Characteristic approach to the soliton resolution

Maciej Maliborski

University of Vienna

joint work with Piotr Bizoń and Bradley Cownden
arXiv:2112.11249

September 27, 2022 @ Summer School 2022: Geometric dispersive PDEs

Characteristic approach to the soliton resolution

- Soliton resolution
- Characteristic approach
- Results

"(...) numerical simulations can 'provide insight into deep and fundamental properties of a mathematical model and lead to the discovery of completely new phenomena.'" – Gennady El

in "Landmarks-Computer Simulations Led to Discovery of Solitons" by David Lindley

Soliton resolution (1)

Global-in-time generic solutions of nonlinear dispersive wave equations (on unbounded domains)

$$\partial_t U = F(U)\,,$$

resolve for $t \to \infty$ into

$$U \sim \sum_i Q_i \quad + \text{radiation}\,.$$

Asymptotic simplification: the set of possible endstates is 'smaller' than the set of initial data.

- Physical evidence: solitons in optical fibers, or black hole mergers.
- Recent rigorous results: equivariant wave maps $\mathbb{R}^{2+1} \to \mathbb{S}^2$ [Duyckaerts et al. '21] , [Jendrej&Lawrie '21] and the $4+1$ Yang-Mills equation [Jendrej&Lawrie '21] ; also radial solutions of the energy critical nonlinear wave equation [Kenig '21] . Abstract results.
- **This work: another evidence supporting the conjecture. Advantage of using the characteristic foliations. A toy-model with interesting dynamics.**

Soliton resolution (2)

Model: the equivariant YM equation in $4+1$ dimensions (critical case)

$$W_{tt} = W_{rr} + \frac{1}{r} W_r + \frac{2}{r^2} W(1-W^2)\,.$$

- Spatial domain: $r \geq 1$, with $W(t, r=1) = 0$. Remark: corner conditions.
- Conserved energy

$$E[W] = \frac{1}{2} \int_1^\infty \left(W_t^2 + W_r^2 + \frac{(1-W^2)^2}{r^2} \right) r dr.$$

- Finite energy solutions $|W(t,\infty)| = 1$.
- Global existence.
- The half-kink $Q(r) = \frac{r^2-1}{r^2+1}$, a global minimizer of energy, a natural candidate for an attractor.
- Coherent structures for large energies in the asymptotic region possible, feature absent in supercritical dimensions, cf. WM $3+1$ [Bizoń et al. '12], [Lawrie&Schlag '13], [Kenig et al. '14], [Kenig et al. '15].

Soliton resolution (3)

YM equation in $4+1$ dimensions

$$W_{tt} = W_{rr} + \frac{1}{r} W_r + \frac{2}{r^2}\, W(1 - W^2)\,,$$

on **the whole space** $r \geq 0$.

- Scale invariance $W_\lambda(t,r) = W(t/\lambda, r/\lambda)$.
- Energy criticality $E_0[W_\lambda] = E_0[W]$

$$E_0[W] = \frac{1}{2}\int_0^\infty \left(W_t^2 + W_r^2 + \frac{(1-W^2)^2}{r^2} \right) r dr\,.$$

- Kinks $Q_\lambda(r)$ and antikinks $-Q_\lambda(r)$

$$Q_\lambda(r) = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}\,, \qquad E_0[\pm Q_\lambda] = \frac{4}{3}\,,$$

  interpolate between $W = \pm 1$.

Soliton resolution (4)

Theorem [Jendrej&Lawrie '21] : Any finite-energy solution of the $4+1$ YM equation* posed **on the whole space** tends (modulo sign) either to the vacuum $W=1$ or to an alternating chain of N rescaled kinks and antikinks

$$1+\sum_{j=1}^{N}(-1)^{N+j}\left(Q_{\lambda_j(t)}(r)-1\right) .$$

Here $\lambda_j(t)$ are continuous positive functions such that for each $j=1,...,N$

$$\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\to 0 \quad \text{as} \quad \begin{cases} t\to\infty & \text{(for global-in-time solutions)},\\ t\to T & \text{(for blowup at finite time } T\text{)},\end{cases}$$

where $\lambda_{N+1}(t)=t$ (global) or $\lambda_{N+1}(t)=T-t$ (blowup), corresponding to the non-existing self-similar expansion and collapse.

