Matrix Completion Problems

Hannah Entner

March 29, 2020

Contents

Eidesstattliche Erklärung

Ich erkläre hiermit an Eides statt durch meine eigenhändige Unterschrift, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe. Alle Stellen, die wörtlich oder inhaltlich den angegebenen Quellen entnommen wurden, sind als solche kenntlich gemacht.
Ich erkläre mich mit der Archivierung der vorliegenden Bachelorarbeit einverstanden.

Datum, Ort

Unterschrift

1 Introduction

My thesis "Matrix Completion Problems" is based on three papers, the first one of the same title from Monique Laurent[1], the second one *"Poistive Definite Completions of Partial Hermitian Matrices"*, which was published in the book *"Linear Algebra and its Applications"*[2] and the third one *"Algorithmic aspects of vertex elimination on graphs"*[3]. I will denote in each chapter or section on which paper it is based on.

As the title "Matrix Completion Problems" already says, it deals with the completion of given partial matrices, or to find out if there even exists a completion to a described property. The properties, which are interesting, are positive (semi)definite matrices, distance matrices, completely positive matrices, contraction matrices and matrices of a given rank. In my thesis I am going to concentrate on the "Matrix Completion Problems" with positive (semi)definite matrices.
To understand the given problem I want to demonstrate it in a short example.

Example 1.1.

$$\begin{pmatrix} 1 & 1 & ? & 0 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ 0 & ? & 1 & 1 \end{pmatrix}$$

As shown above, you can see what is meant with a partial matrix. The "?" are marking the entries which are not defined. Now we want to know if there is a possibility to complete this matrix to a positive (semi)definite matrix.

Some questions that may occur:

- Are there requirements that I can check to see if there even is a completion?
- Is the completion unique?
- Why can we already tell that it can't be completed to a (semi)definite matrix?
- Is there an algorithm, which can calculate the completion?

Later on these questions will be answered in the context of my thesis. For the derivation of the main theorem, which helps to determine whether a partial matrix is completable or not, I want to start with the most relevant theory.

[1] M. Laurent. Matrix Completion Problems. *Encyclopedia of Optimization.* Springer, Boston, (2008)
[2] R.Grone, C.R. Johnson, E.M. Sá, and H.Wolkowicz. Positive definite completions of partial hermitian matrices. *Linear Algebra and its Applications*, 58:109-124 (1984)
[3] G.S. Leuker, D.J. Rose, R.E. Trajan, Algorithmic aspects of vertex eliminations on graphs, *SIAM J. Comput*, 5(2):266-283 (1976)

2 Definitions and Connections

The first part of this chapter is based on the paper *"Matrix Completion Problems"*[1].

2.1 Positive (semi)definite Matrices

Definition 2.1. A square real symmetric matrix $A \in \mathbb{R}^{(n \times n)}$ or a complex hermitian matrix $A \in \mathbb{C}^{(n \times n)}$, with $n \in \mathbb{N}$, is called:

- **positive semidefinite** $\Leftrightarrow \forall x \in \{\mathbb{R}^{(n\times 1)}, \mathbb{C}^{(n\times 1)}\} : x^\star \cdot A \cdot x \geq 0$
- **positive definite** $\Leftrightarrow \forall x \in \{\mathbb{R}^{(n\times 1)}, \mathbb{C}^{(n\times 1)}\} \setminus \{0\} : x^\star \cdot A \cdot x > 0$

Here $x^\star$ denotes the transposed (in $\mathbb{R}$) or the conjugate transposed (in $\mathbb{C}$) vector of x.

There are some "obvious" properties a matrix has to have in order to be completable. A partial matrix A can only be completed to a positive (semi)definite matrix, if every specified principal submatrix of A is positive (semi)definite. It is also assumed that all diagonal entries are specified. If we have a partial complex matrix A, we suppose that A is partial Hermitian, if a_{ij} is given it is equal to $a^\star_{ji}$ whenever a_{ji} is specified.
To check these properties, I am bringing back the example from above.

Example 2.2.

$$\begin{pmatrix} 1 & 1 & ? & 0 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ 0 & ? & 1 & 1 \end{pmatrix}$$

All diagonal entries are specified and every existing principal submatrix have the shape $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, therefore all the submatrices are positive semidefinite, because for all $x \in \mathbb{R}^{(2\times 1)} \setminus \{0\}$ applies: $x^\star \cdot A \cdot x \geq 0$.

[1] M. Laurent. Matrix Completion Problems. *Encyclopedia of Optimization.* Springer, Boston (2008).

2.2 Graphs and Patterns

The second part of this chapter is based on the paper *"Positive definite completions of partial hermitian matrices"*[2].

Definition 2.3. By $G = (V, E)$ we will denote a finite **undirected graph**. That is the set V of vertices and the set E of edges, which is a subset of the set $\{(x, y) : x, y \in V\}$, consider $(x, y) = (y, x)$ because the graph is undirected.

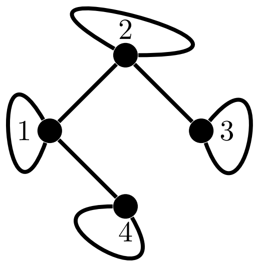
Herewith are some characterizations of graphs, which we are going to need later.

Definition 2.4. Let $G = (V, E)$ denote a graph with $V = \{1, \cdots, n\}$ and $E \subseteq \{(x, y) : x, y \in V\}$, where $n \in \mathbb{N}$.

