SELF-SIMILAR BLOWUP IN WAVE EQUATIONS

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ABSTRACT. These are lecture notes from a mini course given at the *Summer School on Geometric dispersive PDEs* in Obergurgl, Austria, in September 2022.

1. Wave maps

We consider wave maps $U : \mathbb{R}^{1,3} \to \mathbb{S}^3 \subset \mathbb{R}^4$, characterized as solutions to the partial differential equation

$$\partial^\mu \partial_\mu U + (\partial_\mu U \cdot \partial^\mu U)U = 0.$$

We restrict ourselves to *corotational maps* which are of the form

$$U(t,x) = \begin{pmatrix} \sin(u(t,|x|))\frac{x}{|x|} \\ \cos(u(t,|x|)) \end{pmatrix}$$

for an auxiliary function $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. This ansatz turns out to be compatible with the wave maps equation, i.e., when plugging it in, we obtain the single semilinear radial wave equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right) u(t,r) + \frac{\sin(2u(t,r))}{r^2} = 0. \tag{1}$$

The principal goal is to construct global solutions and since this is a wave equation, the natural mathematical setting is to study the *Cauchy problem*, i.e., we prescribe $u(0,\cdot)$ and $\partial_0 u(0,\cdot)$ and try to construct a solution that assumes these data. However,

$$u^T(t,r) := 2\arctan(\tfrac{r}{T-t})$$

for any $T \in \mathbb{R}$ is a wave map on $\mathbb{R}\times[0,\infty)\backslash\{(T,0)\}$, as one checks easily. At $(t,x) = (T,0)$, u^T exhibits a gradient blowup and hence, it is in general not possible to construct global smooth solutions even for the nicest possible data, e.g. smooth and compactly supported. Our goal is to understand the nature of this breakdown and its relevance for generic evolutions.

2. The mode stability problem

We start with the wave maps equation (1) and switch to similarity cordinates

$$\tau = -\log(T-t) + \log T, \qquad \rho = \frac{r}{T-t}$$

or

$$t = T - Te^{-\tau}, \qquad r = Te^{-\tau}\rho,$$

where $T > 0$ is a parameter. Then u satisfies Eq. (1) if and only if $v_T(\tau,\rho) := u(T - Te^{-\tau}, Te^{-\tau}\rho)$ satisfies

$$\left(\partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + \partial_\tau - (1-\rho^2)\partial_\rho^2 + \left(2\rho - \frac{2}{\rho}\right)\partial_\rho\right) v_T(\tau,\rho) + \frac{\sin(2v_T(\tau,\rho))}{\rho^2} = 0. \tag{2}$$

Note that the parameter T does not show up in the coefficients of Eq. (2). We will consider Eq. (2) in the coordinate range $\tau \geq 0$ and $\rho \in [0,1]$, which corresponds to the backward lightcone of the point $(T,0)$ in the "physical" coordinates (t,r).

The blowup solution $u^{T'}(t,r) = 2\arctan(\frac{r}{T'-t})$ transforms into

$$v_T^{T'}(\tau,\rho) := u^{T'}(T - Te^{-\tau}, Te^{-\tau}\rho) = 2\arctan\left(\frac{\rho}{1+(\frac{T'}{T}-1)e^{\tau}}\right).$$

We would like to understand the stability of the family $\{v_T^{T'} : T' > 0\}$. First, let us point out that v_T^T is independent of τ whereas nearby members of the family move away from v_T^T as τ increases. Indeed, if $T' < T$, $v_T^{T'}(\tau,\cdot)$ develops a gradient blowup as $\tau \to \tau_*$, where τ_* is determined by $(\frac{T'}{T}-1)e^{\tau_*} = -1$. On the other hand, if $T' > T$, $v_T^{T'}(\tau,\rho) \to 0$ as $\tau \to \infty$. By these observations, it is expected that the static solution v_T^T is unstable because a "generic" perturbation will push it towards a nearby member of the family. However, such a "push" can be compensated by adapting T. Thus, the instability is "artificial" and caused by the free parameter T in the definition of the similarity coordinates or, on a more fundamental level, by the time-translation symmetry of the wave maps equation. In other words, stability of the blowup means that for any given (small) initial perturbation of u^1, say, there should be a T close to 1 that makes the evolution in similarity coordinates with parameter T converge to v_T^T.

2.1. **Mode solutions.** The most elementary stability analysis consists of looking for *mode solutions*. This means that one plugs in the ansatz

$$v_T(\tau,\rho) = v_T^T(\rho) + e^{\lambda\tau}f(\rho), \qquad \lambda \in \mathbb{C}$$

into Eq. (2) and linearizes in f. This yields the "spectral problem"

$$-(1-\rho^2)f''(\rho) - \frac{2}{\rho}f'(\rho) + 2(\lambda+1)\rho f'(\rho) + \frac{2\cos(2v_T^T(\rho))}{\rho^2}f(\rho) + \lambda(\lambda+1)f(\rho) = 0. \tag{3}$$

Clearly, if there are "admissible" mode solutions with $\operatorname{Re}\lambda > 0$, we expect the solution v_T^T to be unstable. What exactly "admissible" in this context means can only be answered once one has set up the functional analytic framework to study the wave maps evolution. For now we will restrict ourselves to smooth solutions and we will see later that this is the correct class of functions. Furthermore, observe that Eq. (3) has singular points at $\rho = 0$ and $\rho = 1$ and therefore, it is expected that only for special values of λ there will be nontrivial solutions in $C^\infty([0,1])$.

We have already argued that we expect an "artificial" instability of v_T^T. So how does this instability show up in the context of the spectral problem Eq. (3)? To see this, we differentiate the equation

$$\left(\partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + \partial_\tau - (1-\rho^2)\partial_\rho^2 + \left(2\rho - \frac{2}{\rho}\right)\partial_\rho\right) v_T^{T'}(\tau,\rho) + \frac{\sin(2v_T^{T'}(\tau,\rho))}{\rho^2} = 0.$$

with respect to T' and evaluate the result at $T' = T$. This yields

$$\left(\partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + \partial_\tau - (1-\rho^2)\partial_\rho^2 + \left(2\rho - \frac{2}{\rho}\right)\partial_\rho\right) v_*(\tau,\rho) + \frac{2\cos(2v_T^T(\rho))}{\rho^2}v_*(\tau,\rho) = 0$$

with

$$v_*(\tau,\rho) := \partial_{T'}v_T^{T'}(\tau,\rho)\big|_{T=T'} = -\frac{2}{T}e^{\tau}\frac{\rho}{1+\rho^2}.$$

Consequently, the function $\rho \mapsto \frac{\rho}{1+\rho^2}$ solves Eq. (3) with $\lambda = 1$ and this is the mode solution that reflects the expected "artificial" instability. This observation naturally leads to the following definition.

Definition 2.1. We say that the blowup solution u^T is *mode stable* if the existence of a nontrivial $f \in C^\infty([0,1])$ that satisfies Eq. (3) necessarily implies that $\operatorname{Re}\lambda < 0$ or $\lambda = 1$.

The goal of this section is to prove the following theorem.

Theorem 2.2. *The blowup solution u^T is mode stable.*

In what follows we will somewhat imprecisely call $\lambda \in \mathbb{C}$ an "eigenvalue" if Eq. (3) has a nontrivial solution in $C^\infty([0,1])$.