Soliton resolution (5)

Conjecture **[Bizoń et al. '21] :** Any smooth, finite-energy solution $W(t, r)$ of the $4 + 1$ YM equation **subject to the boundary condition** $W(t, 1) = 0$ tends for $t \to \infty$ (modulo sign) either to the half-kink or to the rescaled half-kink plus an alternating chain of N rescaled kinks and antikinks:

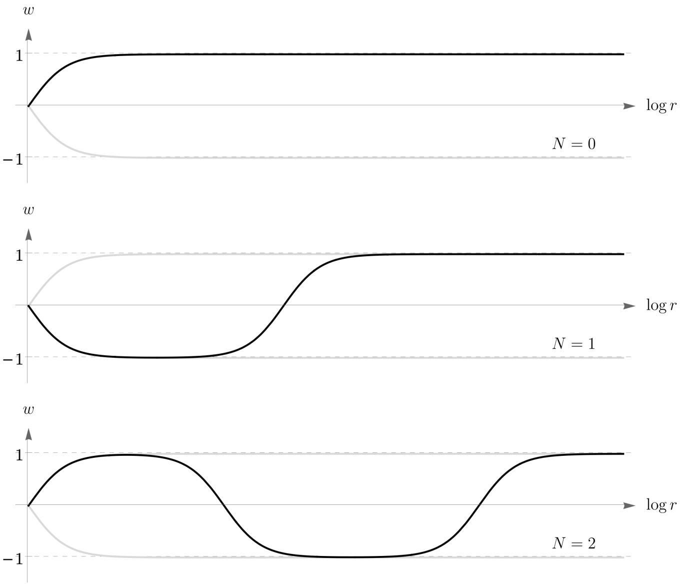
$$\begin{cases} Q(r) & \text{if } N = 0, \\ -Q_{\mu(t)}(r) + \sum_{j=1}^{N} (-1)^{j+1} Q_{\lambda_j(t)}(r) + 1 & \text{if } N \text{ is odd}, \\ Q_{\mu(t)}(r) + \sum_{j=1}^{N} (-1)^{j} Q_{\lambda_j(t)}(r) & \text{if } N \geq 2 \text{ is even}. \end{cases}$$

Here $\lambda_j(t)$ are continuous positive functions such that for each $j = 1, ..., N$

$$\lambda_j(t) \to \infty \quad \text{and} \quad \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \quad \text{as} \quad t \to \infty,$$

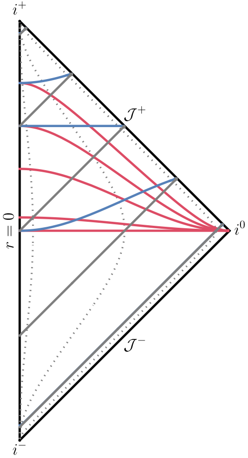
where by convention $\lambda_{N+1}(t) = t$. Also, as $W(t, 1) = 0$ there is $\mu(t) \to 1$ for $t \to \infty$.

Soliton resolution (6)



Characteristic formulation (1)

Flat space in spherical-polar coordinates: $\eta = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$



- Minkowski time slices

$$t = \mathrm{const}\,,$$

- Hyperbolidal slicing, e.g.

$$t - \sqrt{a^2 + r^2} = s = \mathrm{const}\,,$$

$a = \mathrm{const},$

- Null slicing

$$t - r = u = \mathrm{const}\,,$$

$a \to 0.$

Characteristic formulation (2)

- Using $u = t - r$, and $x = r^{-\frac{1}{2}}$ we get

$$-4x\,\mathsf{w}_{xu} + 4\mathsf{w}_u = x^4\,\mathsf{w}_{xx} + x^3\,\mathsf{w}_x + 8x^2\mathsf{w}(1-\mathsf{w}^2)\,,$$
$$\mathsf{w}(u,1) = 0\,, \qquad \mathsf{w}(0,x) = g(x)\,.$$

