Lecture 3
Well-posedness for quasilinear evolution equations -continuous dependence on the initila data &
Enhanced lifespan for nonlinear evolutions

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This is joint work with Daniel Tataru

Continuous dependence on the initial data

• There is one intermediate step in getting the continuous dependence on the initial data: that is to show that one can approximate rough solutions by smooth solutions.

- we start with u_0 and this generates a solution u
- consider a sequence of regularized initial data u_0^h (h being the dyadic frequency scale), and this generates solutions u^h.
- we know that u_0^h converges to u_0 in H^s

Goal: We want to prove u^h converges to u in H^s *uniformly*.

Idea: **Frequency envelopes**

Track the evolution of the energy of sols between dyadic energy shells.

Definition

$\{c_k\} \in \ell^2$ is a frequency envelope for u_0 in H^s if

- $\|P_k u_0\|_{H^s} \leq c_k$
- c_k is slowly varying (to account for energy leakage between nearby shells)

Frequency Envelope

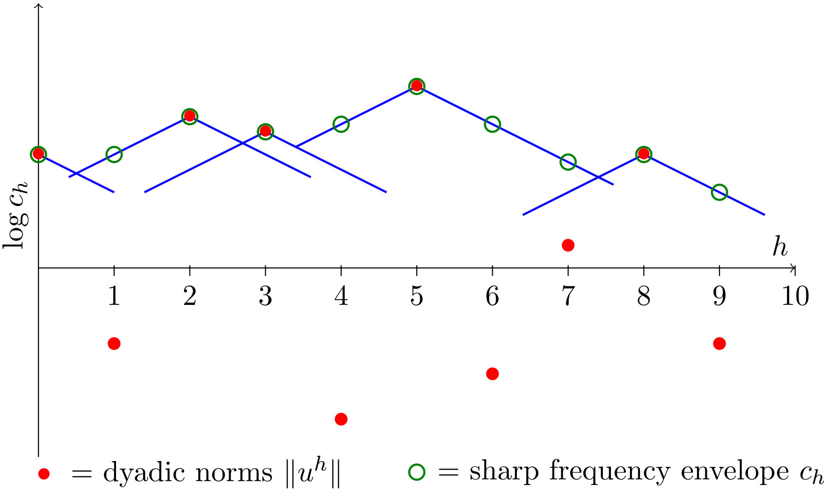

Figure: Construction of sharp frequency envelopes.

Convergence via frequency envelopes

Show that the frequency envelopes carry over from the data to the solution, in two steps:
(i) Energy estimates for each u_h give high freq. bounds

$$\|u^h\|_{L^\infty H^s} \lesssim 1, \qquad \|u^h\|_{L^\infty H^{s+j}} \lesssim 2^{hj} c_h$$

(ii) Energy estimates for differences give low frequency bounds

$$\|u^{h+1} - u^h\|_{L^\infty L^2} \lesssim 2^{-sh} c_h$$

Telescopic sum for $h < k$, using interpolation:

$$\|u^h - u^k\|_{L^\infty L^2} \lesssim 2^{-sh} c_h$$

$$\|u^h - u^k\|_{L^\infty H^s} \lesssim c_{[h,k]}$$

These show uniform convergence first in L^2 and then in H^s! In particular we get $u \in C(H^s)$.

Continuous dependence via frequency envelopes

We take a convergent sequence of data $u_{0n} \to u_0$ in H^s, and corresponding solutions. For each, we also have the regularized solutions u_n^h, and frequency envelopes c_n^h. Then we estimate

$$\|u_n - u\|_{L^\infty H^s} \lesssim \|u_n^h - u^h\|_{L^\infty H^s} + \|u_n^h - u_n\|_{L^\infty H^s} + \|u^h - u\|_{L^\infty H^s}$$

- First term goes to 0 for each h, by difference bounds for regular solutions
- Second and third terms are bounded by $c_n^{>h}$ respectively $c^{>h}$.

Key fact: Since $u_{0n} \to u_0$, it follows that the corresponding envelopes can be chosen to satisfy $c_n^h \to c^h$ in ℓ^2, which gives

$$\lim_{h\to\infty} c_n^{>h} = 0 \qquad \text{uniformly in } h.$$

Then we successively let $n \to \infty$ and then $h \to \infty$.

