] & \Pi(X) \end{tikzcd}$$

Proof. Because i_k, j_k are inclusions the above diagram commutes. Let $F_k : \Pi(X_k) \to \mathcal{G}$ be functors into a groupoid such that $F_1\Pi(i_1) = F_2\Pi(i_2)$. We have to show that there exists a unique functor $F : \Pi(X) \to \mathcal{G}$ such that $F_1 = F\Pi(j_1)$ and $F_2 = F\Pi(j_2)$. Consider a path $u : I \to X$ that represents a morphism $[u] \in \Pi(X)$ from $u(0) \to u(1)$. By 2.34 we can choose a decomposition

$$0 = t_1 < t_2 < ... < t_{n+1} = 1$$

such that $\forall\, i \in (1, n) : u|_{(t_i,t_{i+1})} = u_i$ is included in some X_k. We get a decomposition:

$$u = u_n \circ u_{n-1} \circ ... \circ u_1$$

Consider a function $\gamma_n : \{1, ..., n\} \to \{1, 2\}$ such that $u_i \in X_{\gamma_n(i)}$, then

$$[u] = \Pi(j_{\gamma_n(n)})([u_n]) \circ\, ... \,\circ \Pi(j_{\gamma_n(1)})([u_1])$$

We want to show that there exists a unique functor F that doesn't depend on the decomposition or γ_n such that:

$$F([u]) = F_{\gamma_n(n)}([u_n]) \circ\, ... \,\circ F_{\gamma_n(1)}([u_1])$$

It is important to note that in this case the compositions $\circ$ above always exist for a viable decomposition of $[u]$. To see this consider the composition $F_{\gamma_n(i+1)}([u_{i+1}]) \circ F_{\gamma_n(i)}([u_i])$. If $\gamma_n(i) = \gamma_n(i + 1)$ we can simply apply that $F_{\gamma_n(i+1)}([u_{i+1}]) \circ F_{\gamma_n(i)}([u_i]) = F_{\gamma_n(i)}([u_{i+1}] \circ [u_i])$. If $\gamma_n(i) \neq \gamma_n(i+1)$, $u(t_{i+1})$ must lie in the intersection $X_1 \cap X_2$ since u_i must fully lie in $X_{\gamma_n(i)}$ and u_{i+1} must fully lie in $X_{\gamma_n(i+1)}$. It follows that there is a path w containing $u(t_{i+1})$ that fully lies in $X_1 \cap X_2$. We can modify the decomposition of u in such a way that between every change in $im(u)$ from X_1 to X_2 and X_2 to X_1 there is such a w. Since $F_1\Pi(i_1) = F_2\Pi(i_2)$ on $[w] \in \Pi(X_1 \cap X_2)$ and $F_k([u] \circ [w]) = F_k([u]) \circ F_k([w])$, the compositions always exist. We will show that F is a well defined functor.

- By commutativity of
$$F_1\Pi(i_1) = F_2\Pi(i_2)$$
F doesn't depend on γ_n for every n.

- Given any viable decomposition of u we can obtain any other viable decomposition by splitting and joining paths with a suitable γ_n. So we just have to argue that F yields the same result after splitting and joining paths, this then implies that F doesn't depend on the decomposition. To join paths, all of them have to lie fully in one of the X_k for the new decomposition to be viable. The same is true for splitting paths. Using $F_k([u \circ w]) = F_k([u] \circ [w]) = F_k([u]) \circ F_k([w])$ we get the desired result.

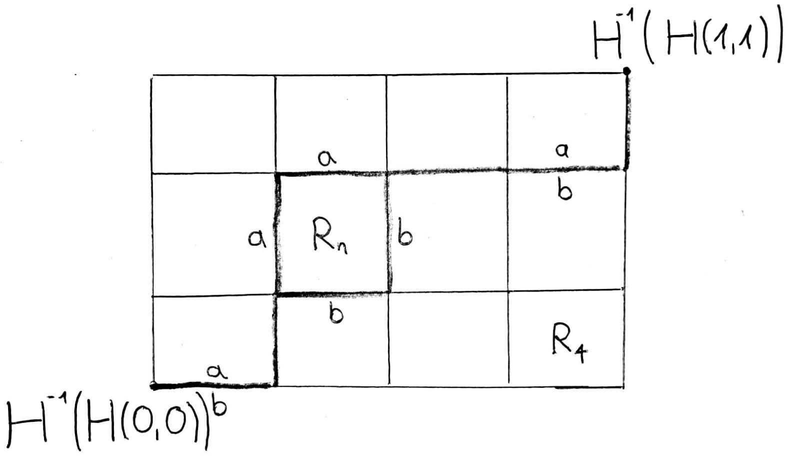
- if u,v are paths and $u \circ v$ is defined,