- Smooth solutions for smooth initial data $g(x)$ of finite energy ($g(0) = 1$). No corner conditions.
- The asymptotic expansion near $x = 0$ [Chruściel et al. '06, '11]

$$\mathsf{w}(u,x) = 1 + \underbrace{c_1(u)}_{\substack{\text{radiation}\\\text{coefficient}}}\, x + \underbrace{c_2(u)}_{\substack{\text{the}\\\text{Newman-Penrose}\\\text{constant}}}\, x^2 + \underbrace{c_3(u)}_{\dot{c}_3 = \frac{15}{8} c_1}\, x^3 + \cdots\,,$$

non-uniform expansion [Bizoń&Friedrich '13]: for large n the $|c_n(u)|$ grows polynomially with u. Remark: numerical difficulties.

Characteristic formulation (3)

- The local conservation law → the energy loss formula

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}u} = -\dot{c}_1^2(u)\,, \quad \mathcal{E}[\mathsf{w}] := \int_0^1 \left(\frac{1}{4}\,\mathsf{w}_x^2 + \frac{1}{x^2}(1-\mathsf{w}^2)^2 \right) x\,dx\,.$$

- The energy $\mathcal{E}$ is non-increasing and bounded from below, there exists

$$\mathcal{E}_\infty = \lim_{u\to\infty} \mathcal{E}[\mathsf{w}(u,\cdot\,)] \geq \frac{2}{3}\,.$$

- For the endstate of the evolution, which according to the Conjecture is a non-radiative superposition of the half-kink and N kinks/anti-kinks, there is

$$\mathcal{E}_\infty = \lim_{u\to\infty} \mathcal{E}[\mathsf{w}(u,\cdot)] = \frac{2}{3} + \frac{4}{3}N\,.$$

- **We study the relaxation to the $N=0$ and $N=1$ attractors. Remark: no convincing numerical evidence of $N \geq 2$ states (unstable configurations?).**

Linearized dynamics around $N = 0$ attractor (1)

- With $\mathrm{w} = q(x) + x f(u,x)$ we get

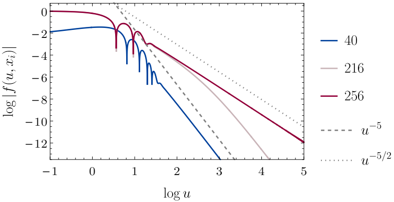
$$f_{ux} + \frac{1}{4}\partial_x(x^3 f_x) - U(x) f - 6q(x)x^2 f^2 - 2x^3 f^3 = 0\,, \qquad f(u,1) = 0\,,$$

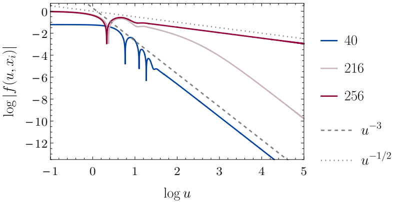
with $U(x) = \frac{15}{4}x + \mathcal{O}(x^5)$.

- Late time behaviour

$$f(u,x) \sim \begin{cases} u^{-5} & (x > 0, u \to \infty)\,, \\ u^{-5/2} & (x = 0, u \to \infty)\,. \end{cases}$$

for generic solutions with vanishing NP constant. Slower decay rate for data with nonzero NP constant.





Linearized dynamics around $N = 0$ attractor (2)

- With $y = x^2$ and $f(u,x) = e^{su}\, v(y)$ we get

$$2sv' + (y^2 v')' - V(y)v = 0, \quad V(y) = \frac{15}{4} - \frac{48 + 5y^2}{(1+y^2)^2}\, y^2\,, \quad v(1) = 0\,.$$

- Various methods of solving the eigenvalue problem. Here we follow the approach of [Leaver '90]

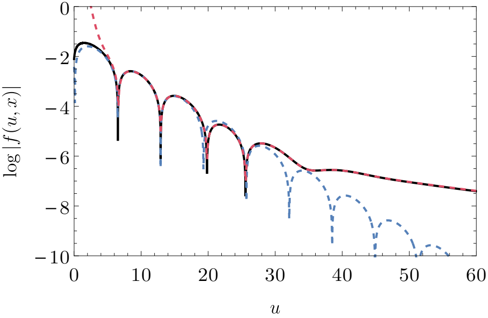
$$v(y) = \sum_{j\geq 1} a_j (1-y)^j\,, \qquad a_1 = 1\,,$$

absolutely convergent for $y \in (0,1]$.

