Stable blowup for the supercritical hyperbolic Yang–Mills equations

Irfan Glogić
University of Vienna

Summer School: Geometric dispersive PDEs
Obergurgl, 27. September 2022

Adapted from:
Recent trends in nonlinear dispersive equations
Będlewo, 05. May 2022

FWF
Der Wissenschaftsfonds.

Overview

- Introduction of the hyperbolic Yang-Mills system
- Regularity - blowup dichotomy, previous results
- Equivariance reduction, the Bizoń-Biernat solution and conjecture
- Main result: Stability of the Bizoń-Biernat solution
- Strategy of the proof: Cauchy problem in similarity variables
- Non-self-adjoint spectral analysis - The mode stability problem
- Future outlook

Introduction

Consider the connection 1-forms $A = (A_0, A_1, \ldots, A_d)$ with

$$A_\alpha : \mathbb{R}^{1+d} \mapsto \mathfrak{so}(d, \mathbb{R}), \quad \alpha = 0, 1, ..., d,$$

where $\mathfrak{so}(d, \mathbb{R})$ is a set of real skew-symmetric $(d \times d)$-matrices.

Covariant derivative acting on $\mathfrak{so}(d, \mathbb{R})$-valued functions:

$$\mathbf{D}_\alpha := \partial_\alpha + [A_\alpha, \cdot].$$

Curvature tensor $F = F[A]$ given by:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta].$$

Yang-Mills action functional:

$$\mathcal{S}[A] := \int_{\mathbb{R}^{1+d}} \text{tr}(F_{\alpha\beta} F^{\alpha\beta}).$$

Associated Euler-Lagrange equations

$$\mathbf{D}^\alpha F_{\alpha\beta} = 0$$

which are called the hyperbolic Yang-Mills equations.

Regularity vs. Breakdown

The Yang-Mills system is nonlinear. Does it admit blowup? If yes, what are the possible blowup mechanisms? Which ones are generic?

Useful heuristic about the existence of blowup is given by the relation between the nonlinear scaling $A \mapsto A_\lambda$, and the conserved energy $\mathcal{E}[A]$:

$$A_\lambda(t,x) := \lambda^{-1} A(t/\lambda, x/\lambda), \quad \lambda > 0,$$

$$\mathcal{E}[A](t) := - \sum_{0 \leqslant \alpha < \beta \leqslant d} \int_{\mathbb{R}^d} \mathrm{tr}(F_{\alpha\beta}^2(t,x))\, dx.$$

Since $\mathcal{E}[A_\lambda](t) = \lambda^{d-4} \mathcal{E}[A](t/\lambda)$, the YM equations are energy-subcritical if $d \leqslant 3$, energy-critical if $d = 4$, and energy-supercritical if $d \geqslant 5$.

Blowup heuristics: Subcritical equations are globally regular, but for supercritical ones, although solutions with sufficiently small initial data exist for all times, large data solutions blowup in finite time.

(Some) previous results

$d = 3$: Global regularity holds. Choquet-Bruhat-Christodoulou '81, Eardley-Moncrief '82, Klainerman-Machedon '95.

$d \geqslant 4$: Blowup exists, and can be observed in a subclass of equivariant solutions

$$A_0 \equiv 0, \quad A_k^{ij}(t,x) = \sigma_k^{ij}(x) u(t,|x|) = (\delta_k^j x^i - \delta_k^i x^j) u(t,|x|),$$

for which the Yang-Mills system reduces to a radial $(d+2)$-dimensional semilinear wave equation in $u = u(t,r)$

$$u_{tt} - u_{rr} - \frac{d+1}{r} u_r = (d-2)(3u^2 - r^2 u^3). \qquad (*)$$

$d = 4$ (energy-critical): Blowup for $(*)$ observed numerically by Bizoń-Tabor '01. Rigorous constructions by Krieger-Schlag-Tataru '09 and Raphaël-Rodnianski '12.

(Some) previous results

$d \geqslant 5$ (energy-supercritical): A self-similar solution to $(*)$ exists in $d = 5, 7, 9$ Cazenave-Shatah-Tahvildar-Zadeh '98.

Bizoń '02 found this solution in closed form in $d = 5$

$$u_{0,T}(t,r) := \frac{1}{(T-t)^2} \frac{8}{3\rho^2 + 5}, \quad \rho = \frac{r}{T-t}, \quad T > 0,$$

and Bizoń-Tabor '01 conjectured that it drives the generic blowup.

Theorem (Donninger '14, Costin-Donninger-G.-Huang '16)

The solution $u_{0,T}$ is nonlinearly stable under small equivariant perturbations.