$$F([u] \circ [v]) = F([u]) \circ F([v])$$

.

- F maps constant paths to zeros of G since both F_1 and F_2 do so.

This leaves to show that $[u] = [v]$ implies $F([u]) = F([v])$. We will use an induction argument. Let $H : I \times I \to X$ be a homotopy of paths $H(0,t), H(1,t)$ from u to v . There exists $n \in \mathbb{N}$ such that H sends each subsquare $R_k = (i/n, (i+1)/n] \times [j/n, (j+1/n)]$ into one of the sets X_k. This follows from 2.34 and the fact that H is continuous. If we consider two edge paths u, u' who's inverse images a, b differ on a subsquare $R_k \subset I \times I$ as indicated in the following figure:



then by construction of the squares R_k both a and b are fully mapped into the same set X_k. Now because F_1, F_2 are well defined functors $F_k([u]) = F_k([u'])\,, k \in \{1,2\}$. So

$$F([u]) = F([u'])$$

With changes like these we can inductively pass from $H(0,s) = u$ to $H(1,s) = v$. □

Theorem 5.2. (The Theorem of Seifert and van Kampen) Let $X = X_1 \cup X_2$ be a topological space where $X_1 \cap X_2 = X_{12} \neq \emptyset$ and $X_1, X_2, X_{1,2}$ are open and path connected Let $i_k : X_{12} \to X_k, j_k : X_k \to X$ be inclusions. The following diagram is a pushout in the category of groups:

$$\begin{tikzcd}
\pi_1(X_{12}) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(X_1) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_2) \arrow[r, "\pi_1(j_2)"'] & \pi_1(X)
\end{tikzcd}$$

Proof. If Z is path connected there is a retraction functor $r : \Pi(Z) \to \pi_1(Z, z)$ onto a subcategory with the single object x We choose a morphism $[u_x] \in \Pi(Z)$ from x to z with $u_z = id$. hence r assigns $u_y v u_x^{-1} : z \to z$ to a morphism $v : x \to y$. We apply this to $Z = X_{12}, X_1, X_2$ and X $x_0 = z$ and choose a morphism $u_x \in \Pi(Z)$ if x is contained in Z. We choose r_1 and r_2 in such a way that $r_1 i_1 = r_{12} i_1$, $r_2 i_2 = r_{12} i_2$. Furthermore with $t_k : \Pi(X_k) \to \Pi(X)$ we get the retraction r by demanding: $r t_k = t_k r_k$. This is possible because every path in $\Pi(X_{12})$ is included in both $\Pi(X_1)$ and $\Pi(X_2)$. Having constructed the restrictions r_k, r from $\Pi(X)$ to $\pi_1(X)$ we can apply 2.12 and the proof is finished. □

Theorem 5.3. (Alternative formulation) If X is the union of open path connected sets X_1, X_2 and if $X_1 \cap X_2$ is path connected then the homomorphism

$$\phi : \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X)$$

is surjective and there is an isomorphism

$$\pi_1(X) \simeq (\pi_1(X_1) * \pi_1(X_2))/N$$

where N is the normal subgroup generated by all elements of the form

$$\pi_1(i_1)([u])\pi_1(i_2)([u])^{-1}$$

for

$$[u] \in \pi_1(X_1 \cap X_2)$$

Remark 5.4. Note that in the future, N will refer to the set of generators of N and N as a normal subgroup equivalently. To see why this makes sense consider the formulation in terms of the presentation of groups. For this refer to Proposition 2.48. Given the two fundamental groups $\pi_1(X_1) = \langle S_1 | R_1 \rangle, \pi_1(X_2) = \langle S_2 | R_2 \rangle$ and R_N the set of generators of N. $\pi_1(X)$ is given by $\pi_1(X) = \langle S_1 \cup S_2 | R_1, R_2, R_N \rangle$.