- The eigenvalues s_n (QN frequencies) selected by the condition that the series is absolutely convergent at $y = 0$; the corresponding solutions $f_n(u,y)$ (QN modes) are purely outgoing.
- One pair of complex conjugate frequencies

$$s \approx -0.364322 \pm 0.476858i\,.$$

Linearized dynamics around $N = 0$ attractor (3)



Remark: single quasinormal mode (explicit solution) for perturbation of $\mathrm{w} = 1$.

Nonlinear dynamics around $N = 0$ attractor (1)

- Initial data

$$\mathsf{w}(0,x) = 1 + bx^4 - (1+b)x^6 + \underbrace{x(1-x)}_{\text{NP}\neq 0},$$

  For $-7.7295 \lesssim b \lesssim 2.5933$ there is $\mathcal{E} < 2$, so $N = 0$ is the only possible attractor.

- Nonlinear tails for vanishing NP case

$$f(u,x) = (\mathsf{w} - q(x))/x \sim \begin{cases} u^{-4} & (x > 0, u \to \infty) \\ u^{-3/2} & (x = 0, u \to \infty) \end{cases}$$

  Radiation coefficient $c_1(u) \sim u^{-3/2}$. Cf. results on nonlinear tails at the interior for semilinear wave equations in high even spatial dimensions [Agemi et al. '94] , [Hintz&Vasy '15] .

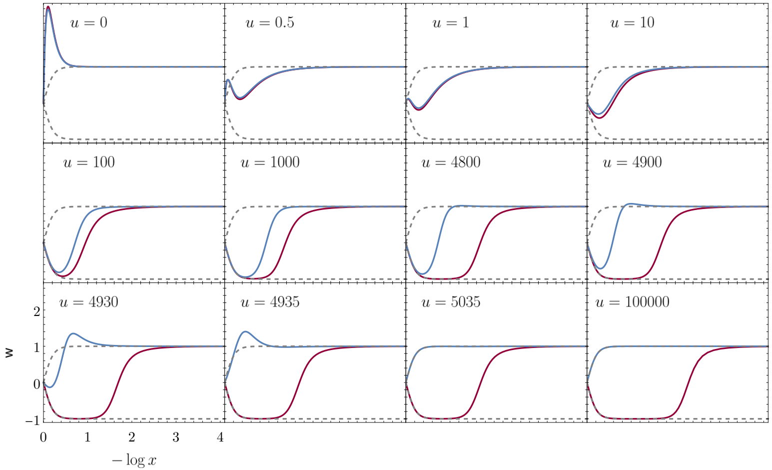
- Nonlinear tails for data with non-zero NP identical to the linear case.

Nonlinear dynamics around $N = 1$ attractor (1)

Evolution of initial data with energy $\mathcal{E} > 2$.

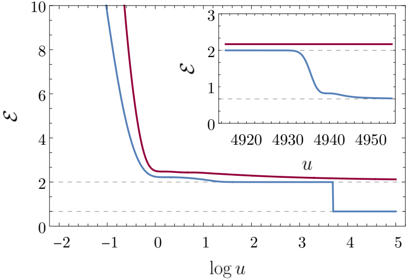
- For moderate (positive) values of b: fast convergence to $q(x)$ (the $N = 0$ attractor).
- For larger values of b: formation of the superposition of the anti-half-kink and the expanding kink (the $N = 1$ attractor).
- At $b = b_0 \approx 12.458$: transition between $N = 0$ and $N = 1$ attractors.
- Subcritical ($b = b_0 - \varepsilon$, $0 < \varepsilon \ll 1$) evolution: $N = 1$ attractor for intermediate times, but the expansion stops and eventually the kink gets annihilated (burst of energy).