2.2. **Fuchsian classification.** To begin with, we would like to understand better which problem we are actually facing. The term in Eq. (3) involving the cosine turns out to be a rational function. Indeed, we have

$$2\cos(2v_T^T(\rho)) = 2\cos(4\arctan(\rho)) = 2\frac{1-6\rho^2+\rho^4}{(1+\rho^2)^2}$$

and thus, Eq. (3) has the (regular) singular points $0, \pm 1, \pm i, \infty$ and by switching to the independent variable ρ^2, the number of singular points can be reduced to four: $0, 1, i$, and ∞. This means that Eq. (3) is a Fuchsian differential equation of Heun type. The normal form for a Heun equation reads

$$g''(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right] g'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} g(z) = 0$$

where $\alpha, \beta, \gamma, \delta, \epsilon, a, q \in \mathbb{C}$. Around each of the singular points there exist two linearly independent local solutions. The interesting question then is how local solutions around different singular points are related to each other. This is known as the *connection problem* and unfortunately, for Heun equations this problem is open. If we had only three regular singular points, we would be dealing with a *hypergeometric differential equation* for which the connection problem has been solved in the 19th century. This indicates that the spectral problem we are trying to solve is potentially hard.

2.3. **Frobenius analysis.** Now we turn to a more quantitative analysis and first recall Frobenius' theory for Fuchsian equations of second order. These are equations over the complex numbers of the form

$$f''(z) + p(z)f'(z) + q(z)f(z) = 0 \tag{4}$$

where p and q are given functions and f is the unknown.

Theorem 2.3. *Let $R > 0$ and let $p, q : \mathbb{D}_R \setminus \{0\} \to \mathbb{C}$ be holomorphic. Suppose that the limits*

$$p_0 := \lim_{z\to 0}[zp(z)], \qquad q_0 := \lim_{z\to 0}[z^2 q(z)]$$

exist and let $s_\pm \in \mathbb{C}$ satisfy $P(s_\pm) = 0$, where

$$P(s) := s(s-1) + p_0 s + q_0$$

is the indicial polynomial. *Let $\operatorname{Re} s_+ \geq \operatorname{Re} s_-$. Then there exists a holomorphic function $h_+ : \mathbb{D}_R \to \mathbb{C}$ with $h_+(0) = 1$ and such that $f : \mathbb{D}_R \setminus \{0\} \to \mathbb{C}$, given by $f(z) = z^{s_+}h_+(z)$, satisfies Eq. (4). Furthermore, if $s_+ - s_- \notin \mathbb{N}_0$, there exists a holomorphic function $h_- : \mathbb{D}_R \to \mathbb{C}$ with $h_-(0) = 1$ and such that $f(z) = z^{s_-}h_-(z)$ is another solution of*

Eq. (4) *on* $\mathbb{D}_R \setminus \{0\}$. *Finally, if* $s_+ - s_- \in \mathbb{N}_0$, *there exist* $c \in \mathbb{C}$ *and a holomorphic function* $h_- : \mathbb{D}_R \to \mathbb{C}$ *with* $h_-(0) = 1$ *such that*

$$f(z) = z^{s_-} h_-(z) + cf(z)\log z$$

is another solution of Eq. (4) *on* $\mathbb{D}_R \setminus (-\infty, 0]$.

Idea of proof. The idea is to plug in a generalized power series ansatz $z^\sigma \sum_{k=0}^\infty a_k z^k$ and to determine σ and the coefficients $(a_k)_{k\in\mathbb{N}_0}$ by comparing powers of z. The convergence of the corresponding series is then shown by a simple induction. The second solution can be obtained by the reduction of order ansatz. We skip the details because this is a standard result that can be found in many textbooks, see e.g. [2] for a modern account. □

Slightly re-arranged, Eq. (3) reads

$$f''(\rho) + 2\frac{1-(\lambda+1)\rho^2}{\rho(1-\rho^2)} f'(\rho) - \left[V(\rho) + \frac{\lambda(\lambda+1)}{1-\rho^2}\right] f(\rho) = 0. \tag{5}$$

with

$$V(\rho) := 2\frac{1-6\rho^2+\rho^4}{\rho^2(1-\rho^2)(1+\rho^2)^2}$$

and the indicial polynomial at $\rho = 0$ reads $s(s-1) + 2s - 2$ with zeros 1 and -2. As expected, there is only one smooth solution around $\rho = 0$ and it behaves like ρ. At $\rho = 1$ the indicial polynomial is given by $s(s-1) + \lambda s = 0$ with zeros 0 and $1 - \lambda$. Again, there is only one smooth solution around $\rho = 1$ (the cases $\lambda \in \{0, 1\}$ require some extra care). Thus, our goal is to show that the local solution that is smooth around $\rho = 0$ is necessarily nonsmooth at $\rho = 1$ if $\operatorname{Re}\lambda \geq 0$ (and $\lambda \neq 1$).

2.4. **Supersymmetric removal.** The case $\lambda = 1$ is special and we already know that this is an "eigenvalue". In order to proceed, it is necessary to "remove" it. This can be achieved by a factorization procedure that has its origin in supersymmetric quantum mechanics (hence the name). In our setting, the procedure is as follows. First, we introduce an auxiliary function g by $f(\rho) = p(\rho)g(\rho)$, where we choose p in such a way that the resulting equation for g has no first-order derivative. Indeed, inserting the above ansatz into Eq. (5) yields the condition

$$p'(\rho) = -\frac{1-(\lambda+1)\rho^2}{\rho(1-\rho^2)} p(\rho)$$

which is satisfied by e.g. $p(\rho) = \rho^{-1}(1-\rho^2)^{-\frac{\lambda}{2}}$. Plugging the ansatz

$$f(\rho) = \rho^{-1}(1-\rho^2)^{-\frac{\lambda}{2}} g(\rho)$$

into Eq. (5) yields

$$g''(\rho) - V(\rho)g(\rho) = \frac{\lambda(\lambda-2)}{(1-\rho^2)^2} g(\rho). \tag{6}$$

Recall that the function $\rho \mapsto \frac{\rho}{1+\rho^2}$ solves Eq. (5) with $\lambda = 1$. Thus,

$$g_0(\rho) := (1-\rho^2)^{\frac{1}{2}} \frac{\rho^2}{1+\rho^2}$$

satisfies

$$g_0''(\rho) - V(\rho)g_0(\rho) = -\frac{1}{(1-\rho^2)^2} g_0(\rho).$$

Motivated by this, we rewrite Eq. (6) as

$$g''(\rho) + \left[\frac{1}{(1-\rho^2)^2} - V(\rho)\right] g(\rho) = \frac{(\lambda-1)^2}{(1-\rho^2)^2} g(\rho).$$

This resembles a spectral problem for a Schrödinger operator with a ground state g_0. Observe that g_0 has no zeros on $(0,1)$ and we have the factorization

$$\begin{aligned}\left(\partial_\rho + \frac{g_0'(\rho)}{g_0(\rho)}\right)\left(\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}\right) &= \partial_\rho^2 - \partial_\rho\left(\frac{g_0'(\rho)}{g_0(\rho)}\right) - \frac{g_0'(\rho)^2}{g_0(\rho)^2} = \partial_\rho^2 - \frac{g_0''(\rho)}{g_0(\rho)} \\ &= \partial_\rho^2 + \left[\frac{1}{(1-\rho^2)^2} - V(\rho)\right].\end{aligned}$$