Proposition 5.5. 5.2 holds for any number of intersecting open path connected sets X_i where $\bigcap_{i \in I} X_i$ is non empty.

Proof. Consider an increasing sequence of adjunction spaces $Y_{i+1} = Y_i \cup X_{i+1}$, where the attaching map for every i is given by $f_i : X_{i+1} \cap Y_i \to Y_i$. We construct a diagram $F : I \to TOP$ in the category of topological spaces where $F(i) = Y_i$. We choose morphisms in this diagram to be inclusions of topological spaces. For the topologies on X_i we simply use the subspace topology from X. The colimit of this diagram is X (the proof is rather trivial and will be left out). Since π_1 is a functor, by applying π_1 to the diagram F we obtain another diagram $P : I \to GRP\,, P(i) = \pi_1(F(i)) = \pi_1(Y_i)$ where morphisms exist in P if and only if morphism exist in F. Note that by 5.2 we can inductively define the fundamental groups of $F(i)$ by applying the theorem in the following way:

$$\begin{tikzcd}
\pi_1(Y_i \cap X_{i+1}) \arrow[r, "\pi_1(i_1)"] \arrow[d, "\pi_1(i_2)"'] & \pi_1(Y_i) \arrow[d, "\pi_1(j_1)"] \\
\pi_1(X_{i+1}) \arrow[r, "\pi(j_2)"'] & \pi_1(Y_{i+1})
\end{tikzcd}$$

If we construct $P(i)$ in this way then, remembering 5.4, we can write the sequence of groups as $P(i+1) = ((*_{(1,..,i)}\pi_1(X_i)/\bigcup_{(1,..,i)} N_j) * \pi_1(X_{i+1}))/N_{i+1} = *_{(1,...,i+1)}\pi_1(X_j)/\bigcup_{(1,...i+1)} N_j$. Where the N_j are the sets of generators of the normal subgroups specified in 5.3. We want to show that, under these conditions, P has a colimit with $colim(P) = \pi_1(colim(F))$. First we show that the colimit exists. For (H, α) to be a cocone in P, the following diagram has to commute whenever $i \leq j$:

$$\begin{tikzcd}
P(i) \arrow[r, "\pi_1(i)"] \arrow[dr, "\alpha_i"'] & P(j) \arrow[d, "\alpha_j"] \\
& H
\end{tikzcd}$$

Consider the smallest normal subgroup N that contains all generators of the subgroups N_i. By construction of the diagram P, (H, α) can only be a cocone if $N \subseteq ker(\alpha_i)$ for every i. Consider the group $*_{i \in I} P(i)/N$ together with a morphism $\gamma_i : P(i) \to *_{i\in I} P(i)/N$ induced by the inclusion of topological spaces. Since $ker(\gamma_i) \subseteq N \subseteq ker(\alpha_i)$ for all i, we can find a λ such that $a_i = \lambda \circ \gamma_i$ for all i. To show uniqueness of λ consider a loop $u : I \to X$ with $[u] \in *_{i\in I} P(i)/N$ then, by compactness of I, there exists an j such that u lies in $F(j)$ and thus $\gamma_j^{-1}([u]) \in P(j)$. So $\alpha_j(\gamma_j^{-1}([u])) = \lambda([u]) = \tilde{\lambda}([u])$. Since this must hold for any u, λ is unique. It follows that $(*_{i\in I} P(i)/N, \lambda)$ is the unique colimit of P. Consider the map $\phi : colim(P) \to \pi_1(X)$. We need to show that ϕ is an isomorphism. Let $u : I \to X$ be a loop in X. Since u is continuous and I is compact, its image in X is compact. Therefore there exists a space Y_i, which is the union of finitely many X_i, such that u is contained in Y_i. Thus ϕ is surjective. Consider two loops u, v where $[u] = [v]$ in X via a homotopy $H : I \times I \to X$. The image of H is by compactness contained in some Y_i, thus $[v] = [u]$ in $\pi_1(Y_j)$ for all $i \leq j$. It follows that ϕ is bijective. $\square$

Remark 5.6. If we define the adjunction spaces on any possible sequences $(X_i)_{i\in I}$ together with the diagram where morphism are inclusions, then we can use the result from the proof for finite intersections, the respective diagram induced by the functor π_1 and the same compactness argument as before to show that the theorem also holds if any finite intersection $\bigcap_{i \in J \subset I} X_i$ is non empty.

