Lecture 1
Well-posedness for quasilinear evolution equations

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Introduction and the general problem

Question: Proving local well-posedness for a nonlinear PDE.

- Two track structure, where we will broadly discuss ideas for a general problem, and in parallel implement these ideas on a simple, classical concrete example.
- General problem: nonlinear partial differential equation i.e. a first order system in time.

$$\begin{cases} u_t = N(u), \\ u(0) = u_0 \in H^s, \end{cases}$$

  - u a scalar or a vector valued function belonging to a scale of either real or complex Sobolev spaces ($H^s := H^s(\mathbb{R}^n)$ - in this lecture)
  - u_0 some intial data: $u(0, t) = u_0(x)$
  - The nonlinearity N represents a nonlinear function of u and its derivatives,

$$N(u) = N(\{\partial^\alpha u\}_{|\alpha| \leq k}).$$

Model Problem

Our model problem will be a classical first order symmetric hyperbolic system, *quasilinear** PDE in $\mathbb{R} \times \mathbb{R}^n$

$$\begin{cases} \partial_t u = \mathcal{A}^j(u)\partial_j u \\ u(0) = u_0 \end{cases}$$

- u takes values in $\mathbb{R}^m$, i.e., $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$
- the $m \times m$ matrices $\mathcal{A}^j$ are symmetric, and smooth as functions of u
- the order of the nonlinearity N is $k = 1$
- and the scale of Sobolev spaces to be used is indeed the Sobolev scale.

Note: For now, we do not care if the problem we are looking at is dispersive or not.

The local well-posedness question

[Enhanced] Hadamard local well-posedness in Sobolev spaces

$$u(0) \in H^s$$

- existence of solutions u in the class $C(0,T;H^s)$
- uniqueness of solutions, either directly or as unique limits of smooth solutions
- continuous dependence in H^s, i.e. continuity of the data to solution map
$$H^s \ni u(0) \to u \in C(0,T;H^s)$$
- weak Lipschitz dependence, i.e. for two H^s solutions u and v we have the difference bound
$$\|u-v\|_{C(0,T;H^{s_0})} \lesssim \|u(0)-v(0)\|_{H^{s_0}}$$

Related question : long time behavior

- Extended lifespan of solutions for small data

First, comments on T:

- T is the lifespan of solutions!
- In the previous slides T is the best time of existence of solutions.
- In principle: $T = T(u_0)$.
- Continuous dependence on the initial data: perturbed initial data leads to a solution that will exists for almost as long as $T(u_0)$.
- $T(u_0)$ is a lower semicontinuous function of u_0:

$$u_0^n \to_{H^s} u_0 \quad \hookrightarrow \quad \liminf_{n\to\infty} T(u_0^n) \geq T(u_0).$$

Second, comments on the LWP:

- Step 1 look for WP in high Sobolev spaces
- Step 2 look for WP in low Sobolev spaces. How low?
- Step 3 scaling symmetry leads to hard WP threshold: H^{s_c}

Thirdly, comments on the further problems:

- good bounds from below for the lifespan of the solutions?
- blow up criteria?

Low regularity WP: the scaling threshold

What is the lowest value of s for which an equation is Hadamard well-posed for initial data $u(0)$ in the Sobolev space H^s?

Scaling symmetry:

$$u(x,t) \to u(\lambda x, \lambda t)$$

Scaling threshold: Critical Sobolev space

$$s_c = \frac{n}{2}$$

Open question: Are symmetric hyperbolic system of equations well-posed in $\mathcal{H}^s$ for all $s > s_c$?

Ultimate goal of these lectures:

Theorem

Symmetric hyperbolic system of equations are locally WP in H^s for

$$s > s_c + 1 = n/2 + 1.$$

Nonlinearity: disect and analyze

Nonlinearity in quasilinear problems: you cannot think at it as perturbative.

Example of a nonlinearity: a paraproduct approach

$$uv = T_u v + T_v u + \Pi(u, v)$$

$T_u v$ is low frequency is on u and the high frequency on v. Thus we have lh, hl and hh. In high regularity setting, the hh interaction term (i.e Π) one gains from the derivatives on u and v and since the output is at lower frequencies, one can use all of those derivatives. Thus, $\Pi(u, v)$ is better behaved.