Nonlinear dynamics around $N = 1$ attractor (2)



Snapshots of subcritical (blue) and supercritical (red) evolutions.

Nonlinear dynamics around $N = 1$ attractor (3)



Energies of subcritical (blue) and supercritical (red) evolutions.

Nonlinear dynamics around $N = 1$ attractor (4)

Asymptotic dynamics based on the **method of collective coordinates** [Manton '82] .

- The $N = 1$ attractor

$$W(t,r) = 1 - Q_{\mu(t)}(r) + Q_{\lambda(t)}(r), \qquad \mu^2(t) = (\lambda^2(t) - 1)/(\lambda^2(t) + 3)\,.$$

- From the Lagrangian $L = \frac{1}{2}\int_1^\infty \left(W_t^2 - W_r^2 - \frac{(1-W^2)^2}{r^2}\right) r dr$, we get, in the limit of large λ, an effective 1D problem ($\lambda(0) \gg 1$, $\dot{\lambda}(0) > 0$)

$$\left(\frac{4}{3} - \frac{32}{\lambda^4}\right)\dot{\lambda}^2 + \left(2 - \frac{16}{\lambda^2}\right) = E_{\text{eff}} = \text{const.}$$

  with a turning point for $E_{\text{eff}} < 2$, or scattering for $E_{\text{eff}} \geq 2$.

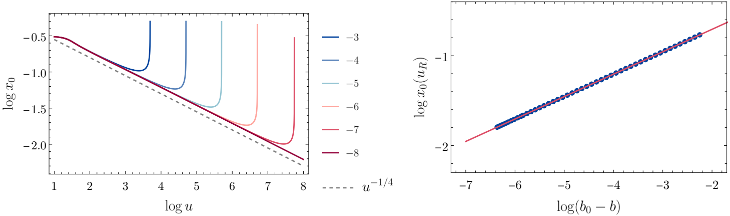
- **This approach fails in the shrinking phase; no radiation. We conjecture $\dot{\lambda} \to 0$, but here $\dot{\lambda}(\infty) > 0$ for $E_{\text{eff}} > 2$.**

Nonlinear dynamics around $N = 1$ attractor (5)

The expanding phase of *subcritical* solutions ($b = b_0 - \varepsilon$):

- If $x_0(u)$ is the zero of $\mathrm{w}(u, x)$ and u_R the return time, then

$$u_R \sim \varepsilon^{-1}\,, \quad x_0(u_R) \sim \varepsilon^{1/4}\,.$$



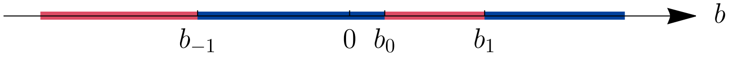
- In terms of the original variables and comparing with the ansatz $N = 1$ we get

$$t_R \sim \varepsilon^{-1}\,, \quad \lambda(t_R) \sim \varepsilon^{-1/2}\,,$$

in agreement with the ODE approximation $\lambda(t_R) \sim 1/\sqrt{2 - E_{\text{eff}}}\,$.

Nonlinear dynamics around $N = 1$ attractor (6)

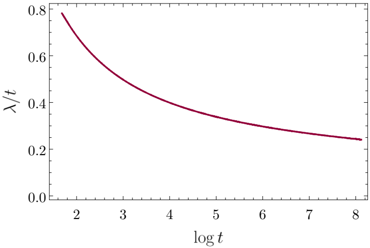
- For large $|b|$ the endstate flips back and forth between the $N = 0$ and $N = 1$ attractors.



We conjecture that there are infinitely many critical values $b_{n \in \mathbb{Z}}$ ($b_1 \approx 47.904$, $b_{-1} \approx -53.944$) at which the curve of initial data intersects the $N = 0$ and $N = 1$ basins of attraction.

