Lecture 4: Low regularity solutions for 2-d gravity waves

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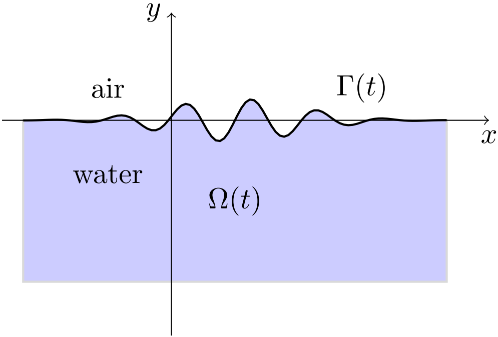
This is joint work with Albert Ai and Daniel Tataru

Water Waves



- Water flows inside the fluid domain (infinite depth)
- Free boundary motion

The Euler equation

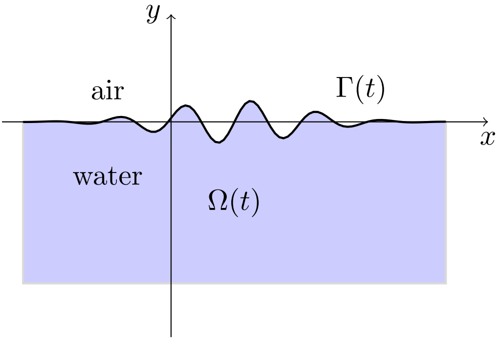
Fluid motion in an open set:

- Euler vs. Navier-Stokes

$$\begin{cases} (\partial_t + v \cdot \nabla)v + \nabla p = -g\mathbf{j} + \mu \Delta v & \text{(Newton's law)} \\ \nabla \cdot v = 0 & \text{(incompressibility)} \end{cases}$$

- $v = v(t, x, y)$ fluid velocity
- $p = p(t, x, y)$ fluid pressure
- g = gravity
- μ = viscosity (resistance to shear stress)
- Euler: $\mu = 0$ (inviscid)

Boundary conditions



Boundary conditions on $\Gamma(t)$:

$$\begin{cases} \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup \Gamma(t) & \text{(kinematic)} \\ p = -2\sigma H \quad \text{on } \Gamma(t) & \text{(dynamic)} \end{cases}$$

H = mean curvature of the boundary, σ = surface tension ($= 0$)

Vorticity and irrotational flows (water waves)

Vorticity = instantaneous rotation of a fluid

$$\omega = \nabla \times v \qquad \text{(curl of } v\text{)}$$

For solutions to Euler equations, ω satisfies a transport equation:

$$(\partial_t + v \cdot \nabla)\omega = (\omega \cdot \nabla)v$$

Irrotational fluid:

- $\omega = 0$ (propagated along the flow)
- Velocity potential

$$v = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega(t)$$

uniquely determined by its values on the free boundary.

Key idea: The fluid equation reduces to an equation of motion for the free boundary ! [Zakharov '68]

Water waves in Eulerian coordinates

Velocity potential

$$v = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega(t)$$

Dynamic boundary condition (Bernoulli law):

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy + p = 0 \quad \text{on } \ \Gamma(t)$$

Equations reduced to the boundary in Eulerian formulation:

- $\eta =$ elevation, $\Gamma(t) = \{y = \eta(t,x)\}$
- $\psi = \phi_{|\Gamma(t)}$

$$\begin{cases} \partial_t\eta - G(\eta)\psi = 0 \\ \partial_t\psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\nabla\psi + G(\eta)\psi)^2}{1+|\nabla\eta|^2} = 0. \end{cases}$$

$$G(\eta) = \text{ Dirichlet to Neuman operator}$$

Choices of coordinates

Choice of coordinates = gauge freedom

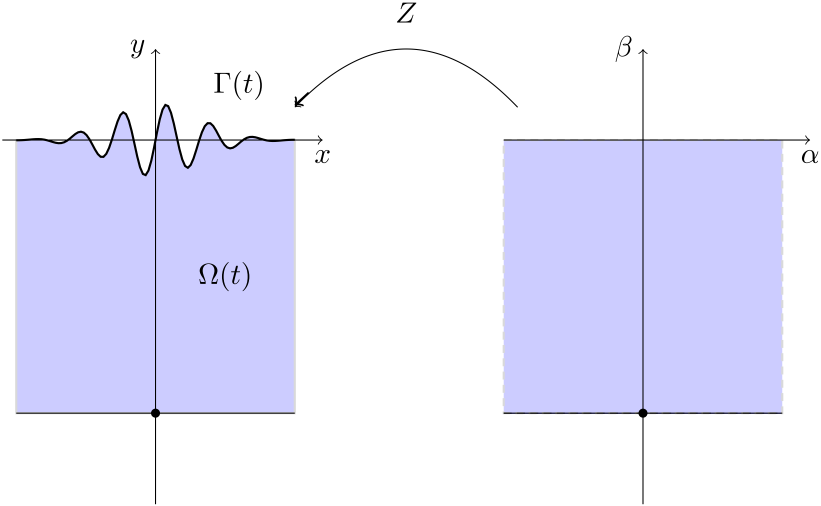
Eulerian coordinates (x, t): Particles are moving in a fixed frame. Flat geometry.