- A **loop** is an edge, which connects a vertices to itself: $(x, x) \in E$ is a loop for $x \in V$.
- A **clique** is a subset $C \subset V$ with $(x, y) \in E$ for all $x, y \in C$.
- A **cycle** in G is a sequence of pairwise distinct vertices $\gamma = (v_1, \cdots, v_s)$ having the property that $(v_1, v_2), (v_2, v_3), \cdots, (v_{s-1}, v_s), (v_s, v_1) \in E$, where $s \geq 3$ is referred to as the length of the cycle.
- A **chord** of the cycle γ is an edge $(v_i, v_j) \in E$ where $1 \leq i < j \leq s, (i, j) \neq (1, s)$, and $\mid i - j \mid \geq 2$.
- The cycle γ is **minimal**, if any other cycle in G has a vertex not in γ, or equivalently, γ has no chord.
- An **ordering** of G is a bijection $\alpha : V \to \{1, \cdots, n\}$, and y is said to follow x if $\alpha(x) < \alpha(y)$.
- G is called a **band graph**, if there exists an ordering α of G and an integer m, $2 \leq m \leq n$, such that $(x, y) \in E$ if and only if $\mid \alpha(x) - \alpha(y) \mid \leq m - 1$.
- A Graph G is **chordal**, if there are no minimal cycles of length ≥ 4. An alternative characterization is that every cycle of length ≥ 4 has a chord.
- An **induced subgraph** of $G = (V, E)$ is of the form $H = (U, F)$, where $U \subseteq V$ and $F = \{(x, y) \in E : x, y \in U\}$.

[2]R.Grone, C.R. Johnson, E.M. Sá, and H.Wolkowicz. Positive definite completions of partial hermitian matrices. *Linear Algebra and its Applications*, 58:109-124 (1984).

Example 2.5. Let $G = (V, E)$ be a finite undirected graph with set of vertices $V = \{1, 2, 3, 4\}$ and set of edges $E = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (4, 4)\}$. If we draw the graph now, we can easily check if it is a chordal graph or not.



This graph is chordal, because there are no cycles of the length ≥ 4 as induced subgraphs. Actually there are no cycles at all.

In the following I will explain, how the matrices and graphs can be connected and how it will lead us to my main theorem that I want to show in my thesis.

2.3 Matrix $\leftrightsquigarrow$ Graph

Definition 2.6. The set of positions corresponding to the specified entries of a partial matrix A is known as the **pattern** of A. If A is a $n \times n$ partial matrix, its pattern can be represented by a finite undirected graph $G = (V, E)$, where we allow that E may contain loops. We assume that $V = \{1, \cdots, n\}$ and an edge is in E if and only if the entry a_{ij} is specified (so a_{ij} is defined if and only if a_{ji} is).

Definition 2.7. Define a partial matrix with pattern $G = (V, E)$ as $A = [a_{ij}]_G$. A **completion** of the partial matrix $A = [a_{ij}]_G$ is a $n \times n$ matrix $M = [m_{ij}]$ which satisfies $m_{ij} = a_{ij}$ for all $(i, j) \in E$. M is a positive (semi)definite completion of $A = [a_{ij}]_G$ if and only if M is a completion of $A = [a_{ij}]_G$ and M is positive (semi)definite.

Definition 2.8. Suppose $G = (V, E)$ is a subgraph of $J = (V, E')$ $(E \subset E')$. A partial matrix $[b_{ij}]_J$ is said to extend a partial matrix $[a_{ij}]_G$, if $b_{ij} = a_{ij}$ for all $(i, j) \in E$. Therefore a completion of a partial matrix $A = [a_{ij}]_G$ is an extension of A associated with the complete Graph on V. So we state the Graph $G = (V, E)$ is **completable** (nonnegative-completable) if and only if any positive (semi)definite partial matrix $A = [a_{ij}]_G$ has a positive (nonnegative) completion.

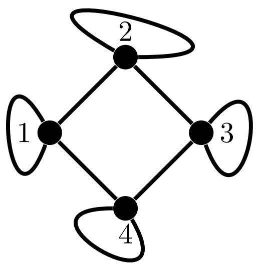
Let us bring back the example from the beginning and look at its pattern.

Example 2.9.

$$\begin{pmatrix} 1 & 1 & ? & 0 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ 0 & ? & 1 & 1 \end{pmatrix}$$

Here the pattern can be represented by the graph $G = (V, E)$ with set of vertices $V = \{1, 2, 3, 4\}$ and set of edges $E = \{(1,1),(1,2),(1,4),(2,2),(2,3),(3,3),(3,4),(4,4)\}$.

As we draw the graph, we can check if it is chordal or not.



In the illustration above we can see that the graph has a minimal cycle of the length ≥ 4 and therefore the graph is not chordal.

Now we have seen how matrices can be interpreted by graphs, in order to check if the graph is chordal or not, which is a very important fact in the next chapter. And since we already assumed, that all the diagonal entries are specified in the partial matrices, we can leave out the loops, that the graphs are not too confusing. Later on we will show why we can do that.

3 The positive (semi)definite Completion Problem

This chapter is based on the paper *"Positive definite completions of partial hermitian matrices"* [1].

In the previous chapter we got to know some properties that can be assumed, so that there could exist a completion of the partial matrix. In the following part we always consider a partial Hermitian Matrix A with entries only specified on a subset of the positions. Moreover we can assume that in the partial matrix all the diagonal elements are specified. In the next step we want to know, if there are any essential assumptions to the partial matrix in order to know if a completion exists and if yes, how we can find such a completion. Therefore one of the main goals of this chapter is to proof the following theorem.

3.1 Existence of a Completion

Theorem 3.1. *The Graph $G = (V, E)$ is completable if and only if the graph G is chordal.*

Before we can proof this theorem, we have to consider some lemmas.