5.1 Applications

Example 5.7. We can use a different approach to find the fundamental group of a S^1 without the theory of covering spaces, by defining a groupoid G and applying Brown's theorem to show an isomorphism $G \simeq \Pi(S^1)$. We use complex number notation for S^1. Let $X_1 = S^1 \backslash 1, X_2 = S^1 \backslash {-1}$ Let G be constructed as follows:

- $Obj(G) = S^1$
- $G(a, b) = \{(a, t) \in S^1 \times \mathbb{R} |\, a exp(2it\pi) = b\}$
- $id_a = (a, 0) \in G(a, a)$
- $(b, s) \in G(b, c), (a, t) \in G(a, b) :\ (b, s) \circ (a, t) = (a, s + t) \in G(a, c)$

with inclusions $i_k : X_{12} \to X_k$ and $j_k : X_k \to S^1$. The sets X_k are simply connected. Therefore there exists a single morphism $(a, b)_k : a \to b$ between two objects a, b. There are bijective maps $f_1 :]0, 1[\to X_1 : t \to exp(2it\pi)$, $f_2 :] - 1/2, 1/2[\to X_2 : t \to exp(2it\pi)$.We define functors $\gamma_k : \Pi(X_k) \to G$ through the identity on objects and by $\gamma_k(a, b) = (a, f_k^{-1}(b) - f_k^{-1}(a))$. Now there is a functor $F : G \to \Pi(S^1)$ which is the identity on objects and sends morphisms $(a, t) \in G(a, b)$ to a class of paths $I \to S^1$, $s \to a exp(2its\pi)$ from a to b. Since F is a functor, the following diagram is commutative :

$$\begin{tikzcd}
\Pi(X_{12}) \arrow[r, "\Pi(i_1)"] \arrow[d, "\Pi(i_2)"'] & \Pi(X_1) \arrow[d, "\gamma_1"] \arrow[ddr, bend left, "\Pi(j_1)"] & \\
\Pi(X_2) \arrow[r, "\gamma_2"] \arrow[drr, bend right, "\Pi(j_2)"] & G \arrow[dr, dotted, "F"] & \\
& & \Pi(S^1)
\end{tikzcd}$$

Now we show that F is an isomorphism. We apply 5.1 to the pair (γ_1, γ_2)

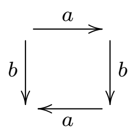
$$\begin{tikzcd}
\Pi(X_{12}) \arrow[r, "\Pi(i_1)"] \arrow[d, "\Pi(i_2)"'] & \Pi(X_1) \arrow[d, "\Pi(j_1)"'] \arrow[ddr, bend left, "\gamma_1"] & \\
\Pi(X_2) \arrow[r, "\Pi(j_2)"] \arrow[drr, bend right, "\gamma_2"'] & \Pi(S^1) \arrow[dr, dotted, "\gamma"] & \\
& & G
\end{tikzcd}$$

and obtain a unique functor $\gamma : \Pi(S^1) \to G$, so $F\gamma = id$. To show that $\gamma F = id$, note that the morphisms of G are generated by the images of γ_1 and γ_2. Given $(a,t) \in G(a,b)$, choose a decomposition $t = t_1 + ... + t_m$ such that $|t_r| < 1/2$ for each r. Set $a_0 = a$ and $a_r = aexp(2i(t_1 + ... + t_r)\pi)$. Then $(a,t) = (a_{m-1}, t_m) \circ ... \circ (a_0, t_1)$ in the G. Since $|t_r| < 1/2$ there exists a function $k : 1, ..., m \to \{0,1\}$ such that $a_{r-1}exp(2it_r s\pi) \in X_{k(r)}$ for $s \in I$. Then $(a_{r-1}, t_r) = \gamma_{k(r)}(a_{r-1}, a_r)$. Thus $G(a,b)$ is generated by morphisms in the images of the γ_k i.e. the images of γ. The set of automorphism of the object 1 in G are given by $\{(1,n)|\, n \in \mathbb{Z}\}$ where $F(1,n)$ is the loop $t \to exp(2int\pi)$. With the composition of morphisms, $\{(1,n)|\, n \in \mathbb{Z}\}$ forms a group that is isomorphic to $\mathbb{Z}$. Thus we have determined the fundamental group of a circle as the automorphisms group of the object 1 in $\Pi(S^1)$.