Lagrangian coordinates (X, t): Frame moves along particle trajectories. Curved geometry.

$$(\partial_t + \nabla \cdot v)X = 0$$

Holomorphic coordinates (α, t): (2-d only) Both particles and frame move. Conformally flat geometry.

Holomorphic (conformal) coordinates



The conformal map

Holomorphic (conformal) coordinates

Holomorphic coordinates:

$$Z : \{\Im z \leq 0\} \rightarrow \Omega(t), \qquad \alpha + i\beta \rightarrow Z(\alpha + i\beta)$$

Boundary condition at infinity:

$$Z(\alpha) - \alpha \rightarrow 0 \;\; (nonperiodic) \qquad Z(\alpha) - \alpha \; periodic \;\; (periodic)$$

Free boundary parametrization:

$$Z : \mathbb{R} \rightarrow \Omega(t), \qquad \alpha \rightarrow Z(\alpha)$$

Variables:

- Perturbation of zero solution:

$$W = Z - \alpha$$

- Holomorphic velocity potential ($v = \nabla \phi$):

$$Q = \phi + i\theta$$

Water waves in holomorphic coordinates

[Ovsiannikov, Zakharov & al, Wu, Hunter-Ifrim-Tataru]

- P - Projection onto negative frequencies

Fully nonlinear equations for *holomorphic* variables (W, Q):

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha + P\left[|R|^2\right] - igW = 0. \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2, \qquad R = \frac{Q_\alpha}{1 + W_\alpha}.$$

Conserved energy (Hamiltonian):

$$E(W, Q) = \int \Im(Q\bar{Q}_\alpha) + \frac{1}{2}g\left(|W|^2 - \Re(\bar{W}^2 W_\alpha)\right) d\alpha \approx \|W\|^2_{L^2} + \|Q\|^2_{\dot{H}^{\frac{1}{2}}}$$

Alinhac's "good variable"

Idea: diagonalize the principal (transport) part of the equation.
Good variables for differentiated equation (Hunter-Ifrim-Tataru '14):

$$\left(\mathbf{W} = W_\alpha, R = \frac{Q_\alpha}{1 + W_\alpha}\right).$$

Differentiated equation [with omitted projections]:

$$\begin{cases} (\partial_t + b\partial_\alpha)\mathbf{W} + \dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}} R_\alpha = G(\mathbf{W}, R) \\ (\partial_t + b\partial_\alpha)R - i\dfrac{(1+a)\mathbf{W}}{1+\mathbf{W}} = K(\mathbf{W}, R) \end{cases}$$

where

$$b = 2\Re P\left[\frac{R}{1+\mathbf{W}}\right], \qquad a = 2\Im P[R\bar{R}_\alpha] = \int |1_{|D|>h} R|^2 dh$$

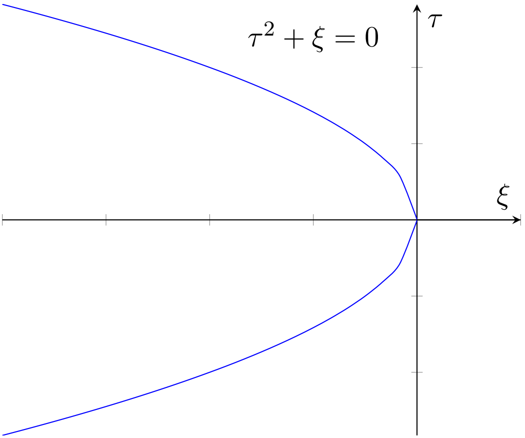
Taylor coefficient: $a \geq 0$, necessary for well-posedness.

