Lecture 2
Well-posedness for quasilinear evolution equations

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Differences of solutions

Differences of solutions - we need to track these differences in order to get a uniqueness statement. To understand differences of solutions, we need to understand the linearized equation i.e energy estimates for it.

- One can use one of the following two equations (for the general problem):

$$(1) \quad v_t = DN(u)v$$
$$(2) \quad v_t = T_{DN(u)}v + F^{lin}(u)v$$

- For the model problem:

$$(1) \quad v_t = \mathcal{A}^j(u)\partial_j v + D\mathcal{A}^j(u)v\partial_j u$$
$$(2) \quad v_t = T_{\mathcal{A}^j(u)}\partial_j v + \cdots + T_{D\mathcal{A}^j(u)v}\partial_j u$$

where the red term in the linearized paradifferential equation (2) is part of $F^{lin}(u)v$,

- Energy estimates can be obtained for any of these equations. We choose the linearization or the paradifferential equation associated depending on the regularity we want to work at.
- In Lecture 1 we derived the energy estimates for the linearization for the model problem in L^2.

Discussion on what we know and why we need more ?

Our energy estimates for the linearized equation for the model problem

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq_A B\|v\|_{L^2}^2$$

By Gronwall's inequality this gives control on the L^2 bound of v (we want to think of v as a difference of solutions).

Question: Why do we need an energy estimate for the paradifferential linearized equation?

Answer: Because the estimates for the linearized problem have limitations. The term $T_{D\mathcal{A}^j(u)v}\partial_j u$ has a derivative on the high frequency term ! So when you want to look at H^s solutions, then $\partial_j u$ is at most in H^{s-1} regularity, regardless of what we multiply by.

Remark

*Thus the estimate above applies to solutions of the linearized model problem in H^σ but **only** for $0 \leq \sigma \leq s-1$.*

Energy estimates for differences of solutions for the model problem

Assume we have two solutions for the model problem u_1 and u_2 and denote by $v := u_1 - u_2$. Subtract the two equations and get an equation very similar to the linearized equation:

$$v_t = A^j(u_1)\partial_j v + (A^j(u_1) - A^j(u_2))\partial_j u_2$$

- $A^j(u_1) - A^j(u_2)$ - we can pull out a v factor
- Same energy estimates as the one for the linearized equations will be obtained. The difference is that the control norms are now

$$A := A_1 + A_2 \quad B := B_1 + B_2.$$

- Some people tend to use some unbalanced control norms- prefer one norm in favour of other; e.g. use A_1 rather than A_2 but this is not interesting here.

An immediate consequence of this difference of solutions equation and its energy estimate is

Corollary

Uniqueness follows right away once these two solutions have the same initial data (by Gronwall).

Energy estimates for difference of solutions for the general problem

Remark

If one looks at the general problem it might not be very easy to do energy estimates for difference of solutions:

- *for example when the problem has some geometrical structure.*
- *concretely, in the water waves case you see this geometrical structure playing a role when you fail to use the diagonal variable.*

Strategy: think of the linearized equation energy estimates as implying the difference bounds.

Sketch: Rather than measure directly the difference of two solutions (say u_1 and u_2), consider a continuous family of solutions between u_1 and u_2, and their corresponding initial data u_{01} and u_{02}. So, we join these two solutions via a family of initial data

$$u_{01} \leftarrow u_{0h} \rightarrow u_{02}, \quad h \in [1,2].$$

Initial data u_{0h} generates solutions u_h. Hence, to compare u_1 to u_2 at later times we go through u_h : $u_2(t) - u_1(t) = \int_0^1 \frac{d}{dh} u_h(t)\, dh$.

Comments on:

$$u_2(t) - u_1(t) = \int_0^1 \frac{d}{dh} u_h(t)\, dh.$$

- $\frac{d}{dh} u_h(t)$ solves the linearized equation!
- If we have energy estimates for the linearized equation, then this leads to energy estimates for difference of solutions !

Remark

Energy estimates for the linearized equation $\Rightarrow$ Bounds for difference of solutions.

For now we can conclude we have **two kinds of energy estimates:**

- EE for the full equation, in all Sobolev spaces H^σ for $\sigma \geq 0$.
- EE for the linearized equation, best thought in L^2 (sometimes you need to vary things a bit: in water waves $\dot{H}^{\frac{1}{4}}$).