Our task is to **rewrite** the equation:

$$u_t = N(u) \quad (*)$$

Linearisation of $N(u)$ is $DN(u)$, and acts like a linear operator

$$v \to DN(u)v.$$

Thus, the new equation we will look at is

$$u_t = T_{DN(u)} u + F(u), \quad \text{[think of } F(u) \text{ as the perturbative component].}$$

The principal part of eqn $(*)$ is essentially a *linear equation*

$$u_t = T_{DN(u)} u$$

Associated equations for the general problem-part I

Paradifferential equation : uncouple the role of the variable and of the coefficients

$$\begin{cases} w_t = T_{DN(u)} w \\ w(0) = w_0. \end{cases}$$

Frequency localized equation: associated equation that arises from localizing in frequency:

$$\begin{cases} w_{kt} = T_{DN(u_{<k})} w_k \\ w_k(0) = w_{k_0}. \end{cases}$$

$w_k = P_k w$ is the Littlewood-Paley projection at frequency 2^k.

Note:

- This is a linear eq for the high frequency w (ie. for w_k)
- $w_k \to u_k$ so at leading order u at high frequencies solves a linear eq, and in this linear eq the coefficients depend on lower frequencies of u. Thus, we have a linear eq with variable coefficients where high frequency solution is moving on a low frequency background.

Associated equations for the general problem-part II

Major role in the analysis of any PDE is played by the linearized equation!

Linearized equation:

$$\begin{cases} v_t = DN(u)v \\ v(0) = v_0. \end{cases}$$

- $DN(u)$ is a differential operator of order k
- v is the linearized variable

Linearized Paradifferential equation:

$$\begin{cases} v_t = T_{DN(u)}v + F^{lin}(u)v \\ v(0) = v_0. \end{cases}$$

- $F^{lin}(u)v$ linear operator in v.
- $F^{lin}(u)v$ is not a Π term ! Here we also have $T_v DN(u)$ which comes form the paradifferential decomposition, and should be included in the leading part based on the Bony's decomposition philosophy.
- Not a clear and concise recipe to write the linearized equation in a paradifferential way! There are multimple choices. EG: for water waves.

Associated equations for the model problem: I

Model problem

$$\begin{cases} \partial_t u = \mathcal{A}^j(u)\partial_j u \\ u(0) = u_0 \end{cases}$$

- $N(u) = \mathcal{A}(u)\partial u$
- $DN(u)v = \mathcal{A}(u)\partial v + D\mathcal{A}(u)v\partial u$

Paradifferential equation:

$$\begin{cases} \partial_t w = T_{\mathcal{A}(u)}\partial w + T_{D\mathcal{A}(u)\partial u} w \\ w(0) = w_0 \end{cases}$$

- First term in the above decomposition is of order 1, and the second term is of order 0, hence perturbative.

The linearized equation:

$$\begin{cases} \partial_t v = \mathcal{A}(u)\partial v + D\mathcal{A}(u)v\partial u \\ v(0) = v_0 \end{cases}$$

- First term in the above decomposition is of order 1, and the second term is of order 0, hence perturbative.

Associated equations for the model problem: II

The linearized paradifferential equation:

$$\begin{cases} \partial_t w = T_{\mathcal{A}(u)} \partial w + T_{D\mathcal{A}(u)\partial u} w + F^{lin}(u) w \\ w(0) = w_0 \end{cases}$$

- First term in the above decomposition is of order 1, and the second term is of order 0, hence perturbative.
- $F^{lin}(u)w$ is not always harmless; but it should be perturbative.

Local well-posedness strategy

Several steps:

1. Energy estimates - it is useful to start with this step, and get good energy estimates (even though at this point the existence of solutions is unknown).
2. Existence of solutions - you looks to see that indeed you have solutions that satisfy the good energy estimates derived.
3. Uniqueness - difference bounds for solutions (sometimes hard) $\Leftrightarrow$ can be related to *estimates for the linearized problem*. And since you can think of this step via energy estimates, you can move it before step **2**.
4. Continuous dependence - last step in the Hadamard style WP theory. This is where we actually split from the **semilinear case**. Our agenda
   - Frequency envelopes bounds - better H^s bounds for the solutions than the energy estimates. Says where the energy gets distributed.
   - cont. dependence will follow.