Consequently, Eq. (6) can be written as

$$(1-\rho^2)^2\left(\partial_\rho + \frac{g_0'(\rho)}{g_0(\rho)}\right)\left[\left(\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}\right) g(\rho)\right] = (\lambda-1)^2 g(\rho).$$

The trick is now to apply the operator $\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}$ to this equation. In terms of

$$\widetilde{g}(\rho) := \left(\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}\right) g(\rho),$$

the resulting equation reads

$$\left(\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}\right)\left[(1-\rho^2)^2\left(\partial_\rho + \frac{g_0'(\rho)}{g_0(\rho)}\right)\widetilde{g}(\rho)\right] = (\lambda-1)^2 \widetilde{g}(\rho).$$

Note that

$$\left(\partial_\rho - \frac{g_0'(\rho)}{g_0(\rho)}\right) g_0(\rho) = 0,$$

i.e., the solution that comes from the artificial instability gets annihilated by this transformation. Finally, we write $\widetilde{f}(\rho) = \rho^{-1}(1-\rho^2)^{1-\frac{\lambda}{2}}\widetilde{g}(\rho)$ and the equation turns into

$$-(1-\rho^2)\widetilde{f}''(\rho) - \frac{2}{\rho}\widetilde{f}'(\rho) + 2(\lambda+1)\rho\widetilde{f}'(\rho) + \frac{2(3-\rho^2)}{\rho^2(1+\rho^2)}\widetilde{f}(\rho) + \lambda(\lambda+1)\widetilde{f}(\rho) = 0, \tag{7}$$

which has the exact same structure as Eq. (3) but with a different "potential". Based on the above, we have the following correspondence result.

Lemma 2.4. *Let $\lambda \in \mathbb{C} \setminus \{1\}$ and suppose that there exists a nontrivial $f \in C^\infty([0,1])$ that satisfies Eq. (3). Then there exists a nontrivial $\widetilde{f} \in C^\infty([0,1])$ that satisfies Eq. (7).*

Proof. Given f, we set

$$\widetilde{f}(\rho) := \rho^{-1}(1-\rho^2)^{1-\frac{\lambda}{2}}\left(\partial_\rho - \frac{2-3\rho^2-\rho^4}{\rho(1-\rho^2)(1+\rho^2)}\right)\left[\rho(1-\rho^2)^{\frac{\lambda}{2}} f(\rho)\right]$$

and since

$$\frac{g_0'(\rho)}{g_0(\rho)} = \frac{2-3\rho^2-\rho^4}{\rho(1-\rho^2)(1+\rho^2)},$$

the above derivation shows that $\widetilde{f}$ is nontrivial (here $\lambda \neq 1$ is used) and satisfies Eq. (7). The fact that $\widetilde{f} \in C^\infty([0,1])$ follows by inspection because $f(\rho)$ behaves like ρ near 0. □

2.5. **Transformation to standard Heun form.** Eq. (7) is again of Heun-type. To see this, we first observe that the indicial polynomial of Eq. (7) at $\rho = 0$ is $s(s-1)+2s-6$ with zeros 2 and -3. At $\rho = 1$ we have, as with the original equation, $s(s-1)+\lambda s$ with zeros 0 and $1-\lambda$. In order to obtain the standard Heun form, one of the indices at each of the singular points must equal zero. Thus, we introduce new variables by writing $\widetilde{f}(\rho) = \rho^2 \widehat{f}(\rho^2)$. Then $\widetilde{f}$ satisfies Eq. (7) if and only if $\widehat{f}$ satisfies the Heun equation

$$\widehat{f}''(x) + \left(\frac{7}{2x} + \frac{\lambda}{x-1}\right)\widehat{f}'(x) + \frac{1}{4}\frac{(\lambda+3)(\lambda+2)x + \lambda^2 + 5\lambda - 2}{x(x-1)(x+1)}\widehat{f}(x) = 0.$$

The domain we are interested in is $x \in [0,1]$ (which corresponds to $\rho \in [0,1]$). However, the fact that the singularity at $x = -1$ has the same distance from 0 as the singularity at $x = 1$ spoils our analysis. For this reason, we need to move it, which is possible by the Möbius transform $x \mapsto \frac{2x}{x+1}$, which maps 0 to 0, 1 to 1, -1 to ∞, and ∞ to 2. Then 1 is the only singularity within distance 1 from 0. Upon writing

$$\widehat{f}(x) = \left(2 - \frac{2x}{x+1}\right)^{1+\frac{\lambda}{2}} g\left(\frac{2x}{x+1}\right),$$

we finally arrive at the Heun equation

(8)

$$g''(z) + \left(\frac{7}{2z} + \frac{\lambda}{z-1} + \frac{1}{2(z-2)}\right)g'(z) + \frac{1}{4}\frac{(\lambda+4)(\lambda+2)z - (\lambda^2+12\lambda+12)}{z(z-1)(z-2)}g(z) = 0.$$

Tracing back the above derivation, we obtain the following lemma.

Lemma 2.5. *Let $\lambda \in \mathbb{C} \setminus \{1\}$ and suppose that there exists a nontrivial $f \in C^\infty([0,1])$ that satisfies Eq. (3). Then there exists a nontrivial $g \in C^\infty([0,1])$ that satisfies Eq. (8).*

2.6. **The recurrence relation.** The indicial polynomial of Eq. (8) at $z = 0$ is $s(s-1)+\frac{7}{2}s$ with zeros 0 and $\frac{5}{2}$. At $z = 1$ we have $s(s-1)+\lambda s = 0$ with zeros 0 and $1-\lambda$. Thus, a solution $g \in C^\infty([0,1])$ is analytic around both $z = 0$ and $z = 1$. This means that such a solution can be represented by a power series centered at $z = 0$ with radius of convergence larger than 1 (the "next" singularity apart from $z = 1$ is at $z = 2$). At this point it becomes clear why it was necessary to remove the singularity at -1 for the argument to work. Thus, the idea is to insert a power series ansatz, obtain a recurrence relation for the coefficients and then prove that the radius of convergence equals 1 if $\operatorname{Re}\lambda \geq 0$.