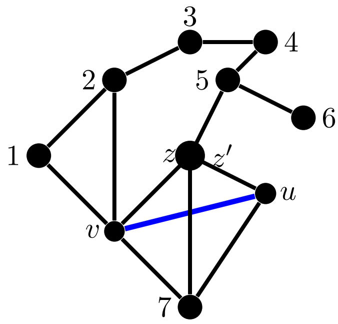
Lemma 3.2. *A Graph G has no minimal cycle of the length exactly 4 if and only if the following holds:*

> *For any pair of vertices u and v with $u \neq v, (u, v) \notin E$ the graph $G + (u, v)$ has a unique maximal clique, which contains both u and v (that is if C and C' are both cliques in $G + (u, v)$, which contain u and v, then so is $C \cup C'$).*

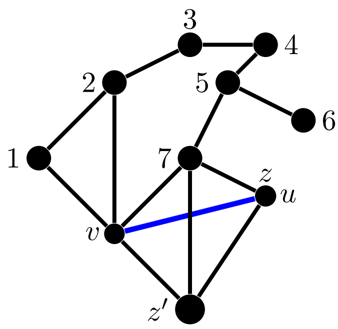
Proof. "$\Rightarrow$" Let G be a graph without minimal cycles of length 4, and randomly choose $u, v \in V$ with $u \neq v, (u, v) \notin E$, and C, C' are as described above. We will show that $C \cup C'$ is a clique in $G + (u, v)$. Let $z \in C$ and $z' \in C'$, we show that (z, z') is an edge of $G + (u, v)$.
We now distinguish three cases:

[1] R.Grone, C.R. Johnson, E.M. Sá, and H.Wolkowicz. Positive definite completions of partial hermitian matrices. *Linear Algebra and its Applications*, 58:109-124 (1984).

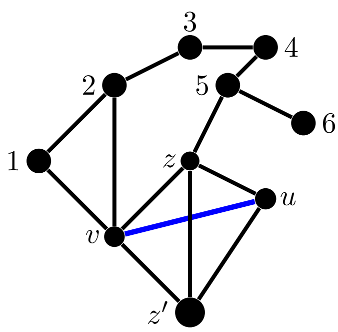
- $z = z'$: This case is trivial, therefore (z, z') is an edge of $G + (u, v)$.



- $(z, z') \cap (u, v) \neq \emptyset$: If we chose $u = z$, and since u and z' are both vertices in the clique C', it is trivial, that (z, z') is an edge of $G + (u, v)$. If we choose $v = z$, $u = z'$ or $v = z'$, you can handle it analogically to this case.

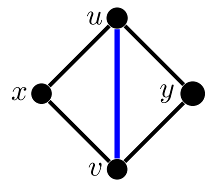


- u, v, z, z' distinct: If we draw the graph now, we can see that (z, u, z', v) is a cycle of $G + (u, v)$, and also of G, but G has, as we assumed in the beginning, no minimal cycles of length 4. Therefore either (u, v) or (z, z') must be an edge of G. Since (u, v) is no edge in G, as we assumed, we now know that (z, z') is an edge of G and therefore also an edge of $G + (u, v)$.



So in all the cases it is clear, that $C \cup C'$ is a clique in $G + (u, v)$.

"⇐" We assume that $\gamma = (x, u, y, v)$ is a minimal cycle of G, so the edges (x, y) and (u, v) are not in G. If we choose $C = \{x, u, v\}$ and $C' = \{y, u, v\}$ as cliques in $G + (u, v)$, which contain u and v, $C \cup C'$ is not a clique of $G + (u, v)$, since (x, y) is not an edge of G or $G + (u, v)$. Therefore G can not have a minimal cycle of the length exactly 4, with $C \cup C'$, a clique, in $G + (u, v)$.



□

We can assume, that a chordal graph holds Lemma 3.2..

Proposition 3.3. *Define L to be the set of vertices of a graph $G = (V, E)$, with $V = \{1, \cdots, n\}$, which have loops and we assume without loss of generality that $L = \emptyset$ or $L = \{1, \cdots, k\}$ for some $1 \leq k \leq n$.*
G is completable (nonnegative-completable) if and only if the graph on L induced by G is completable (nonnegative-completable).

Proof (completable case). Let G' denote the graph on L induced by G.
"⇒" G is completable and therefore G' is completable, since any cliques must be contained in L and because of these there will not be any new fully defined submatrices in G', if we complete G.
"⇐" G' is completable and we want to find a $n \times n$ matrix, which is positive definite. If a general matrix exists, we can say, that G is also completable. We define A as

$$A(x) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & x \cdot I + H \end{pmatrix},$$

where A_{11} is a $k \times k$ matrix, which is the positive completion of a partial matrix with pattern G' and H is a $n - k \times n - k$ Hermitian matrix, then $A(x)$ is positive definite for sufficiently large positive x and therefore G is completable. □

In view of Proposition 3.3. we can assume, that $L = \{1, \cdots, n\}$.

Proposition 3.4. *G is completable if and only if G is nonnegative-completable.*

Proof. "⇒" Assume that G is completable and that $A = [a_{ij}]_G$ is a nonnegative partial matrix with pattern G. Let $A_k = A + (1/k)I$, so that A_k is a positive partial matrix with pattern G for each $k = 1, 2, \cdots$. Let M_k be a positive completion of A_k. Since $L = V$, the sequence $\{M_k\}$ is bounded, and therefore has a convergent subsequence. The limit of this subsequence will then be a nonnegative completion of A.

"$\Leftarrow$" Assume that G is nonnegative-completable and that $A = [a_{ij}]_G$ is a positive partial matrix with pattern G. Choose $\epsilon > 0$ such that $A - \epsilon I$ is still a positive partial matrix with pattern G. Let M be a nonnegative completion of $A - \epsilon I$. Then $M + \epsilon I$ is a positive completion of A. $\square$

In view of Proposition 3.4. we will henceforth only use the term "completable".