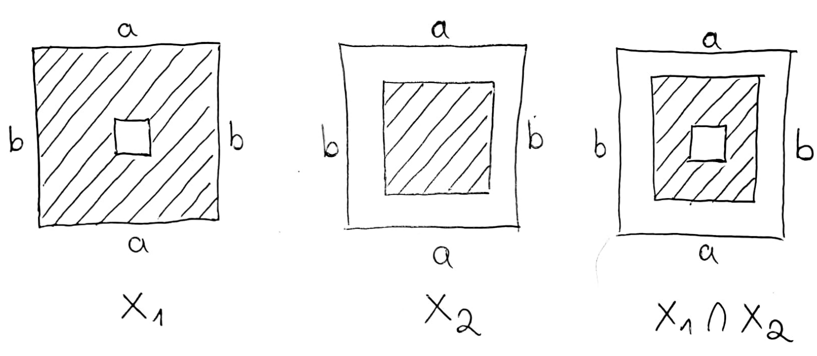
Example 5.8. If $X = X_1 \cup X_2$, X_1, X_2 are path connected and $X_1 \cap X_2$ is simply connected then $\pi_1(X) = \pi_1(X_1) * \pi_1(X_2)$ where $*$ denotes the free product. There are several applications for this, here are two examples:

- If a sequence of X_i is connected through a basepoint that has a contractible neighbourhood then $\pi(\bigvee_{i=1}^n X_i) \simeq *_n \pi_1(X_i)$ If we apply this to a bouquet of circles, we see that its fundamental group is isomorphic to $*_n\mathbb{Z}$.
- Removing a point from a manifold: Given an n-dimensional manifold M with $n > 2$ that is homeomorphic to D^n and $U_p \subset M$ a neighbourhood of $p \in M$. Then M is a pushout of $M \backslash \{p\}$ and U_p, so $\pi_1(M) = \pi_1(U_p) *_{\pi_1(U_p \backslash \{p\})} \pi_1(M \backslash \{p\})$.

Example 5.9. (Compact surfaces) We can compute the fundamental group of compact surfaces by starting with a construction identifying sides of a polygon. For example the Klein bottle X would be obtained from the rectangle



by gluing opposite sites as indicated by the arrows. To compute the fundamental group of X, we draw a smaller square X_1 inside the one above and let X_2 be the frame around it.



Now $\pi_1(X_2) \simeq \{e\}$, $\pi_1(X_2 \cap X_2) \simeq \mathbb{Z}$ and $\pi_1(X_1) \simeq \mathbb{Z} * \mathbb{Z}$. The first two statements are pretty clear, for the last one note that X_1 deformation retracts onto the edges in the above figure. If we glue the rectangle as indicated by the arrows, all the vertices become the same point. Thus, we get two loops connected by a point i.e a bouquet of two circles. In the previous example we saw that its fundamental group is isomorphic to $\mathbb{Z} * \mathbb{Z} = \langle [a], [b] | \emptyset \rangle$. By 5.2,

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(X_1 \cap X_2)} \pi_1(X_2)$$

Given 5.3, we now have to compute N. Let $[u] \in \pi_1(X_1 \cap X_2)$ be a loop (if u is nullhomotopic then $j_1(u)$ and $j_2(u)$ are nullhomotopic). Then $\pi(i_1)([u]) = e$. This implies that $\pi(j_2)([u]) = \pi(j_1)([u]) = e$ in $\pi_1(X)$. Now $\pi(j_2)([u]) = [abab^{-1}]$ as indicated by the arrows in the first figure above. So $[abab^{-1}] \in \pi_1(X_1 \cap X_2)$ becomes $e \in \pi_1(X)$. Therefore

$$N = \pi_1(i_1)([u])\pi_1(i_2)([u])^{-1} = \{[abab^{-1}]\}$$
$$\pi_1(X) \simeq \langle [a], [b] | [abab^{-1}] \rangle$$

Proposition 5.10. Every group G is the fundamental group of some topological space.