Question: Why we can only expect continuous dependence on the initial data? Look at the full frequency localized equation.
Note: In semilinear problems we expect to have Lipschitz depend. on the initial data. In quasilinear problems we can **only** hope to have cont. depend. on the initial data.

1. Energy estimates

Energy estimates means that we track the H^s norm of the u.

- s here is the Sobolev regularity we want to solve our problem (this discussion is for the model problem, but it can be generalized)
- Also want to control not only lower norms if u but also higher norms: H^σ norms of u where $\sigma > 0$ and $\sigma > s$. Tracking the H^σ norm of the solution means an estimates as follows, and then apply Gronwall's ineq

$$\frac{d}{dt}\|u\|^2_{H^\sigma} \leq C(\|u\|_{H^s})\|u\|^2_{H^\sigma}$$

Remark

In general $\|u\|^2_{H^\sigma}$ is not the right thing to track unless the problem is linear. For nonlinear problems one needs to find a better adapted energy, call it $E^\sigma(u)$ that should be equivalent with the linear energy $\|u\|^2_{H^\sigma}$, so that no info would get lost.

Remark

For local WP theory: controlling C by $\|u\|_{H^s}$ should be good enough. For longer times C should capture the effect of nonlinear interactions. Better if we can control C using pointwise measurements of the solutions.

Control parameters for water waves

For water waves we will have two control norms - pointwise norms:

- A - is the scale invariant size of the solutions
- B - will be the replacement for C above, and it would be a pointwise norm.

The energy estimates bound above would be replaced it by finding energy functionals E^σ with the following properties:

- Norm equivalence
$$E^\sigma(u) \approx_A \|u\|^2_{H^\sigma}$$
- Propagation of the norm
$$\frac{d}{dt}E^\sigma(u) \lesssim_A B\|u\|^2_{H^\sigma}$$

Note: The estimate is for more than quadratic nonlinearities in the equation, and this explains the use of this scale invariant norm A.

About the linearized equation

Note: The discussion above carries over to the **linearized equation**.

Note: Understanding the linearized equation is the core of any PDE.

One decides where to study this equation: for most cases one does it in L^2, but it can be at any regularity level, below the regularity we would get the LWP result:

$$\frac{d}{dt}E^{0,lin}(v) \lesssim_A B\|v\|_{L^2}^2, \qquad \text{here } \sigma = 0$$

Remark

One can think of the core equation as being the ***paradifferential linearized equation****, and obtain bounds for that equation. However, the two equations are somewhat interchangeable. For water waves we used both equations.*

Model problem

$$\begin{cases} \partial_t u = \mathcal{A}(u)\partial u \\ u(0) = u_0 \end{cases}$$

Control parameters:

$$A := \|u\|_{L^\infty}, \quad B := \|\nabla u\|_{L^\infty}$$

Linearized equation:

$$\begin{cases} \partial_t v = \mathcal{A}(u)\partial v + D\mathcal{A}(u)v\partial u \\ v(0) = v_0 \end{cases}$$

L^2 - energy estimates, and here is where $\mathcal{A}$ being symmetric matters:

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq C(A)B\|v\|_{L^2}^2$$

Full equation: $\partial^k u =: w$ will solve an equation of the form

$$\partial_t w = \mathcal{A}(u)\partial w + G(u)$$

- $G(u)$ has all the mixed derivatives, and want to see it as being a perturbative term
- G contains $k+1$ derivatives
- G is a multilinear expression but none if its factors carries $k+1$ derivatives; it is included in $\mathcal{A}(u)\partial w$.

L^2 - energy estimates:

$$\frac{d}{dt}\|w\|_{L^2}^2 \leq B\|w\|_{L^2}^2 + \|w\|_{L^2}\|G\|_{L^2}.$$

So we need the bound for G

$$\|G(u)\|_{L^2} \leq_A B\|u\|_{H^k}$$

Sample term $\partial^2 u\, \partial^{k-1} u$

$$\|\partial^2 u\, \partial^{k-1} u\|_{L^2} \leq \|\partial^2 u\|_{L^p}\|\partial^{k-1} u\|_{L^q}$$

Here interpolate between the energy norm $\|\partial^k u\|_{L^2}$ and $B = \|\partial u\|_{L^\infty}$

Done for the day !