Concretely, from Frobenius' theory we know that there exists a solution

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

to Eq. (8), where the power series has radius of convergence at least 1. Thus,

$$g'(z) = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$

and

$$g''(z) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n.$$

By inserting this into Eq. (8), rewritten as

$$z(z-1)(z-2)g''(z) + \left[\tfrac{7}{2}(z-1)(z-2) + \lambda z(z-2) + \tfrac{1}{2}z(z-1)\right] g'(z) + \tfrac{1}{4}\left[(\lambda+4)(\lambda+2)z - (\lambda^2+12\lambda+12)\right] g(z) = 0,$$

we obtain

$$\begin{aligned} 0 =& (z^3 - 3z^2 + 2z)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n \\ &+ [(\lambda+4)z^2 - (2\lambda+11)z + 7]\sum_{n=0}^{\infty}(n+1)a_{n+1}z^n \\ &+ \tfrac{1}{4}\left[(\lambda+4)(\lambda+2)z - (\lambda^2+12\lambda+12)\right]\sum_{n=0}^{\infty}a_n z^n \end{aligned}$$

and balancing the powers of z, we find

$$\begin{aligned} 0 =& \sum_{n=-1}^{\infty}\left[7(n+2)a_{n+2} - \tfrac{1}{4}(\lambda^2+12\lambda+12)a_{n+1}\right] z^{n+1} \\ &+ \sum_{n=0}^{\infty}\left[2(n+2)(n+1)a_{n+2} - (2\lambda+11)(n+1)a_{n+1} + \tfrac{1}{4}(\lambda+4)(\lambda+2)a_n\right] z^{n+1} \\ &+ \sum_{n=1}^{\infty}\left[-3(n+1)na_{n+1} + (\lambda+4)na_n\right] z^{n+1} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n+1}. \end{aligned}$$

By setting $a_{-1} = 0$, we can start all sums at $n = -1$ and we arrive at the recurrence relation

$$a_{n+2} = A_n(\lambda)a_{n+1} + B_n(\lambda)a_n \tag{9}$$

for $n \in \{-1\} \cup \mathbb{N}_0$ and with

$$\begin{aligned} A_n(\lambda) &:= \frac{12n^2 + (8\lambda+56)n + \lambda^2 + 20\lambda + 56}{8n^2 + 52n + 72} \\ B_n(\lambda) &:= -\frac{4n^2 + (4\lambda+12)n + \lambda^2 + 6\lambda + 8}{8n^2 + 52n + 72}. \end{aligned}$$

In order to start the recurrence, we choose the initial condition $a_0 = 1$. This freedom comes from the fact that we are solving a linear differential equation with a one-parameter family of solutions.

2.7. **Properties of the coefficients.** As a first and easy observation we can now rule out the existence of polynomial solutions.

Lemma 2.6. *Let* $\operatorname{Re}\lambda \geq 0$ *and suppose that* $g : \mathbb{C} \to \mathbb{C}$ *is a polynomial that satisfies Eq.* (8)*. Then* $g = 0$.

Proof. Since g is a polynomial, there exists and $N \in \mathbb{N}_0$ and coefficients $(a_n)_{n=0}^N$ such that

$$g(z) = \sum_{n=0}^{N} a_n z^n.$$

Furthermore, by the above, the coefficients a_n satisfy the recurrence relation Eq. (9). Now observe that $B_n(\lambda) = 0$ if and only if $\lambda \in \{-2(n+1), -2(n+2)\}$ and thus, $B_n(\lambda) \neq 0$

for all $n \in \mathbb{N}_0$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. This implies that

$$a_n = -\frac{A_n(\lambda)}{B_n(\lambda)} a_{n+1} + \frac{1}{B_n(\lambda)} a_{n+2}$$

for all $n \in \mathbb{N}_0$ and since $a_n = 0$ for all $n > N$, we conclude that $a_n = 0$ for all $n \in \mathbb{N}_0$. Consequently, $g = 0$. □

Next, we turn to the asymptotic behavior of the coefficients. More precisely, we are interested in the convergence radius of the series $\sum_{n=0}^{\infty} a_n z^n$ and thus, we need to understand the asymptotic behavior of the ratio $\frac{a_{n+1}}{a_n}$. To begin with, we fix notation.

Definition 2.7. For $\lambda \in \mathbb{C}$ the sequence $(a_n(\lambda))_{n\in\mathbb{N}_0}$ is defined recursively by $a_{-1}(\lambda) = 0$, $a_0(\lambda) = 1$, and

$$a_{n+2}(\lambda) = A_n(\lambda)a_{n+1}(\lambda) + B_n(\lambda)a_n(\lambda)$$

for $n \in \{-1\} \cup \mathbb{N}_0$.

Lemma 2.8. *Let* $\operatorname{Re}\lambda \geq 0$. *Then we have*

$$\lim_{n\to\infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} \in \{\tfrac{1}{2}, 1\}.$$

Proof. We have

$$\lim_{n\to\infty} A_n(\lambda) = \tfrac{3}{2}, \qquad \lim_{n\to\infty} B_n(\lambda) = -\tfrac{1}{2}$$

and $s^2 - \frac{3}{2}s + \frac{1}{2} = 0$ if and only if $s \in \{\frac{1}{2}, 1\}$. Consequently, by Poincaré's theorem on difference equations (Theorem 4.3) we either have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \in \{\tfrac{1}{2}, 1\}$$

or there exists an $N \in \mathbb{N}_0$ such that $a_n = 0$ for all $n \geq N$, but the latter is excluded by Lemma 2.6. □

If $\lim_{n\to\infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} = \frac{1}{2}$, the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(\lambda)z^n$ equals 2 and in particular, $\sum_{n=0}^{\infty} a_n(\lambda)z^n$ is a solution to Eq. (8) that belongs to $C^\infty([0,1])$. This is precisely the case we want to rule out. Consequently, our goal is to show that $\lim_{n\to\infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} = 1$.

Whenever $a_n(\lambda) \neq 0$, we write $r_n(\lambda) := \frac{a_{n+1}(\lambda)}{a_n(\lambda)}$. With this notation, the recurrence relation Eq. (9) reads

$$r_{n+1}(\lambda) = \frac{a_{n+2}(\lambda)}{a_{n+1}(\lambda)} = \frac{A_n(\lambda)a_{n+1}(\lambda) + B_n(\lambda)a_n(\lambda)}{a_{n+1}(\lambda)} = A_n(\lambda) + \frac{B_n(\lambda)}{r_n(\lambda)}$$

for $n = 0, 1, 2\ldots$ and we keep in mind that this is only defined as long as $r_n(\lambda) \neq 0$. Furthermore, we have the initial condition

$$r_0(\lambda) = \frac{a_1(\lambda)}{a_0(\lambda)} = a_1(\lambda) = A_{-1}(\lambda)a_0(\lambda) + B_{-1}(\lambda)a_{-1}(\lambda) = A_{-1}(\lambda) = \frac{\lambda^2 + 12\lambda + 12}{28}.$$

2.8. **The quasi-solution.** Rephrased in terms of r_n, our goal is to show that $\lim_{n\to\infty} r_n(\lambda) = 1$ if $\operatorname{Re}\lambda \geq 0$. The idea is now to achieve this by means of a *quasi-solution*

$$\widetilde{r}_n(\lambda) := \frac{\lambda^2}{8n^2 + 33n + 28} + \frac{5\lambda}{5n + 16} + \frac{5n + 6}{5n + 13}, \qquad n \in \mathbb{N},$$

which is supposed to approximate r_n well enough. More precisely, we intend to show that

$$\left|\frac{r_n(\lambda)}{\widetilde{r}_n(\lambda)} - 1\right| \leq \frac{1}{3}$$

for all $n \in \mathbb{N}$ and hence, we must have $\lim_{n\to\infty} r_n(\lambda) = 1$ because by Lemma 2.8, the only other possibility is $\lim_{n\to\infty} r_n(\lambda) = \frac{1}{2}$ which is not compatible with the above bound as $\lim_{n\to\infty} \widetilde{r}_n(\lambda) = 1$. In particular, this estimate implies that $r_n(\lambda) \neq 0$ for all $n \in \mathbb{N}$ and a posteriori we see that the recursion for r_n is defined for all $n \in \mathbb{N}_0$. Note also that our argument provides the necessary wiggle room for feasible estimates. Indeed, it is not necessary to prove that $\lim_{n\to\infty} \frac{r_n(\lambda)}{\widetilde{r}_n(\lambda)} = 1$ directly, which would be much harder if not impossible.