From the Hadamard LWP definition we have two components

1. Energy estimates.
3. Uniqueness.

Existence of Solutions: I

We have no solution to start with, and the goal is to construct some approximate solutions, which eventually will converge to a solution.

Issue: Approximate solutions might not be in the actual space of solutions. Some geometry of the problem could get lost.

Fixes: Still, there are mathematical tools that can help in showing existence of solutions. I will present two methods:

- **Method 1** - iterative construction
- **Method 2** - time discretization idea

Method 1

a) start with $u^{(0)} = u_0$ which is just a constant function

b) construct $u^{(n)}$ approximate solutions so that $u^{(n)} \to u$ ('converges'). Construction scheme:

- Bad idea (loses derivatives): $\partial_t u^{(n+1)} = N(u^{(n)})$
- Better idea, but not much better (look at the linearized equation which we know has EE, and models difference of sols)[Nash-Moser]

$$\partial_t(u^{(n+1)} - u^{(n)}) = DN(u^{(n)})(u^{(n+1)} - u^{(n)}) - (\partial_t u^{(n)} - N(u^{(n)}))$$

Method 1

Question: What is a good scheme that does not lose derivatives?

The idea comes from the paradifferential equation. For the model problem this is

$$u_t = T_{DN(u)}u + F(u)$$

• $F(u)$ plays a perturbative role and hence the good scheme is to set the equation for $u^{(n+1)}$ to be

$$u_t^{(n+1)} = T_{DN(u^{(n)})}u^{(n+1)} + F(u^{(n)})$$

Note:

- $F(u^{(n)}) \to F(u^{(n+1)})$ - easier to have $u^{(n+1)}$ solve a linear equation.
- $DN(u^{(n)})$ - again, linear equation for $u^{(n+1)}$. So at each step there is a linear equation to solve.

This construction works if we manage to get:

- uniform bounds for $u^{(n)}$ in H^s.
- convergence of this sequence of approximate solution to a solution in some weaker topology: say L^2.

Method 1

Remark

You cannot get uniform convergence in H^s, as this would imply the uniformly continuous dependence on the initial data and this is a quasilinear problem...

So now let's come back to what we wanted our construction to satisfy

- the uniform bounds: (i) they are coming from the paradifferential equation which you know is well-posed in all Sobolev spaces ✓, and (ii) we have good bounds for $F(u^{(n)})$, meaning $F : H^s \to H^s$ (Moser estimates), and this is consistent with thinking of F as a perturbative term that satisfies nice bounds in all Sobolev spaces and nice Lipschitz bounds.✓
- we want convergence via L^2 bounds. So we want to compare two consecutive iterations that start with the same initial data:

$$(u^{(n+1)} - u^{(n)})_t = T_{DN(u^{(n)})}(u^{(n+1)} - u^{(n)}) \to \text{ paradiff eq WP in } L^2$$
$$+ F(u^{(n)}) - F(u^{n-1}) \to F \text{ is Lipschitz}$$
$$+ T_{DN(u^{(n)}-DN(u^{(n-1)})}u^{(n-1)} \to \text{delicate, but easy in } L^2$$

Method 1 -Comments

- The red term can be estimated in terms of $\|u^{(n)} - u^{(n-1)}\|_{L^2}$
- Then we get the bounds for the equation above (Exercise)

$$\frac{d}{dt}\|u^{(n+1)} - u^{(n)}\|_{L^2} \leq C\|u^{(n+1)} - u^{(n)}\|_{L^2} + \|u^{(n)} - u^{(n-1)}\|_{L^2}$$

  Last term comes from purple and red terms.
- Apply Gronwall's ineq on time interval $[0, T]$:

$$\begin{aligned}\|u^{(n+1)} - u^{(n)}\|_{L^\infty[0,T:L^2]} &\leq C\int_0^T \|u^{(n)} - u^{(n-1)}\|_{L^2} \\ &\leq CT\|u^{(n)} - u^{(n-1)}\|_{L^\infty[0,T:L^2]}\end{aligned}$$

  Observe we use same topology $L^\infty L^2$ for differences of solutions, and C is a fixed but potentially large constant depending on H^s norm of the solution. Choosing $T \ll 1$ we gain smallness

$$\leq \gamma\|u^{(n)} - u^{(n-1)}\|_{L^\infty[0,T:L^2]}, \quad \gamma < 1, T \ll 1$$

- $(u^{(n+1)} - u^{(n)})_n$ decays exponentially (via telescopic sum)

Method 1 - conclusions

We get that

- $(u^{(n)})_n$ is bounded in $L^\infty H^s$
- $u^{(n)} \to u$ in $L^\infty L^2$

These two statements imply

$$u^{(n)} \to u \text{ in } L^\infty H^\sigma, \quad 0 \leq \sigma < s$$
$$u \in L^\infty H^s$$

Note: Missing the conclusion $u \in CH^s$ - not an issue you want to address right away. Later we will see that solutions can be uniformly approximated by smooth solutions and smooth solutions will be continuous with values in H^s.