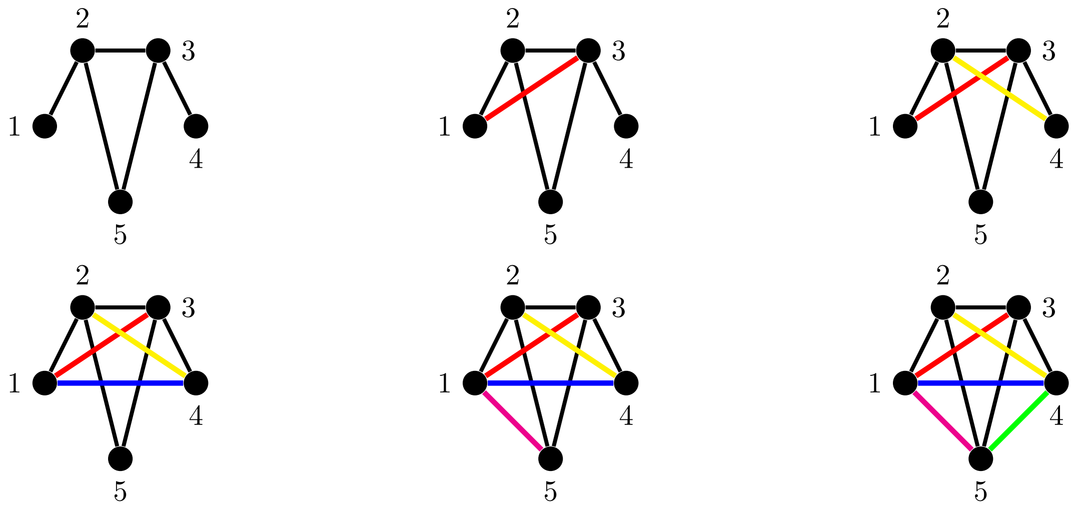
Lemma 3.5. *Let $G = (V, E)$ be chordal, then there exists a sequence of chordal graphs*

$$G_i = (V, E), \qquad i = 0, 1, \cdots, s.$$

such that $G_0 = G$, G_s is the complete graph, and G_i is obtaining by adding an edge to G_{i-1} for all $i = 1, \cdots, s$.

Let's see with an example, if it generally is possible to construct such a sequence.

Example 3.6. Let $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (3, 4), (2, 5), (3, 5)\}$ be chordal.



If we add the red edge $e_0 = (1, 3)$, we get the chordal graph $G_1 = G_0 + e_0$ and we continue the sequence of chordal graphs by adding edges $G_i = G_{i-1} + e_{i-1}$ to the prior graph for $i = 2, 3, 4, 5$. The choice of the edge, which we add next, is not arbitrarily, because, for example, in the first step, we could have chosen any edge except for the blue one. If we would have chosen the blue one, the graph would not have been chordal any more. Therefore we have to check in each step, if the graph is still chordal by adding the edge we chose.

It would be nice to know if there is an algorithm to decide which edge to choose first or if it does not really matter. So to proof the Lemma 3.5, we need some background from the Paper *"Algorithmic aspects of vertex elimination on graphs"*[2]. In this paper they are researching about a perfect elimination ordering on graphs. Therefore there is an algorithm given, which finds a minimal ordering and, as stated in the paper, every minimal ordering is perfect.

Definition 3.7. Let $G = (V, E)$ be a graph with $V = \{1, \cdots, n\}$ and E be the set of edges. A **lexicographic ordering** on a graph is an ordering, where the vertices of the graph are numbered from n to 1. During the search, each vertex v has an associated *label* consisting of a set of numbers selected from $\{1, 2, \cdots, n\}$, in decreasing order. Given two labels $L_1 = [p_1, p_2, \cdots, p_k]$ and $L_2 = [q_1, q_2, \cdots, q_l]$, we define $L_1 < L_2$ if, for some j, $p_i = q_i$ for $i = 1, 2, \cdots, j-1$ and $p_j < q_j$ or if $p_i = q_i$ for $i = 1, 2, \cdots, k$ and $k < l$. $L_1 = L_2$ if $k = l$ and $p_i = q_i$, $1 \leq i \leq k$.

To find minimal orderings and perfect orderings, we use a lexicographic ordering scheme.

Algorithm lexicographic ordering

begin

assign the label $\emptyset$ to all vertices;

for $i := n$ **step** -1 **until** 1 **do begin**

select:

pick an unnumbered vertex v with largest *label*;

comment assign v the number i;

$\alpha(i) := v$

update:

for each unnumbered vertex w such that there is a chain $(v = v_1, v_2, \cdots, v_{k+1} = w)$ with v_j unnumbered and $label(v_j) < label(w)$ for $j = 2, 3, \cdots, k$ **do** add i to $label(w)$;

end end

For the unordered graph $G = (V, E)$ this algorithm constructs an ordering α and a label $L(v)$ given by the final value of $label(v)$ for each $v \in V$.

Definition 3.8. For any vertex v with label $L(v)$, let $L_i(v) = L(v) \cap \{n, n-1, \cdots, i+1\}$ be the **value of the** $label(v)$ in the algorithm above, just before the number i is assigned to a vertex. (Note: $L_n(v) = \emptyset$ for all $v \in V$)

[2]G.S. Leuker, D.J. Rose, R.E. Trajan, Algorithmic aspects of vertex eliminations on graphs, *SIAM J. Comput*, 5(2):266-283 (1976).

Lemma 3.9. *Let $G = (V, E)$ be a graph with lexicographic ordering α and labels $L(v)$, $v \in V$.*

(i) If $L_i(v) < L_i(w)$, then $L_j(v) < L_j(w)$ for all $1 \leq j \leq i$.

(ii) $L_i(w) \leq L_j(w)$ for all $j \leq i$.