The quasi-solution we use comes out of the blue and finding it involves a bit of art indeed. However, there are some rules of thumb. It is a natural first attempt to look for a quasi-solution that is quadratic in λ because both $A_n(\lambda)$ and $B_n(\lambda)$ are quadratic polynomials in λ. Then, with the help of a computer algebra system, one can look at the first few terms of the sequences $(r_n(0))_{n\in\mathbb{N}_0}$, $(\frac{1}{2}(r_n(1) - r_n(-1)))_{n\in\mathbb{N}_0}$, and $(\frac{1}{2}(r_n(1) - 2r_n(0) + r_n(-1)))_{n\in\mathbb{N}_0}$ and fit simple rational functions in n. Sometimes some additional tweaking is necessary.

Lemma 2.9. *We have $\widetilde{r}_n(\lambda) \neq 0$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$.*

Proof. Since $\widetilde{r}_n(\lambda)$ is completely explicit, the proof consists of solving a quadratic equation and is left as an exercise. □

Corollary 2.10. *Let $n \in \mathbb{N}$ and $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Then the functions*

$$r_1 : \overline{\Omega} \to \mathbb{C}, \qquad \frac{1}{\widetilde{r}_n} : \overline{\Omega} \to \mathbb{C}$$

are continuous and holomorphic on Ω.

Definition 2.11. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ and $n \in \mathbb{N}_0$ we set

$$\delta_n(\lambda) := \frac{r_n(\lambda)}{\widetilde{r}_n(\lambda)} - 1$$

as well as

$$\epsilon_n(\lambda) := \frac{A_n(\lambda)\widetilde{r}_n(\lambda) + B_n(\lambda)}{\widetilde{r}_n(\lambda)\widetilde{r}_{n+1}(\lambda)} - 1$$

and

$$C_n(\lambda) := \frac{B_n(\lambda)}{\widetilde{r}_n(\lambda)\widetilde{r}_{n+1}(\lambda)}.$$

Note carefully that ϵ_n and C_n are explicit.

Lemma 2.12. *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. Then the functions δ_n satisfy the recurrence relation*

$$\delta_{n+1}(\lambda) = \epsilon_n(\lambda) - C_n(\lambda)\frac{\delta_n(\lambda)}{1 + \delta_n(\lambda)}$$

for all $n = 0, 1, 2, \ldots$ as long as $1 + \delta_n(\lambda) \neq 0$.

Proof. This follows straightforwardly by inserting the definition of δ_n and by taking into account that r_n satisfies the recurrence relation $r_{n+1} = A_n + \frac{B_n}{r_n}$. □

Next, we provide quantitative bounds on the functions in play.

Lemma 2.13. *We have the bounds*

$$|\delta_1(it)| \leq \tfrac{1}{3}, \qquad |\epsilon_n(it)| \leq \tfrac{1}{12}, \qquad |C_n(it)| \leq \tfrac{1}{2}$$

for all $n \in \mathbb{N}_0$ and all $t \in \mathbb{R}$.

Proof. These are all bounds on explicit expressions and they are left as an exercise. □

By the Phragmén-Lindelöf principle, the bounds on the imaginary axis extend to the whole complex right half-plane.

Lemma 2.14. *We have the bounds*

$$|\delta_1(\lambda)| \leq \tfrac{1}{3}, \qquad |\epsilon_n(\lambda)| \leq \tfrac{1}{12}, \qquad |C_n(\lambda)| \leq \tfrac{1}{2}$$

for all $n \in \mathbb{N}_0$ *and all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq 0$.

Proof. Let $n \in \mathbb{N}_0$. By construction and Corollary 2.10, the functions $\delta_1, \epsilon_n, C_n$ are continuous on the closed complex right half-plane and holomorphic on the open right half-plane. Furthermore, since δ_1, ϵ_n, and C_n are rational functions, there exists a $K_n > 0$ such that

$$|\delta_1(\lambda)| + |\epsilon_n(\lambda)| + |C_n(\lambda)| \leq K_n e^{|\lambda|^{\frac{1}{2}}}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$. Consequently, Lemma 2.13 and the Phragmén-Lindelöf principle (Lemma 4.2) yield the claim. □

Now we can conclude the proof of mode stability by a simple induction.

Lemma 2.15. *We have the bound*

$$|\delta_n(\lambda)| \leq \tfrac{1}{3}$$

for all $n \in \mathbb{N}_0$ *and all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq 0$.

Proof. By Lemma 2.14 the claim holds for $n = 1$. Assuming that it holds for n, we find, again by Lemma 2.14,

$$|\delta_{n+1}(\lambda)| \leq |\epsilon_n(\lambda)| + |C_n(\lambda)| \frac{|\delta_n(\lambda)|}{1 - |\delta_n(\lambda)|} \leq \tfrac{1}{12} + \tfrac{1}{2}\frac{\frac{1}{3}}{\frac{2}{3}} = \tfrac{1}{3}$$

and the claim follows inductively. □

This concludes the proof of the mode stability and Theorem 2.2 is established.

3. Functional analytic setup

Next, we develop the functional analytic setup for studying the stability of the wave maps blowup.

3.1. **Wave propagators.**

To begin with, we recall the standard wave propagators. Our convention for the Fourier transform is

$$(\mathcal{F}f)(y) := \int_{\mathbb{R}^d} e^{-2\pi i y \cdot x} f(x)dx,$$

initially defined on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and by duality extended to the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Definition 3.1 (Wave propagators)**.** For $f \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\begin{aligned} \cos(t|\nabla|)f &:= \mathcal{F}^{-1}(\cos(2\pi t|\cdot|)\mathcal{F}f) \\ \frac{\sin(t|\nabla|)}{|\nabla|}f &:= \mathcal{F}^{-1}\left(\frac{\sin(2\pi t|\cdot|)}{2\pi|\cdot|}\mathcal{F}f\right). \end{aligned}$$

Recall that if $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$u(t,\cdot) := \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$$

is the unique solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u(0,x) = f(x), \quad \partial_0 u(0,x) = g(x) & \text{for } x \in \mathbb{R}^d \end{cases}.$$

Furthermore, recall the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \||\cdot|^s \mathcal{F}f\|_{L^2(\mathbb{R}^d)}, \qquad s > -\tfrac{d}{2}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The wave propagators behave very well with respect to these norms.