Nonlinear dynamics around $N = 1$ attractor (7)

- *Supercritical* evolution: what is the asymptotic speed of the kink? Numerical evidence $\lambda(t)/t \to 0$:



- Work in progress. For the $4+1$ YM equation in the whole space logarithmic correction to the self-similar blowup [Bizoń et al. '04] , [Raphaël&Rodnianski '12]

$$\lambda(t) \sim \frac{2}{3} \frac{T-t}{\sqrt{-\log(T-t)}} \, .$$

Characteristic approach to the soliton resolution

- Another evidence supporting the soliton resolution conjecture.
- Advantage of using the characteristic foliations.
- A toy-model with interesting dynamics.

- Can be extended to corotational WM equation (work in progress).
- YM/WM equation on a wormhole (joint work in progress with Piotr Bizoń and Jacek Jendrej)

Corotational wave maps on a wormhole (1)

A map $X : M \mapsto N$ from a spacetime $(M, g_{\alpha\beta})$ into a Riemannian manifold (N, G_{AB}) is the wave map if it is a critical point of the action

$$S[X] = \int_M g^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^\beta G_{AB} \,.$$

- Domain M: the wormhole with metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (r^2 + a^2)\mathrm{d}\phi^2$.
- Target $N = \mathbb{S}^2$ with the round metric $\mathrm{d}s^2 = \mathrm{d}u^2 + \sin^2 u \, \mathrm{d}\theta^2$.
- Corotational ansatz: $u = u(t, r)$ and $\theta = k\phi$, where $k \in \mathbb{N}$.
- The wave map system reduces to the single semilinear wave equation

$$u_{tt} = u_{rr} + \frac{r}{r^2 + a^2} u_r - \frac{k^2}{2} \frac{\sin 2u}{r^2 + a^2} \,.$$

Corotational wave maps on a wormhole (2)

- The length scale a removes the singularity at $r = 0$ and consequently the solutions are globally regular in time ($a = 1$).
- The equation is $1+1$ dimensional ($-\infty < r < \infty$), yet it inherits strong dispersive decay from the original $2+1$ dimensional problem.
- Conserved energy

$$E[u] = \frac{1}{2}\int_{-\infty}^{\infty}\left(u_t^2 + u_r^2 + k^2\frac{\sin^2 u}{r^2+1}\right)\sqrt{r^2+1}\,\mathrm{d}r\,.$$

- Finite energy requires that $u(t,-\infty) = m\pi$, $u(t,\infty) = n\pi$ ($m, n \in \mathbb{Z}$). We choose $m = 0$ so n determines the topological sector (degree).
- **Our aim is describe the asymptotic behavior of solutions for $t \to \infty$.**

Corotational wave maps on a wormhole (3)

- With $r = \sinh x$
$$\cosh^2 x\, u_{tt} = u_{xx} - \frac{k^2}{2} \sin 2u\,.$$
- For static solutions $u(t,x) = Q(x)$ of degree $n = 1$ there holds the Bogomolnyi identity
$$E[Q] \geq \frac{1}{2} \int_{-\infty}^{\infty} \left(Q' - k \sin Q\right)^2 \mathrm{d}x + 2k\,,$$
which is saturated on the kink solution $Q(x) = 2 \arctan\left(e^{kx}\right)$.
- By translation, we have a one-parameter family of kinks $Q_c(x) = Q(x-c)$ (however, translation in x is not the symmetry of the full equation).
- There are no static solutions of degree $n \geq 2$.

Corotational wave maps on a wormhole (4)

- Solutions of degree $n = 0$ with energy less than $2E[Q]$ tend asymptotically to zero.
- Solutions of degree $n \geq 1$ with energy less than $(n+2)E[Q]$ tend asymptotically to

$$\sum_{j=1}^{n} Q\left(x - c_j(s)\right) , \quad |c_{j+1}(s) - c_j(s)| \to \infty \, .$$

- For higher energies the kink-antikink pairs can be created.
- The asymptotic behavior of solutions at the threshold of the kink-antikink creation is well described by collective coordinates [Manton '82] .
- Work in progress ...