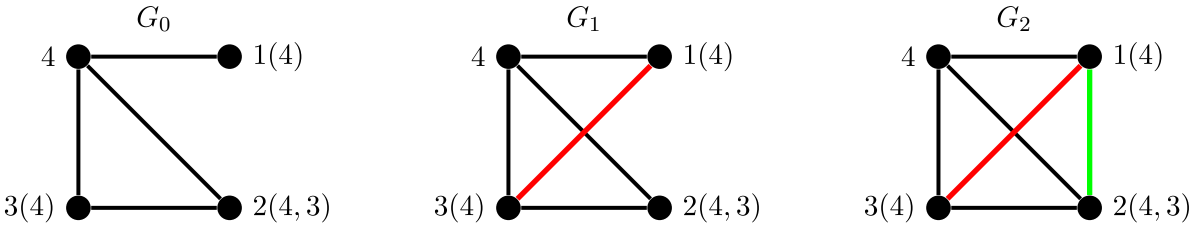
(iii) If $\alpha^{-1}(w) = j < \alpha^{-1}(v) = i$, then either $L_i(w) < L_i(v)$ and $L(w) < L(v)$, or $L_i(w) = L_i(v)$ and $L(v) \leq L(w)$.

(iv) If $L(w) < L(v)$ with $\alpha^{-1}(w) = j$ and $\alpha^{-1}(v) = i$, then either $j < i$ with $L_i(w) < L_i(v)$, or $i < j$ with $L_j(w) = L_j(v)$.

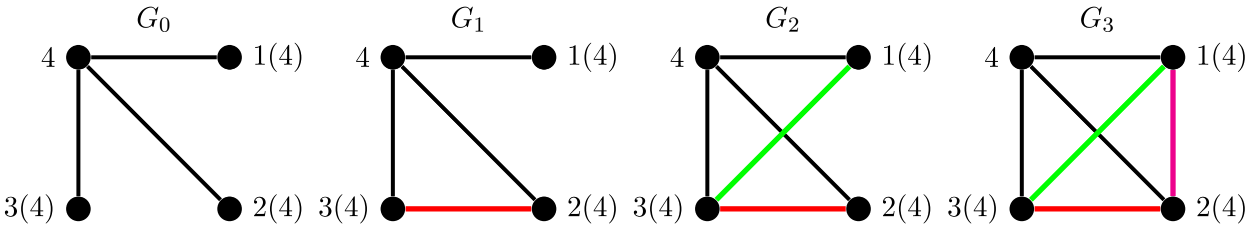
(v) $j \in L(w)$ if and only if $\alpha^{-1}(w) < j$ and there exists a $v = \alpha(j)$, w chain $[v = v_1, v_2, \cdots, v_{k+1} = w]$ such that $L_j(v_i) < L_j(w)$ and $\alpha^{-1}(v_i) < j$, $2 \leq i \leq k$.

Proof. The proofs of (i)-(v) are straightforward. Properties (i) and (ii) follow from the definition of the labels and the order relation; (iii) and (iv) summarize the statement *select* of the algorithm above. Property (v) follows from (i), (ii) and statement *update* of the algorithm and it means that the updated labels produced by one execution of statement update depend only on the old labels and not on the order of updating. □

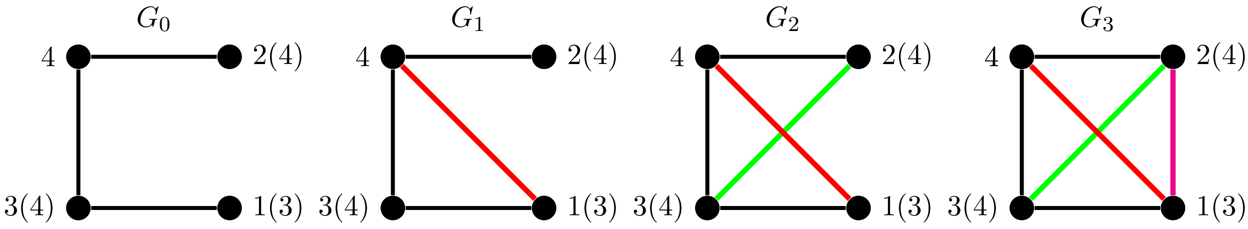
Proof Lemma 3.5. Let $G(V, E)$ be a chordal graph with $V = \{1, \cdots, n\}$. We want to construct a sequence of chordal graphs $[G_0, G_1, \cdots, G_s]$, with G_s the complete graph and $G = G_0$. We will prove the Lemma with induction. For $n = 1, 2, 3$ it is trivial, because there are just chordal graphs on 3 or less vertices. We check $n = 4$ if the given graph is almost complete with just one edge missing to be complete, ($s = 1$) is also trivial because every complete graph is chordal. For $s = 2$ we could have a graph like this, with the minimal ordering from the algorithm:



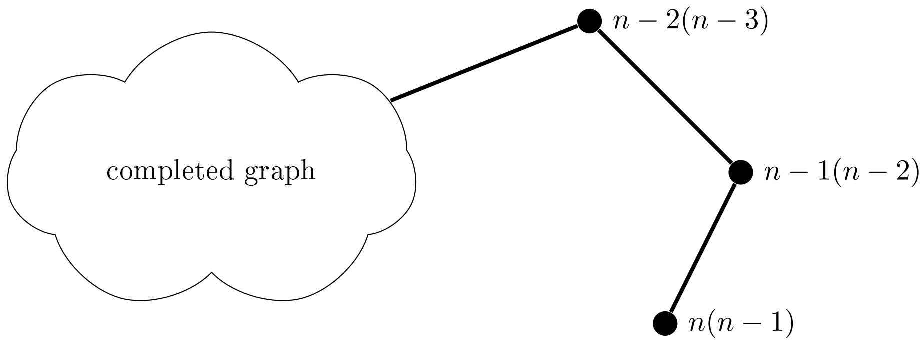
So we are using the ordering to select which edge to add first. For $s = 3$ our graphs could somehow look like this:



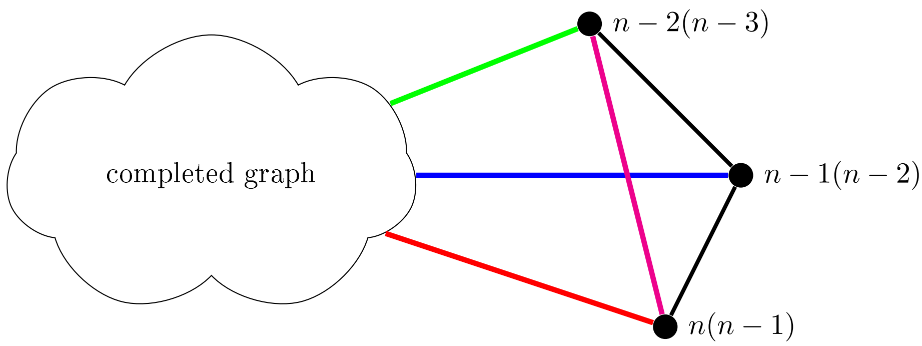
In this case you can "choose" in the algorithm, which vertex you give a number first, but also with $n = 4$ and $s = 3$, there is another possibility for the initial graph, which makes it not as random as the case before:



As we checked all possibilities for $n = 4$, we now can assume, that it is correct for all integer until $n - 1$. We check the lemma for n. Let's assume that we have a chordal graph given, that is completed for all its vertices except for $\{n-2, n-1, n\}$.



First, we can connect all the vertices from the completed part of the graph with edges to the three vertices on the outside and the graph will still be chordal. Next, we only have to concentrate on the vertices on the outside. If we add the edges just as before, the lemma is proofed.



$\square$

The following lemmas are based on the paper *"Positive definite completions of partial hermitian matrices"* [3].

Lemma 3.10. *Let $G' = (V', E')$ denote a subgraph of $G = (V, E)$ induced by a subset $V' \subset V$. If G is completable, then so is G'.*

Proof. Let $A' = [a_{ij}]_{G'}$ be a partial nonnegative matrix. Define a partial nonnegative matrix $A = [a_{ij}]_G$ via

$$a_{ij} = \begin{cases} a'_{ij} & if \quad (i,j) \in E' \\ 0 & if \quad (i,j) \in E \setminus E' \end{cases}$$

Since G is completable (Proposition 3.3), regarding the previous Definitions there is a nonnegative completion M of A . Now let M' be the principal submatrix of M corresponding to rows and columns indexed by V'. Therefore M' is the completion of A' and G' is also completable. □

Lemma 3.11. *There is a unique k-by-k positive semidefinite matrix $A = [a_{ij}]$, which satisfies*

$$a_{ij} = 1, \qquad all \quad |i - j| \leq 1. \tag{3.1}$$

Namely, A is the matrix of all 1's.

Proof. Since the existence of the positive semidefinite matrix of all 1's is trivial, we only need to show the uniqueness. For $k = 2$ with property (3.1) A is already the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and therefore the Lemma holds. For $k = 3$ we have with (3.1):

$$A = \begin{pmatrix} 1 & 1 & ? \\ 1 & 1 & 1 \\ ? & 1 & 1 \end{pmatrix},$$

now we consider that all principal submatrices have to be positive semidefinite, if we cross out the first or third row and column we have the case $k = 2$. Next we have to check, what we could exchange the "?" with, so that the matrix is positive semidefinite, denote the "?" as x:

$$det(A) = det \begin{pmatrix} 1 & 1 & x \\ 1 & 1 & 1 \\ x & 1 & 1 \end{pmatrix} = 1 + 2x - x^2 - 2 = -x^2 + 2x - 1$$

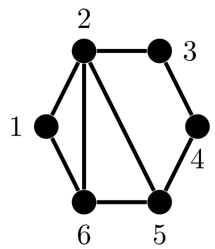
The determinant has to be ≥ 0 and if it is > 0, the matrix would be positive definite. Therefore we have the exact result $x = 1$, because in every other case the determinant would be < 0 and the lemma holds for $k = 3$. Assume that $k \geq 4$ and that the $k \times k$ matrix A satisfies (3.1).

[3] R.Grone, C.R. Johnson, E.M. Sá, and H.Wolkowicz. Positive definite completions of partial hermitian matrices. *Linear Algebra and its Applications*, 58:109-124 (1984).

$$\begin{matrix} 1 & 1 & ? & \cdots & \cdots & ? \\ 1 & \ddots & \ddots & \ddots & ? & \vdots \\ ? & \ddots & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & ? \\ \vdots & ? & \ddots & \ddots & \ddots & 1 \\ ? & \cdots & \cdots & ? & 1 & 1 \end{matrix}$$

Let $1 \leq i < j \leq k$, $\mid i-j \mid \geq 2$. The 3×3 principal submatrices of A corresponding to rows and columns i, $i+1$, j are positive semidefinite and satisfies (3.1). Hence $a_{ij} = 1 = a_{ji}$, since we already checked the case $k = 3$. And therefore the unique positive semidefinite matrix is the one of all 1's. □

Proof Theorem 3.1. (only if). We consider that there exists a completion of the partial matrix, so let $G = (V, E)$ be a completable graph. To obtain a contradiction, we assume that G has a minimal cycle γ of length ≥ 4. For example the graph could look like this:



We may assume without loss of generality that $\gamma = \{1, \cdots, k\}$. With the example of the graph above, we have the minimal cycle $\gamma = \{2, 3, 4, 5\}$. Let G' be the graph on $\{1, \cdots, k\}$ induced by G and consider the $k \times k$ partial matrix $A' = [a'_{ij}]_{G'}$ defined by