Paper available on arxiv: Local well posedness for quasilinear problems: a primer (joint with D. Tataru) https://arxiv.org/pdf/2008.05684.pdf

Enhanced lifespan for nonlinear evolutions

Tools use to obtain enhanced lifespan of existence of solutions for nonlinear evolutions:

- Normal forms
- Quasilinear modified energy method
- Wave packets analysis

How are these methods related to low regularity well-posedness results?

Generic lifespan bounds

- What can be said about long time existence of solutions of a nonlinear PDE assuming it is known that the data is sufficiently small and smooth ?
- We must consider the nonlinear interactions more carefully:

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-1}$ (quadratic lifespan)

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-2}$ (cubic lifespan)

Normal forms and long time existence

(i) Normal form method (Shatah '85): transform an equation with a quadratic nonlinearity into one with a cubic one via a normal form transformation,

$$u \to v = u + B(u, u) + \text{higher}$$

(ii) The first consideration is that of three-wave resonances: when two linear waves may combine nonlinearly to create another linear wave feeding back into the system. Resonances of this form will correspond to solutions to an algebraic system.

(iii) If the set of resonances is nonempty, the second consideration is the direction in which linear waves may travel (interact transversally or parallel). Some localization of the data is required.

Examples:

1. The first easy example to look at is

$$\begin{cases} i\partial_t u + au = u^2 \\ u(0,x) = u_0(x), \end{cases}$$

   where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, a is just a non zero real constant, and u_0 is assumed to be smooth.

Dispersion relation: $\omega(\xi) = -a$

Normal form

$$u \to v = u - \frac{1}{a}u^2 \quad \Rightarrow \quad iv_t + av = \frac{2}{a}u^3.$$

Note: The bilinear form $B(u,u) := 1/au^2$ is bounded; in particular this assures us that whatever information we obtain on the v-equation can be transfered back to the u-equation. In fact, we have global solutions because one can fully eliminate the nonlinearity with a nonlinear normal form transformation.

Examples:

2. Another example is

$$\begin{cases} i\partial_t u + A(D)u = u^2 \\ u(0,x) = u_0(x), \end{cases}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $A(D)$ is a real Fourier multiplier with symbol $A(\xi)$, and u_0 is assumed to be smooth.

Dispersion relation:

The linear operator $L := i\partial_t + A(D)$, and the symbol associated to it is $p(\tau, \xi) = -\tau + A(\xi)$. The dispersion relation is

$$\omega(\xi) = \tau = A(\xi)$$

Quadratic resonances

Solving the following system will tell us if the quadratic nonlinearity is removable or not. This is called the three-wave resonances system:

$$\begin{cases} \omega(\xi_1) + \omega(\xi_2) = \omega(\xi_3) \\ \xi_1 + \xi_2 = \xi_3, \end{cases}$$

where ξ_1 and ξ_2 represent the input frequencies and ξ_3 the output frequency. Interactions occur whenever we can find a solution. In this case the quadratic nonlinearity is nonremovable.

Cubic resonances

Solving the following system will tell us if the quadratic nonlinearity is removable or not. This is called the four-wave resonances system:

$$\begin{cases} \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) = \omega(\xi_0) \\ \xi_1 + \xi_2 + \xi_3 = \xi_0, \end{cases}$$

where ξ_1, ξ_2 and ξ_3 represent the input frequencies and ξ_0 the output frequency. Interactions occur whenever a pair of input frequencies sum to zero, which says that the cubic nonlinearity is nonremovable.

3. The KdV equation

$$\begin{cases} u_t + \frac{1}{3} u_{xxx} = (u^2)_x \\ u(0,x) = u_0(x), \end{cases}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and u_0 is assumed to be smooth.

Dispersion relation: $\omega(\xi) = -\frac{1}{3}\xi^3$

We have resonant interactions at 0: when both the input frequencies are zero.

Normal form

$$u \to v = u + (\partial_x^{-1} u)^2 \quad \Rightarrow \quad v_t - u^2 \partial^{-1} u = v_{xxx}$$

Note: The bilinear form $B(u,u) := (\partial_x^{-1} u)^2$ is unbounded at low frequencies.

4. The Burgers-Hilbert equation

$$\begin{cases} u_t + uu_x = H[u] \\ u(0,x) = u_0(x), \end{cases}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, H is the Hilbert transform and u_0 is assumed to be small and smooth.