Lemma 3.2. *Let $s \geq 0$. Then we have the bounds*

$$\begin{aligned} \|\cos(t|\nabla|)f\|_{\dot{H}^s(\mathbb{R}^d)} &\leq \|f\|_{\dot{H}^s(\mathbb{R}^d)} \\ \left\|\frac{\sin(t|\nabla|)}{|\nabla|}f\right\|_{\dot{H}^{s+1}(\mathbb{R}^d)} &\leq \|f\|_{\dot{H}^s(\mathbb{R}^d)} \end{aligned}$$

for all $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. The proof is left as an exercise. □

3.2. **The wave propagators in similarity coordinates.** Next, we switch to the similarity coordinates

$$\tau = -\log(T-t) + \log T, \qquad \xi = \frac{x}{T-t}$$

or

$$t = T - Te^{-\tau}, \qquad x = Te^{-\tau}\xi.$$

We consider the coordinate range $\tau \geq 0$ and $\xi \in \mathbb{R}^d$. The solution to the wave equation in similarity coordinates is then given by the wave propagators in similarity coordinates.

Definition 3.3. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $T > 0$ we set

$$\begin{aligned} [C_T(\tau)f](\xi) &:= [\cos((T-Te^{-\tau})|\nabla|)f](Te^{-\tau}\xi) \\ [S_T(\tau)f](\xi) &:= \left[\frac{\sin((T-Te^{-\tau})|\nabla|)}{|\nabla|}f\right](Te^{-\tau}\xi) \end{aligned}$$

and by duality we extend these operators to $\mathcal{S}'(\mathbb{R}^d)$.

The crucial observations now is the fact that the wave propagators in similarity coordinates decay exponentially, provided one takes sufficiently many derivatives.

Lemma 3.4. *Let $s \geq 0$. Then we have the bounds*

$$\begin{aligned} \|C_T(\tau)f\|_{\dot{H}^s(\mathbb{R}^d)} &\leq T^{-\frac{d}{2}+s}e^{(\frac{d}{2}-s)\tau}\|f\|_{\dot{H}^s(\mathbb{R}^d)} \\ \|S_T(\tau)f\|_{\dot{H}^{s+1}(\mathbb{R}^d)} &\leq T^{-\frac{d}{2}+s+1}e^{(\frac{d}{2}-s-1)\tau}\|f\|_{\dot{H}^s(\mathbb{R}^d)} \end{aligned}$$

for all $\tau \geq 0$, $T > 0$, and $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. The proof is a simple scaling argument and left as an exercise. □

3.3. **Back to the wave maps equation.** Now we return to the wave maps equation Eq. (1),

$$\left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r\right) u(t,r) + \frac{\sin(2u(t,r))}{r^2} = 0. \tag{10}$$

Even if it may look like at first glance, this is not a standard nonlinear wave equation because of the singularity at $r = 0$. A Taylor expansion shows that smooth solutions of this equation must vanish at $r = 0$. This observation motivates the introduction of the new variable $\widetilde{v}(t,r) := \frac{u(t,r)}{r}$. In terms of $\widetilde{v}$, Eq. (10) reads

$$\left(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\right) \widetilde{v}(t,r) + \frac{\sin(2r\widetilde{v}(t,r)) - 2r\widetilde{v}(t,r)}{r^3} = 0.$$

This is now a proper radial semilinear wave equation with a smooth nonlinearity (observe the cancellation in the numerator) but in 5 rather than 3 spatial dimensions. Thus, it is natural to formulate the problem in terms of the function $v(t,x) := \widetilde{v}(t,|x|)$, where $v(t,\cdot)$ is a radial function on $\mathbb{R}^5$. This leads to the equation

$$(\Box v)(t,x) + \frac{\sin(2|x|v(t,x)) - 2|x|v(t,x)}{|x|^3} = 0,$$

where

$$(\Box v)(t,x) := (\partial_t^2 - \Delta_x)v(t,x)$$

denotes the *d'Alembertian* or *wave operator*. In terms of the function $\widetilde{w}(\tau,\xi) := v(T - Te^{-\tau}, Te^{-\tau}\xi)$, we obtain

$$T^{-2}e^{2\tau}\widetilde{\Box}_{\tau,\xi}\widetilde{w}(\tau,\xi) + \frac{\sin(2Te^{-\tau}|\xi|\widetilde{w}(\tau,\xi)) - 2Te^{-\tau}|\xi|\widetilde{w}(t,x)}{T^3e^{-3\tau}|\xi|^3} = 0,$$

where $T^{-2}e^{2\tau}\widetilde{\Box}_{\tau,\xi}$ is the wave operator in similarity coordinates, i.e.,

$$T^{-2}e^{2\tau}\widetilde{\Box}_{\tau,\xi}\widetilde{w}(\tau,\xi) = (\Box v)(T - Te^{-\tau}, Te^{-\tau}\xi).$$

Note that the coefficients of $\widetilde{\Box}_{\tau,\xi}$ are independent of τ. In order to obtain an autonomous equation, we switch to the variable $w(\tau,\xi) := Te^{-\tau}\widetilde{w}(\tau,\xi)$. This leads to

$$e^{-\tau}\widetilde{\Box}_{\tau,\xi}\left(e^{\tau}w(\tau,\xi)\right) + \frac{\sin(2|\xi|w(\tau,\xi)) - 2|\xi|w(\tau,\xi)}{|\xi|^3} = 0. \tag{11}$$

Recall that the solution of

$$e^{-\tau}\widetilde{\Box}_{\tau,\xi}(e^{\tau}w(\tau,\xi)) = 0$$

is given by

$$e^{\tau}w(\tau,\cdot) = C_T(\tau)w(0,\cdot) + T^{-1}S_T(\tau)[\partial_0 w(0,\cdot) + (\cdot)^j\partial_j w(0,\cdot) + w(0,\cdot)]$$

since $(\partial_0 v)(0,Te^{-\tau}\xi) = [e^{\tau}\partial_\tau + e^{\tau}\xi^j\partial_{\xi^j}](T^{-1}e^{\tau}w(\tau,\xi))|_{\tau=0}$. Note carefully that we gain an additional factor of decay. Recall that we have the static solution

$$w_*(\xi) := \frac{2}{|\xi|}\arctan(|\xi|)$$

which we want to perturb. Thus, we plug in the ansatz $w(\tau,\xi) = w_*(\xi) + \varphi(\tau,\xi)$ and obtain the equation

$$e^{-\tau}\widetilde{\Box}_{\tau,\xi}\left(e^{\tau}\varphi(\tau,\xi)\right) + \frac{2|\xi|\cos(2|\xi|w_*(\xi))\varphi(\tau,\xi) - 2|\xi|\varphi(\tau,\xi)}{|\xi|^3} + N(\varphi(\tau,\xi),\xi), \tag{12}$$

where

$$N(y,\xi) := \frac{\sin(2|\xi|(w_*(\xi)+y)) - \sin(2|\xi|w_*(\xi)) - 2|\xi|\cos(2|\xi|w_*(\xi))y}{|\xi|^3}.$$

Note that $N(y,\xi)$ is quadratic in y and smooth in ξ. Consequently, by introducing the variable

$$\Phi(\tau)(\xi) = \begin{pmatrix} \varphi(\tau,\xi) \\ (\partial_\tau + \xi^j\partial_{\xi^j} + 1)\varphi(\tau,\xi) \end{pmatrix},$$