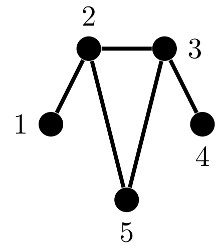
$$\begin{aligned} &a'_{ij} = 1 \qquad if \quad |i-j| \leq 1, \\ &a'_{1k} = -1 (= a'_{k1}) \end{aligned} \tag{3.2}$$

If we continue the example, we have a 4×4 matrix A' and with (3.2) we get:

$$A' = \begin{pmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix}$$

Not only in our example, but also in general it is easy to see, that A' is a partial semidefinite matrix with pattern G' (by checking the specified principal submatrices of order 2). In consideration of Lemma 3.11, A' is not completable to a positive semidefinite matrix, and that's why G' is not completable. However, G' is completable by Lemma 3.10, and we have a contradiction and therefore G has to be chordal. □

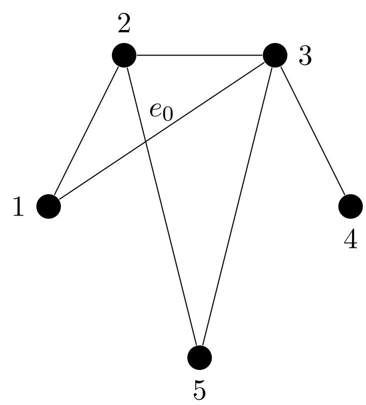
Proof Theorem 3.1. (if). Assume that $G = (V, E)$ is chordal and not the complete graph on $V = \{1, \cdots, n\}$, for example the set of edges could be $E = \{(1, 2), (2, 3), (2, 5), (3, 5), (3, 4)\}$.



Let $G = G_0, G_1, \cdots, G_s$ be a sequence of chordal graphs satisfying the conditions of Lemma 3.5. Let A be any partial positive matrix with pattern G. With the pattern we used above and any partial matrix A could be:

$$A = \begin{pmatrix} 1 & 1 & ? & ? & ? \\ 1 & 1 & 1 & ? & 1 \\ ? & 1 & 1 & 1 & 1 \\ ? & ? & 1 & 1 & ? \\ ? & 1 & 1 & ? & 1 \end{pmatrix}$$

If we can show that a partial matrix $A_1 = [a_{ij}]_{G_1}$ exists, which extends A, then the existence of a positive completion of A will follow by induction.



Let $e_0 = (u, v)$ be the edge of G_1, which is no edge of G. By Lemma 3.2 there is a unique maximal clique C of G_1, which contains both u and v. We may assume without loss of generality that $C = \{1, \cdots, p\}$ and that $u = 1$ and $v = p$, in our case $C = \{1, 2, 3\}$ with $u = 1$ and $v = 3$. For any $z \in \mathbb{C}$, let $A_1(z)$ denote the partial matrix with pattern G_1, which extends A with z in position $(1, p)$ and $\overline{z}$ in position $(p, 1)$, in our case $z = a_{13}$ and $\overline{z} = a_{31}$.

$$A_1(z) = \begin{pmatrix} 1 & 1 & z & ? & ? \\ 1 & 1 & 1 & ? & 1 \\ \overline{z} & 1 & 1 & 1 & 1 \\ ? & ? & 1 & 1 & ? \\ ? & 1 & 1 & ? & 1 \end{pmatrix}$$

Let $M(z)$ be the leading principal $p \times p$ submatrix of $A_1(z)$
Our case:

$$M(z) = \begin{pmatrix} 1 & 1 & z \\ 1 & 1 & 1 \\ \overline{z} & 1 & 1 \end{pmatrix}$$

Since any clique containing $(1, p)$ is a subset of $\{1, \cdots, p\}$, $A_1(z)$ is partial positive if $M(z)$ is a positive matrix. Hence we only need to show that $M(z_0)$ is positive definite for some z_0. The graph on C induced by G has exactly the set of edges

$$\{(i, j) \mid |i - j| \leq p - 2, 1 \leq i \leq j \leq p\}$$

and consequently is a band graph. Therefore there exists z_0 as required.
In our example we can use Lemma 3.11 and know that $z = 1$ for the matrix to be positive semidefinite. □

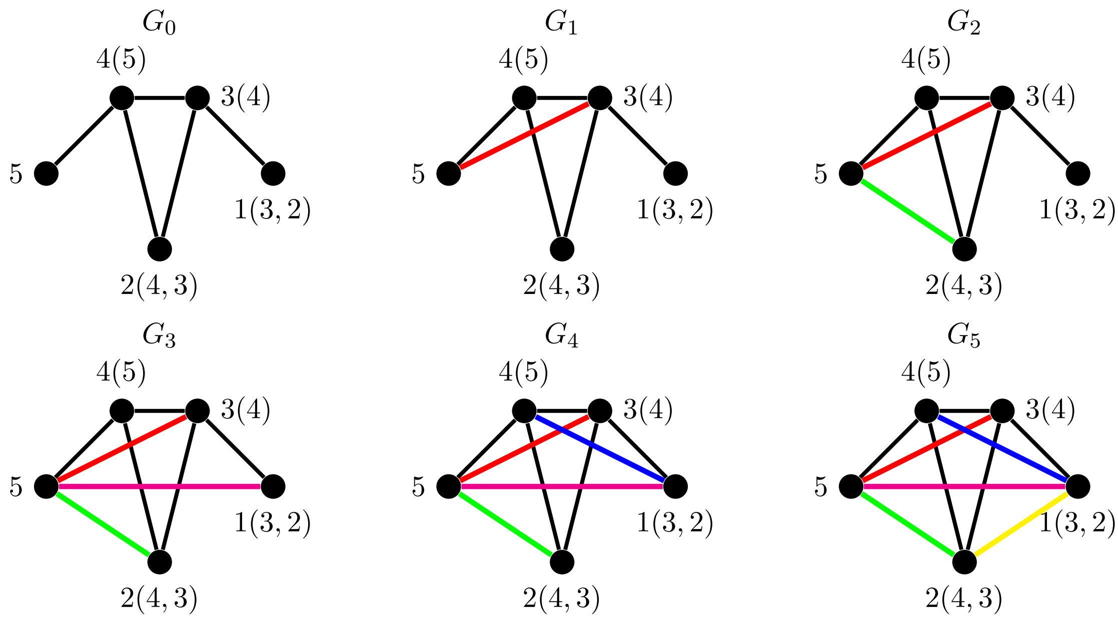
3.2 Example

I would like to conclude my thesis with another example.