Note: Good variable in Eulerian setting: Alazard-Burq-Zuily '11

Linearization around the zero solution

$$\begin{cases} w_t + q_\alpha = 0 \\ q_t - iw = 0 \end{cases}$$

Dispersion relation (characteristic set):



Well-posedness for nonlinear equations

Equation: $u_t = N(u)$ Linearization: $v_t = DN(u)v$

Para-diff: $u_t = T_{DN(u)}u + F(u)$ Linearized: $v_t = T_{DN(u)}v + F_{lin}(u)v$

- Existence of regular solutions
  - Regularization/iteration scheme
- Uniqueness of regular solutions
  - Estimates for differences in a weaker topology
- Rough solutions as unique limits of smooth solutions
  - Lipschitz bounds for linearized equation in a weaker topology
  - Uniform propagation of higher regularity
- Continuous dependence on initial data
  - Lipschitz bounds for linearized equation in a weaker topology
  - Frequency envelopes

Low regularity well-posedness: a primer

Following [Tataru, Bahouri-Chemin '98-00, nonlinear wave eqn.]

Step 1. Energy estimates:

$$\frac{d}{dt}E^s(u) \lesssim \|D^\sigma u\|_{L^\infty} E^s(u), \qquad E^s(u) \approx \|u\|_{H^s}^2$$

- Similar bounds for the linearized equation in H^{s_0} for a fixed s_0.
- Gives well-posedness in H^s if $H^s \subset C^\sigma$.

Step 2. Strichartz estimates:

$$\|D^\sigma u\|_{L^p L^\infty} \lesssim \|u\|_{H^s}$$

- Frequency localized, paradifferential
- Similar bounds for the linearized equation
- Parametrices, dispersion on semiclassical time scales

Low regularity WP for gravity waves

Question: Local well-posedness for $(\mathbf{W}, R) \in H^s \times H^{s+\frac{1}{2}}$.

Scaling: Critical Sobolev space $s_c = \frac{1}{2}$.

Local well-posedness results:

- **'11** $s = 1 + \epsilon$, Alazard-Burq-Zuily [energy estimates]
- **'14** $s = 1$, Hunter-I.-Tataru [energy estimates, 2-d]
- **'15** $s = 1 - \frac{1}{24}$, Alazard-Burq-Zuily [energy + Strichartz]
- **'18** $s = 1 - \frac{1}{8} + \epsilon$, Ai [energy + no loss Strichartz]
- **'19** $s = 1 - \frac{1}{4}$, Ai-I.-Tataru [balanced energy estimates]
- **'22** $s = 1 - \frac{3}{8}$, Ai-I.-Tataru [balanced energy + no loss Strichartz]

The long time existence problem

Objective: Obtain improved lifespan estimates for small data.

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-1}$

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-2}$.

(iii) Normal form method (Shatah '85): transform equation with quadratic nonlinearities into one with cubic ones via a normal form transformation,

$$u \to v = u + B(u, u)$$

- works for nonresonant and null resonant interactions, but
- unbounded for quasilinear problems

Normal form methods for quasilinear pde's

1. Modified energy method (Hunter-Ifrim-Tataru) Modify the energy functional rather than the unknown,

$$E_{lin}(u) \rightarrow E_{NL}(u) = E_{lin}(u) + correction$$

- works for quasilinear problems
- provides an algorithm to compute these energies

2. Normal form flow method (Hunter-Ifrim, Ifrim) Use a normal form based flow to construct a bounded normal form transformation

- works for some quasilinear problems
- most elegant, but problem specific

3. Paradiagonalization (Delort, Alazard-Delort) Combines a partial normal form with a paradifferential symmetrization

- works for some quasilinear problems
- microlocal based approach

Energy estimates for water waves

1. Alazard-Burq-Zuily '11-15, Eulerian, quasilinear energy:

$$\frac{d}{dt}E^s(\mathbf{W}, R) \lesssim \|(\mathbf{W}, R)\|_{C^{\frac{1}{2}} \times C^1} E^s(\mathbf{W}, R)$$

2. Hunter-Ifrim-Tataru '14, holomorphic, modified energy, cubic:

$$\frac{d}{dt}E^s(\mathbf{W}, R) \lesssim A_0 A_{1/2} E^s(\mathbf{W}, R)$$

$$A_\sigma = \|(\mathbf{W}, R)\|_{BMO^\sigma \times BMO^{\sigma+\frac{1}{2}}}, \qquad \sigma = \text{scaling index}$$

3. Ai-Ifrim-Tataru '19, holomorphic, paradifferential modified energy, balanced cubic:

$$\frac{d}{dt}E^s(\mathbf{W}, R) \lesssim A_{1/4}^2 E^s(\mathbf{W}, R)$$

The linearized equation

Original variables: (W, Q), auxiliary variable $Y = \dfrac{W_\alpha}{1 + W_\alpha}$

Material derivative: $D_t = \partial_t + b\partial_\alpha$.