Proof. Choose a presentation $G = \langle S, R \rangle$. Now construct X from $\bigvee S^1$ by attaching 2-cells along the loops specified in R. □

Proposition 5.11. If X is a finite connected graph and T is a maximal tree in X then $\pi_1(X)$ is a free group generated by the the edges in $X \backslash T$.

Proof. Let $e_i, i \in I$ be the edges in $X \backslash T$. Because T is contractible, $\pi_1(T) = \{e\}$. Because T is a maximal tree, it contains all vertices in X. So if we glue an edge e_i onto T, it will always connect two vertices that already lie in T. Since for any vertex v_1, v_2 there is a unique path connecting these points to v_0 in T, we create exactly one loop if we connect v_1, v_2 by an edge e_1. To see this clearly we could deformation retract T onto v_0 to which we connect the edges e_i. Every edge then creates a single loop. Consider the spaces $T \cup e_i$. These are path connected and their pairwise intersection is T. We can inductively define a sequence of adjunction spaces $T_{i+1} = T_i \cup e_{i+1}$ then we set $T \cup e_i = X_i$ and $T_i = Y_i$ and apply 5.2 in the following way:

$$\begin{tikzcd}
\pi_1(Y_i \cap X_{i+1}) \arrow[r, "\pi(i_1)"] \arrow[d, "\pi(i)"'] & \pi_1(Y_i) \arrow[d, "\pi(j)"] \\
\pi_1(X_{i+1}) \arrow[r, "\pi(j)"'] & \pi_1(Y_{i+1})
\end{tikzcd}$$

This shows that $\pi_1(T_i) = *_i Z$. For infinite graphs X we construct the diagrams and colimits as in 5.5 and the proof is finished.

Theorem 5.12. Every subgroup H of a free group F is free.

Proof. Given a free group F of rank n, we know by 5.8, that $F \simeq \pi_1(\bigvee S^1) = B$. By 4.14, for each subgroup $H < F$ there is a covering space $p : E \to B$ with $\pi_1(p)(\pi_1(E)) = H$. Since $\pi_1(p)$ is injective, $\pi_1(E) \simeq H$. By 4.15, H is a connected graph, by 5.11, H is free. □

Bibliography

[1] Tim Netzer.*Algebraische Topologie.* Link: https://www.uibk.ac.at/mathematik/algebra/media/teaching/algebraische-topologie.pdf

[2] Tim Netzer. *Algebra.* Link: https://www.uibk.ac.at/mathematik/algebra/media/teaching/algebra.pdf

[3] Harald Woracek. 2021. *Topology.* Link: https://www.asc.tuwien.ac.at/~woracek/homepage/downloads/lva/2021S_Topologie/Topology.pdf

[4] Carson Collins. 2018. *Covering Space, Graphs, and Groups.* Link: https://math.uchicago.edu/~may/REU2018/REUPapers/Collins.pdf

[5] Tammo tom Dieck. 2008. *Algebraic Topology.* Link: https://www.maths.ed.ac.uk/~v1ranick/papers/diecktop.pdf

[6] Randall R. Holmes. 2019. *Category Theory.* Link: https://web.auburn.edu/holmerr/8970/Textbook/CategoryTheory.pdf

[7] Allen Hatcher. 2001. *Algebraic Topology.* Link: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf

[8] Ronald Brown. 2006. *Topology and Groupoids.* Link: https://groupoids.org.uk/pdffiles/topgrpds-e.pdf

[9] Tiago Mendonça Lucena de Veras, Arthur F. Ramos, Ruy J. G. B. de Queiroz, Anjolina G. de Oliveira. 2019. *An alternative approach the calculation of fundamental groups based on labeled natural deduction.* Link: https://arxiv.org/pdf/1906.09107.pdf

,