3.4. **Semigroup formulation.** Eq. (12) can be written as the first-order system

$$\partial_\tau \Phi(\tau) = \mathbf{L}_0\Phi(\tau) + \mathbf{L}'\Phi(\tau) + \mathbf{N}(\Phi(\tau)),$$

where $\mathbf{L}_0$ is a differential operator with τ-independent coefficients,

$$\left[\mathbf{L}'\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right](\xi) = \begin{pmatrix} 0 \\ -\frac{2\cos(2|\xi|w_*(\xi))-2}{|\xi|^2} f_1(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{16}{(1+|\xi|^2)^2} f_1(\xi) \end{pmatrix},$$

and

$$\mathbf{N}\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right)(\xi) := \begin{pmatrix} 0 \\ N(f_1(\xi),\xi) \end{pmatrix}.$$

Note that by Lemma 3.4, a suitable realization of the operator $\mathbf{L}_0$ generates a semigroup $\mathbf{S}_0$ with the property that

$$\|\mathbf{S}_0(\tau)\|_{\dot{H}^s(\mathbb{R}^5)\times\dot{H}^{s-1}(\mathbb{R}^5)} \leq e^{(\frac{3}{2}-s)\tau}$$

for all $\tau \geq 0$ and $s \geq 1$. In particular,

$$\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \leq e^{-\frac{1}{2}\tau},$$

where $\mathcal{H} := (\dot{H}^2(\mathbb{R}^5)\times\dot{H}^1(\mathbb{R}^5)) \cap (\dot{H}^3(\mathbb{R}^5)\times\dot{H}^2(\mathbb{R}^5))$. The point is that $\mathcal{H}$ is a Banach algebra.

Lemma 3.5. *Let $0 \leq s < \frac{d}{2} < t$ and $H := \dot{H}^s(\mathbb{R}^d) \cap \dot{H}^t(\mathbb{R}^d)$. Then we have*

$$\|fg\|_H \lesssim \|f\|_H \|g\|_H$$

for all $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Proof. Left as an exercise. □

3.5. **Spectral analysis of the generator.** Furthermore, we have a nice compactness property.

Lemma 3.6. *The operator $\mathbf{L}' : \mathcal{H} \to \mathcal{H}$ is compact.*

Proof. This follows from the fact that $\mathbf{L}'$ maps the first component to the second one combined with the decay of the *potential* $\frac{16}{(1+|\xi|^2)^2}$. The details are left as an exercise. □

As a consequence of the bound $\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \leq e^{-\frac{1}{2}\tau}$ we see that the *free resolvent*

$$\mathbf{R}_{\mathbf{L}_0}(\lambda) := (\lambda\mathbf{I} - \mathbf{L}_0)^{-1} = \int_0^\infty e^{-\lambda\tau}\mathbf{S}_0(\tau)d\tau$$

exists provided that $\operatorname{Re}\lambda > -\frac{1}{2}$. By the *Birman-Schwinger principle*, i.e., the identity

$$\lambda\mathbf{I} - \mathbf{L} = [\mathbf{I} - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda\mathbf{I} - \mathbf{L}_0), \qquad \mathbf{L} := \mathbf{L}_0 + \mathbf{L}',$$

we see that $\lambda\mathbf{I} - \mathbf{L}$ is bounded invertible for $\operatorname{Re}\lambda > -\frac{1}{2}$ if and only if $\mathbf{I} - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$ is bounded invertible. Since $\mathbf{L}'$ is compact, one can show, by invoking the *analytic Fredholm theorem*, that $\lambda\mathbf{I} - \mathbf{L}$ is bounded invertible for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\frac{1}{2}$ except for a finite number of eigenvalues. In order to locate these eigenvalues, we need to solve the equation $(\lambda\mathbf{I}-\mathbf{L})\mathbf{f}=\mathbf{0}$. A straightforward computation leads to Eq. (3). This is the connection to the mode stability problem that gives the latter a proper functional analytic interpretation.

Proposition 3.7. *For the spectrum $\sigma(\mathbf{L})$ of $\mathbf{L}$ we have*

$$\sigma(\mathbf{L})\subset\{z\in\mathbb{C}:\operatorname{Re}z\leq-\tfrac{1}{2}\}\cup\{1\}$$

and 1 is an eigenvalue of $\mathbf{L}$ with algebraic multiplicity 1.

Proof. The statement about the spectrum follows from mode stability (Theorem 2.2). The fact that the algebraic multiplicity equals 1 can be proved by ODE methods, by showing that the equation $(\mathbf{I}-\mathbf{L})\mathbf{f}=\mathbf{f}_*$, where $\mathbf{f}_*$ is the eigenfunction associated to the eigenvalue 1, has no solution. □

We define the Riesz or spectral projection associated to the eigenvalue 1 by

$$\mathbf{P}:=\frac{1}{2\pi i}\int_\gamma \mathbf{R}_{\mathbf{L}}(z)dz,$$

where $\gamma:[0,1]\to\mathbb{C}$ is given by $\gamma(t):=1+\frac{1}{2}e^{2\pi it}$. From abstract semigroup theory we can then obtain a sufficiently detailed understanding of the linearized evolution generated by $\mathbf{L}$.

Lemma 3.8. *The operator $\mathbf{L}$ generates a semigroup $\mathbf{S}$. Furthermore, there exists an $\epsilon>0$ and a $C>0$ such that*

$$\|\mathbf{S}(\tau)(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}\leq Ce^{-\epsilon\tau}\|(\mathbf{I}-\mathbf{P})\mathbf{f}\|_{\mathcal{H}}$$
$$\mathbf{S}(\tau)\mathbf{P}\mathbf{f}=e^{\tau}\mathbf{P}\mathbf{f}$$

for all $\tau\geq0$ and $\mathbf{f}\in\mathcal{H}$.

3.6. **The nonlinear problem.** In Duhamel form, the equation we would like to solve reads

$$\Phi(\tau)=\mathbf{S}(\tau)\mathbf{f}+\int_0^\tau \mathbf{S}(\tau-\tau')\mathbf{N}(\Phi(\tau'))d\tau'.$$

Typically, such an equation is solved by a fixed point argument. However, in the present form this is not possible due to the exponential growth of the semigroup on $\operatorname{rg}\mathbf{P}$. Thus, we borrow an idea from dynamical systems theory known as the *Lyapunov-Perron method* and consider instead the equation

$$\Phi(\tau)=\mathbf{S}(\tau)[\mathbf{f}-\mathbf{C}(\mathbf{f},\Phi)]+\int_0^\tau \mathbf{S}(\tau-\tau')\mathbf{N}(\Phi(\tau'))d\tau', \tag{13}$$

where

$$\mathbf{C}(\mathbf{f},\Phi):=\mathbf{P}\mathbf{f}+\mathbf{P}\int_0^\infty e^{-\tau'}\mathbf{N}(\Phi(\tau'))d\tau'$$

is a correction term that stabilizes the evolution. Formally, this term is obtained by applying the spectral projection $\mathbf{P}$ to the original equation. Consequently, the subtraction of $\mathbf{C}(\mathbf{f},\Phi)$ corrects the initial data along the one-dimensional subspace $\operatorname{rg}\mathbf{P}$ on which the linearized evolution grows exponentially. Note, however, that there is a nonlinear self-interaction, i.e., the correction term depends on the solution itself and is not known in advance as would be the case for a linear problem. Nonetheless, by a routine fixed point argument utilizing the Banach algebra property of $\mathcal{H}$, we can show that Eq. (13) has a solution $\Phi\in C([0,\infty),\mathcal{H})$ for any small data $\mathbf{f}$. Finally, by realizing that the data we want to describe depend on T, we see, e.g. by the intermediate value theorem, that there always exists a T that makes the correction term vanish.