Linearized variables (w, q), good variables $(w, r = q - Rw)$.

Linearized equations [with omitted projections]:

$$\begin{cases} D_t w + (1 - \bar{Y})r_\alpha + R_\alpha(1 - \bar{Y})w = G(w, r) \\ D_t r - i(1 + a)(1 - Y)w = K(w, r) \end{cases}$$

Theorem (Ai-Ifrim-Tataru)

Assume that $A_{1/4} \in L^2$. Then the linearized equation is well-posed in $\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}$, and there exists an energy functional $E_{lin}^{1/4}$ such that we have the balanced energy estimates

$$\frac{d}{dt} E_{lin}^{1/4}(w, r) \lesssim A_{1/4}^2 E_{lin}^{1/4}(w, r)$$

The linear paradifferential equation

$$\begin{cases} T^w_{D_t} w + T^w_{1-\bar{Y}} \partial_\alpha r + T^w_{(1-\bar{Y})R_\alpha} w = g, \\ T^w_{D_t} r - i T^w_{1-Y} T^w_{1+a} w = k, \end{cases}$$

- Weyl paradifferential quantization
- Balanced cubic estimates in $L^2 \times H^{\frac{1}{2}}$: modified energy method
- Balanced cubic estimates in $H^s \times H^{s+\frac{1}{2}}$: NF reduction to $s = 0$

Theorem (Ai-Ifrim-Tataru)

There exists a partial normal form transformation $(w, r) \to (\tilde{w}, \tilde{r})$ *s.t. :*
(i) Equivalent norm

$$\|(\tilde{w}, \tilde{r})\|_{\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}} \approx_{A_0} \|(w, r)\|_{\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}}$$

(ii) $(\tilde{w}, \tilde{r})$ *solves paradiff. equation* (20) *with perturbative sources* $(\tilde{g}, \tilde{k})$*:*

$$\|(\tilde{g}, \tilde{k})\|_{\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}} \lesssim_{A_0} A^2_{1/4} \|(w, r)\|_{\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}}$$

A story of two linearizations

$$\text{Original equation} \quad \longleftrightarrow \quad \text{differentiated equation}$$

$$\mathrm{EQ}(W, Q) = 0 \quad \longleftrightarrow \quad \mathrm{DiffEQ}(\mathbf{W}, R) = 0$$

$$\mathrm{LinEQ}(W, Q)[w, r] = 0 \quad \longleftrightarrow \quad \mathrm{LinDiffEQ}(\mathbf{W}, R)[\hat{w}, \hat{r}] = 0$$

$$\mathrm{ParaLinEQ}(W, Q)[w, r] = 0 \quad \leadsto \quad \mathrm{ParaLinDiffEQ}(\mathbf{W}, R)[\hat{w}, \hat{r}] = 0$$

Proposition

Assume $A_{1/4} \in L^2$. Then

a) The $\dot{H}^s \times \dot{H}^{s+\frac{1}{2}}$ well-posedness, $s \geq 1$, for [ParaLinEQ] is equivalent to the $\dot{H}^s \times \dot{H}^{s+\frac{1}{2}}$, $s \geq 0$ well-posedness for [ParaLinDiffEQ].

b) The $\dot{H}^{\frac{1}{4}} \times \dot{H}^{\frac{3}{4}}$ well-posedness for [LinEQ] is equivalent to the $\dot{H}^{-\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}$ well-posedness for [LinDiffEQ].

Balanced normal form analysis

$$U_t + N(U) = 0 \qquad \Longleftrightarrow \qquad (\partial_t + T_{DN(U)})U = F(U)$$

Normal form analysis for terms in $N(U)$:

- Quadratic terms $Q_2(U, U)$
  - Low-high $Q_2(U_{lo}, U_{hi})$, belongs into the paradiff. part.
  - $Q_2(U_{hi}, U_{hi})$, apply quadratic NFT, turns to cubic.
- Cubic terms $Q_3(U, U, U)$
  - Low–low-high $Q_3(U_{lo}, U_{lo}, U_{hi})$, goes into the paradiff. part.
  - Low-high-high $Q_3(U_{lo}, U_{hi}, U_{hi})$, apply quadratic NFT with coeff.
  - High-high-high $Q_3(U_{hi}, U_{hi}, U_{hi})$, perturbative.

Further difficulties:

- Also needed for the linearized equation: symmetry loss e.g. in $Q_3(u_{lo}, U_{med}, U_{hi})$, go to quartic order.
- Also needed for paradiff. equation: a finite number of quadratic NFT do not suffice, replace by exponential para-conjugations.

References

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