Method 2

This method was used in some other contexts not water waves related problems.

It is based on a time discretization idea

$$\begin{array}{ccccc} u_0 & \to & u(\epsilon) & \to & u(2\epsilon)\dots \\ t=0 & \to & t=\epsilon & \to & t=2\epsilon\dots \end{array}$$

What is this scheme? So instead of trying to move this continuously in time you chose a small time step $t = \epsilon$ and at each time ϵ construct $u(\epsilon)$; this leads to a discretized solution.

What is the goal? The goal is to find a function that takes these discrete values at these discrete points, turn it into a continuous function, and if this is good approximation, in the limit, we are going to get a solution (need some compactness: Arzela-Ascoli).

Method 2: Euler type method

Steps to make this scheme work:

- This plays the role of the energy bound in Method 1

$$E^s(u(k+1)\epsilon) \leq (1 + C\epsilon)E^s(u(k)\epsilon)$$

- From one step to another one almost solves the equation

$$u((k+1)\epsilon) = u(k\epsilon) + \epsilon N(u(k\epsilon)) + O(\epsilon^2)$$

It is like discretizing the derivative on the time scale ϵ

Why is this scheme advantageous? Constructing one step in this scheme is all you need!

Let's construct this step : start with u_0 and construct u_1. You need to do it such that

- $E^s(u_1) \leq (1 + C\epsilon)E^s(u_0)$
- $u_1 = u_0 + \epsilon N(u_0) + O(\epsilon^2)$ in a weaker topology (e.g. L^2)

Method 2: Regularization + Euler

- Bad idea: neglect $O(\epsilon^2)$ in the u_1 definition. This loses derivatives.
- Good idea: regularize the data to avoid derivative loss. Explicitly this is done in 2 steps.
  - (i) Regularize $u_0 \to \tilde{u}$
  - (ii) Set $u_1 = \tilde{u} + \epsilon N(\tilde{u})$

Return to the model problem, and see how we achieve

$$E^s(\tilde{u}) \leq (1 + C\epsilon) E^s(u_0)$$

(i) **Regularization:** We choose $\tilde{u} = P_{\leq \epsilon^{-\frac{1}{2}}} u_0$. This implies that energy decreases,

$$\|\tilde{u}\|_{H^s} \leq \|u_0\|_{H^s}$$

while higher regularity is gained:

$$\|\tilde{u}\|_{H^{s+1}} \lesssim \epsilon^{-\frac{1}{2}} \|u_0\|_{H^s}$$

If s is large enough then

$$\|\tilde{u} - u_0\|_{L^2} \leq \epsilon^{s/2} \leq \epsilon^2$$

You gain more than you need and no energy increase.

Method 2: Regularization + Euler, conclusion

(ii) **Iteration step:** We set

$$u_1 = \tilde{u} + \epsilon N(\tilde{u})$$

which immediately gives

$$u_1 = u_0 + \epsilon N(u_0) + O(\epsilon^2)$$

We also need to make sure there is no energy increase from $\tilde{u}$ to u_1. This is like proving energy estimates:

$$\|u_1\|_{H^s}^2 = \|\tilde{u}\|_{H^s}^2 + 2\epsilon\langle\tilde{u}, N(\tilde{u})\rangle_{H^s} + \epsilon^2\|N(\tilde{u})\|_{H^s}^2$$

- The first term on the RHS is the original energy
- The second term on the RHS gets bounded by integration by parts, exactly as in the energy estimates.
- The third term on the RHS is where you can estimate because you have the regularization at your disposal

This gives, as needed,

$$\|u_1\|_{H^s}^2 \leq \|\tilde{u}\|_{H^s}^2 + C\epsilon\|\tilde{u}\|_{H^s}^2 + C\epsilon^2\|\tilde{u}\|_{H^{s+1}}^2 \leq \ (1 + C\epsilon)\|u_0\|_{H^s}^2$$

Done for the day !