Example 3.12. Let A be a partial matrix and we want to complete it to a positive semidefinit matrix.

$$A = \begin{pmatrix} 1 & 2 & ? & ? & ? \\ 2 & 5 & -3 & ? & 4 \\ ? & -3 & 3 & -1 & 0 \\ ? & ? & -1 & 2 & ? \\ ? & 4 & 0 & ? & 10 \end{pmatrix}$$

In the first step we have to check if A is a partial positive semidefinite matrix. It is easy to check that all defined submatrices are positive semidefinit. If we draw the pattern for the partial matrix, which is the same as in the proof above, we also see that the graph is chordal and therefore all the conditions in order dot the partial matrix to b completed are fulfilled. Therefore we can start with constructing the sequence of chordal graphs to get an order in which we can fill up the A.



Now we can start to complete the matrix with the position in the first row and third column and since it has to be symmetric, also the third row and first column. To simplify the process, I have named the rows and columns from a to e. Therefore all the defined submatrices, which will contain this element, have to stay positive semidefinite. Hence we are looking for a number that $M(z)$, as described in the proof above, is positive semidefinite.

$$det(M_{(a,c)}(z)) = \begin{pmatrix} 1 & 2 & z \\ 2 & 5 & -3 \\ z & -3 & 3 \end{pmatrix} = -5z^2 - 12z - 6$$

For the matrix to be positive semidefinite, the determinate has to be ≥ 0, so if we choose $z = -1$ this is given. With that we now have A_1 with pattern G_1.

$$A_1 = \begin{pmatrix} 1 & 2 & -1 & ? & ? \\ 2 & 5 & -3 & ? & 4 \\ -1 & -3 & 3 & -1 & 0 \\ ? & ? & -1 & 2 & ? \\ ? & 4 & 0 & ? & 10 \end{pmatrix}$$

In the next step we fill up the position (a, e), therefore we have to check the submatrix without the fourth row and column, and so on.

$$det(M_{(a,e)}(z)) = \begin{pmatrix} 1 & 2 & -1 & z \\ 2 & 5 & -3 & 4 \\ -1 & -3 & 3 & 0 \\ z & 4 & 0 & 10 \end{pmatrix} = -6z^2 + 24z - 22 \qquad A_2 = \begin{pmatrix} 1 & 2 & -1 & ? & 2 \\ 2 & 5 & -3 & ? & 4 \\ -1 & -3 & 3 & -1 & 0 \\ ? & ? & -1 & 2 & ? \\ 2 & 4 & 0 & ? & 10 \end{pmatrix}$$

$$det(M_{(a,d)}(z)) = \begin{pmatrix} 1 & -1 & z \\ -1 & 3 & -1 \\ z & -1 & 2 \end{pmatrix} = -3z^2 + 2z + 3 \qquad A_3 = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 \\ 2 & 5 & -3 & ? & 4 \\ -1 & -3 & 3 & -1 & 0 \\ 0 & ? & -1 & 2 & ? \\ 2 & 4 & 0 & ? & 10 \end{pmatrix}$$

$$det(M_{(b,d)}(z)) = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -3 & z \\ -1 & -3 & 3 & -1 \\ 0 & z & -1 & 2 \end{pmatrix} = -2z^2 + 2z + 1 \qquad A_4 = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 \\ 2 & 5 & -3 & 1 & 4 \\ -1 & -3 & 3 & -1 & 0 \\ 0 & 1 & -1 & 2 & ? \\ 2 & 4 & 0 & ? & 10 \end{pmatrix}$$

$$det(M_{(d,e)}(z)) = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 \\ 2 & 5 & -3 & 1 & 4 \\ -1 & -3 & 3 & -1 & 0 \\ 0 & 1 & -1 & 2 & z \\ 2 & 4 & 0 & z & 10 \end{pmatrix} = -z^2 + 2 \qquad A_5 = \begin{pmatrix} 1 & 2 & -1 & 0 & 2 \\ 2 & 5 & -3 & 1 & 4 \\ -1 & -3 & 3 & -1 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 2 & 4 & 0 & -1 & 10 \end{pmatrix}$$

A_5 is our completed matrix and as we wanted, the matrix is positive semidefinite.

We have seen some properties and also an algorithm to complete a partial positive semidefinite matrix to a positive semidefinite matrix. Concluding we can say, that we are only able to complete a partial matrix to a nonnegative matrix, if the graph, which is the pattern of the partial matrix, is chordal and this was the main goal to show in my thesis.

4 References

1 M. Laurent. Matrix Completion Problems. *Encyclopedia of Optimization*. Springer, Boston (2008).

2 R.Grone, C.R. Johnson, E.M. Sá, and H.Wolkowicz. Positive definite completions of partial hermitian matrices. *Linear Algebra and its Applications*, 58:109-124 (1984).

3 G.S. Leuker, D.J. Rose, R.E. Trajan, Algorithmic aspects of vertex eliminations on graphs, *SIAM J. Comput*, 5(2):266-283 (1976).