4.1. **The Phragmén-Lindelöf principle.** The Phragmén-Lindelöf principle is an extension of the maximum principle to unbounded domains. There are many different versions and we only present a very basic one that is used in this course. First, recall the fundamental maximum principle from complex analysis.

Lemma 4.1 (Maximum principle)**.** *Let $\Omega \in \mathbb{C}$ be open, connected, and bounded. Suppose that $f : \overline{\Omega} \to \mathbb{C}$ is continuous and that $f|_\Omega : \Omega \to \mathbb{C}$ is holomorphic. Then*

$$|f(z)| \leq \max_{\zeta \in \partial\Omega} |f(\zeta)|$$

for all $z \in \Omega$.

The maximum principle shows that if we want to control a holomorphic function on a bounded domain Ω, it is enough to control it on the boundary. The assumption of boundedness is crucial here. However, under a mild growth condition, the maximum principle extends to unbounded domains and in this situation it goes by the name of Phragmén-Lindelöf. There are many different versions of this principle. We use a very basic one that allows us to bound a function on the complex right half-plane by its values on the imaginary axis.

Lemma 4.2 (Phragmén-Lindelöf principle)**.** *Let $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and suppose that $f : \overline{\Omega} \to \mathbb{C}$ is continuous and that $f|_\Omega : \Omega \to \mathbb{C}$ is holomorphic. Let $M \geq 0$. If*

(1) *$|f(it)| \leq M$ for all $t \in \mathbb{R}$ and*

(2) *there exists a $C \geq 0$ such that $|f(z)| \leq Ce^{|z|^{\frac{1}{2}}}$ for all $z \in \Omega$*

then

$$|f(z)| \leq M$$

for all $z \in \Omega$.

Proof. The proof is very simple and plays the situation back to the standard maximum principle. First, we note that the function $z \mapsto z^{\frac{3}{4}} : \overline{\Omega} \to \mathbb{C}$ is continuous and holomorphic on Ω. Then, for $\epsilon > 0$, we define an auxiliary function $f_\epsilon : \overline{\Omega} \to \mathbb{C}$ by $f_\epsilon(z) := e^{-\epsilon z^{\frac{3}{4}}} f(z)$. Again, f_ϵ is continuous and holomorphic on Ω. Furthermore,

$$|f_\epsilon(z)| = e^{-\epsilon \operatorname{Re} z^{\frac{3}{4}}} |f(z)| = e^{-\epsilon |z|^{\frac{3}{4}} \cos(\frac{3}{4} \arg z)} |f(z)|$$

for all $z \in \overline{\Omega}$ and thus,

$$|f_\epsilon(it)| = e^{-\epsilon |z|^{\frac{3}{4}} \cos(\frac{3}{4}\frac{\pi}{2})} |f(it)| \leq |f(it)| \leq M$$

for all $t \in \mathbb{R}$ and $\epsilon > 0$ because $\eta := \cos(\frac{3}{4}\frac{\pi}{2}) > 0$. Next, we have the bound

$$|f_\epsilon(z)| \leq e^{-\epsilon\eta|z|^{\frac{3}{4}}} |f(z)| \leq Ce^{-\epsilon\eta|z|^{\frac{3}{4}} + |z|^{\frac{1}{2}}} = Ce^{-\epsilon\eta|z|^{\frac{3}{4}}(1 - |z|^{-\frac{1}{4}})} \to 0$$

as $|z| \to \infty$ and thus, $|f_\epsilon(z)| \leq M$ if $|z|$ is sufficiently large. For $R > 0$ we define the domain

$$\Omega_R := \{z \in \mathbb{C} : |z| < R\} \cap \Omega.$$

By the above, f_ϵ is holomorphic on Ω_R, continuous on $\overline{\Omega}_R$, and there exists an $R_\epsilon > 0$ such that $|f_\epsilon(z)| \leq M$ for all $z \in \partial\Omega_R$, provided that $R \geq R_\epsilon$. Consequently, by the maximum principle, $|f_\epsilon(z)| \leq M$ for all $z \in \Omega_R$ and since this argument works for any $R \geq R_\epsilon$, we see that in fact $|f_\epsilon(z)| \leq M$ for all $z \in \Omega$. This yields

$$|f(z)| \leq e^{\epsilon|z|^{\frac{3}{4}}} |f_\epsilon(z)| \leq Me^{\epsilon|z|^{\frac{3}{4}}}$$

for any $z \in \Omega$ and any $\epsilon > 0$ and upon letting $\epsilon \to 0$, we obtain the desired bound. $\square$

4.2. **Asymptotics of difference equations.**

Theorem 4.3 (Poincaré)**.** *Let $p, q : \mathbb{N} \to \mathbb{C}$ and suppose that*

$$p_\infty := \lim_{n\to\infty} p(n), \qquad q_\infty := \lim_{n\to\infty} q(n)$$

exist. Assume further that there exist $z_1, z_2 \in \mathbb{C}$ with $|z_1| > |z_2|$ and such that

$$z_j^2 + p_\infty z_j + q_\infty = 0, \qquad j \in \{1, 2\}.$$

Let $a : \mathbb{N} \to \mathbb{C}$ satisfy

$$a(n+2) + p(n)a(n+1) + q(n)a(n) = 0 \tag{14}$$

for all $n \in \mathbb{N}$. Then either there exists an $n_0 \in \mathbb{N}$ such that $a(n) = 0$ for all $n \geq n_0$ or we have

$$\lim_{n\to\infty} \frac{a(n+1)}{a(n)} \in \{z_1, z_2\}.$$

Idea of proof. In order to understand what is going on, we consider the *limiting equation*

$$a(n+2) + p_\infty a(n+1) + q_\infty a(n) = 0. \tag{15}$$

Then it follows that the functions $n \mapsto z_j^n$ for $j \in \{1, 2\}$ solve this equation simply because

$$z_j^{n+2} + p_\infty z_j^{n+1} + q_\infty z_j^n = z_j^n(z_j^2 + p_\infty z_j + q_\infty) = 0.$$

Consequently, the general solution of Eq. (15) is given by $a(n) = \alpha_1 z_1^n + \alpha_2 z_2^n$, where $\alpha_j \in \mathbb{C}$ can be chosen arbitrarily. Thus, if $\alpha_1 \neq 0$, we can write

$$a(n) = \alpha_1 z_1^n \left[1 + \frac{\alpha_2}{\alpha_1}\left(\frac{z_2}{z_1}\right)^n\right]$$

and $\lim_{n\to\infty} \frac{a(n+1)}{a(n)} = z_1$ follows immediately because $|\frac{z_2}{z_1}| < 1$ by assumption. On the other hand, if $\alpha_1 = 0$, we obviously have $\lim_{n\to\infty} \frac{a(n+1)}{a(n)} = z_2$. Thus, since $p(n)$ and $q(n)$ get arbitrarily close to p_∞ and q_∞ for large n, the proof consists of showing that the above logic is stable under a suitable perturbation argument, see e.g. [1]. $\square$

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