Dispersion relation: $\omega(\xi) = \operatorname{sgn} \xi$

There are no resonant interactions, hence the quadratic nonlinearity is removable.

Normal form

$$u \to v = u + H\left[Hu\, Hu_x\right] \quad \Rightarrow \quad v_t + Q(u) = Hv,$$

where $\mathcal{Q}(u)$ is cubic in u but involves two spatial derivatives.
Note: The bilinear form $B(u,u) := H\left[Hu\, Hu_x\right]$ is unbounded at high frequencies. This is in fact the case with most of quasilinear equations.

Energy estimates for the Burgers-Hilbert equation:

- Straightforward energy estimates applied to the Burgers-Hilbert equation yield

$$\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2 \lesssim \|u_x\|_{L^\infty}\|u\|_{H^k}^2$$

  for $k \geq 2$, which gives a lifespan of smooth solutions of the order ϵ^{-1}, as in the standard local existence theory for quasilinear hyperbolic PDEs.

- Straightforward energy estimates for v-equation yield

$$\frac{d}{dt}\|\partial_x^k v\|_{L^2}^2 \lesssim \|u_x\|_{L^\infty}^2\|u\|_{H^{k+1}}^2.$$

Remark

The v-equation is cubically nonlinear, but there is a loss of derivatives. On the other hand, the u-equation is quadratically nonlinear, but there is no loss of derivatives

Construction of the quasilinear modified energy

- The reason for this disparity in energy estimates is that the H^k-norms of u and v are not comparable:

$$\|\partial_x^k v\|_{L^2}^2 = \|\partial_x^k u\|_{L^2}^2 + 2\langle \partial_x^k u, \partial_x^k H[Hu \cdot Hu_x]\rangle + \|\partial_x^k H[Hu \cdot Hu_x]\|_{L^2}^2$$

- This suggests the quasilinear modified energy

$$E_k(u) = \frac{1}{2}\|\partial_x^k u\|_{L^2}^2 + \langle \partial_x^k u, \partial_x^k H\left[Hu \cdot Hu_x\right]\rangle$$

- The normal form energy is equivalent with the standard H^k energy

$$E_k(u) = \frac{1}{2}\|\partial_x^k u\|_{L^2}^2(1 + O(\|Hu_x\|_{L^\infty})).$$

and we obtain the following estimate

$$\frac{d}{dt}E_k(u) \leq C_{k,\delta}\|u_x\|_{H^{\frac{1}{2}+\delta}}^2 \|u\|_{H^k}^2$$

where $C_{k,\delta}$ is a constant depending on k and $\delta > 0$ only.

Normal forms for quasilinear equations

Key difficulty: The normal form transformation $v = u + B(u,u)$ involves the leading part of the equation, and thus is unbounded and not invertible. One is left with two seemingly incompatible estimates:
(i) A quadratic energy bound

$$\frac{d}{dt} E_k^{nl}(u) \lesssim A E_k^{nl}(u)$$

(ii) A cubic normal form bound

$$\frac{d}{dt} E_k^{lin}(v) \lesssim A^2 E_{k+1}(u)$$

where A is a good linear lower order control norm for u. Further, the norms $E_k^{nl}(u)$ and $E_k^{lin}(v)$ are likely not equivalent.

Question: How to combine the two in a favorable way ?

The modified energy method

Goal: Rather than using a normal form to eliminate quadratic terms in the equation, construct a modified energy for the original equation, compatible with both the quasilinear and the normal form structure,

$$\frac{d}{dt} E_k^{nl,3}(u) \lesssim A^2 E_k^{nl,3}(u)$$

Algorithm to construct the modified energy:

STEP 1: Start with a cubic normal form energy,

$$E_k^{NF,3}(u) = (\text{quadratic} + \text{cubic}) E_k^{lin}(v)$$

STEP 2: Split $E_k^{NF,3}(u)$ into a high frequency part and lower order terms,

$$E_k^{NF,3}(u) = E_{k,high}^{NF,3}(u) + E_{k,low}^{NF,3}(u)$$

STEP 3: Match to cubic order $E_{k,high}^{NF,3}(u) = E_k^{nl}(u) + \text{quartic}$, and define

$$E_k^{nl,3}(u) = E_k^{nl}(u) + E_{k,low}^{NF,3}(u)$$