(Some) previous results

The analogue of solution $u_{0,T}$ exists in all $d \geqslant 5$, and has a closed form expression, Bizoń-Biernat '15:

$$u_T(t,r) := \frac{1}{(T-t)^2}\phi_d\left(\frac{r}{T-t}\right), \quad \phi_d(\rho) = \frac{\alpha(d)}{\rho^2+\beta(d)}, \; T>0.$$

with $\quad \alpha(d) = 2\left(1+\sqrt{\frac{d-4}{3(d-2)}}\right) \quad$ and $\quad \beta(d) = \frac{1}{3}\big(2d-8+\sqrt{3(d-2)(d-4)}\big).$

For the Yang-Mills system, this leads to the equivariant self-similar solution A_T

$$A_T(t,x) := \frac{1}{T-t}\Phi\left(\frac{x}{T-t}\right), \quad \Phi_k^{ij} = \sigma_k^{ij}\phi_d(|\cdot|).$$

Conjecture (Bizoń-Biernat '15)

Similarity profile A_T (i.e. Φ) is the universal attractor for generic large equivariant data evolutions.

Stability of blowup via Φ

Theorem (G. '21)

Let $d \geqslant 5$ be odd. There exists $\varepsilon > 0$ such that for any smooth equivariant initial data $(A(0,\cdot), \partial_t A(0,\cdot))$ for which

$$\|A(0,\cdot) - A_1(0,\cdot)\|_{H^{\frac{d+1}{2}}(\mathbb{B}^d_2)} + \|\partial_t A(0,\cdot) - \partial_t A_1(0,\cdot)\|_{H^{\frac{d-1}{2}}(\mathbb{B}^d_2)} < \varepsilon,$$

the following holds.

- Finite time blowup: There exists T close to 1 and a unique solution $A \in C^\infty(\Gamma_T)$ which blows up at $(T,0)$.
- Blowup profile Φ: The following decomposition holds

$$A(t,x) = \frac{1}{T-t}(\Phi + \varphi)\left(t, \frac{x}{T-t}\right),$$

where

$$\|\varphi(t,\cdot)\|_{H^{\frac{d+1}{2}}(\mathbb{B}^d)} \to 0 \quad \text{as} \quad t \to T^-.$$

In particular,

$$\|\varphi(t,\cdot)\|_{L^\infty(\mathbb{B}^d)} \to 0 \quad \text{as} \quad t \to T^-.$$

Strategy of the proof

Approach: Pass to similarity variables

$$\tau := \log T - \log(T-t), \quad \rho := \frac{r}{T-t}.$$

Backward lightcone of $(T,0)$ is mapped into the infinite unit cylinder

$$(t,r) \in \bigcup_{t\in[0,T)} \{t\} \times [0, T-t] \quad \to \quad (\tau,\rho) \in [0,\infty) \times [0,1]$$

By letting $\psi(\tau,\rho) := (T-t)^2 u(t,r)$, the radial equation $(*)$ becomes

$$\big(\partial_\tau^2 + 5\partial_\tau + 2\Lambda\partial_\tau - \Delta + \Lambda^2 + 5\Lambda + 6\big)\psi = (d-2)(3\psi^2 - \rho^2\psi^3)$$

where $\Lambda u(\rho) := \rho u'(\rho)$.

Self-similar solution u_T becomes static - ϕ_d, and the problem of stability of finite-time blowup via u_T becomes the one of asymptotic stability of the steady state profile ϕ_d.

Strategy: Use semigroup theory.

Abstract Cauchy problem

We define

$$\psi_1(\tau,\rho) := \psi(\tau,\rho), \quad \psi_2(\tau,\rho) := (\partial_\tau + \Lambda + 2)\psi(\tau,\rho).$$

This yields an evolution equation for $\Psi(\tau) := (\psi_1(\tau,\cdot), \psi_2(\tau,\cdot))$:

$$\partial_\tau \Psi(\tau) = \mathbf{L}_0 \Psi(\tau) + \mathbf{N}_0(\Psi(\tau)) \tag{1}$$

where $(\tau,\rho) \mapsto \Psi(\tau)(\rho) : [0,\infty) \times [0,1] \to \mathbb{R}^2$, and

$$\mathbf{L}_0 = \begin{bmatrix} -\Lambda - 2 & 1 \\ \Delta & -\Lambda - 3 \end{bmatrix}, \quad \mathbf{N}_0(\mathbf{u}) = \begin{bmatrix} 0 \\ (d-2)(3u_1^2 - \rho^2 u_1^3) \end{bmatrix}.$$

$\mathbf{L}_0$ the wave operator in similarity coordinates.

We study the system (1) in

$$\mathcal{H} := H^{\frac{d+1}{2}}_{\text{rad}} \times H^{\frac{d-1}{2}}_{\text{rad}} (\mathbb{B}^{d+2}).$$

Well-posedness in similarity variables

Theorem (Existence of the semigroup)

Let $d \geqslant 5$ be odd. Then the operator $\mathbf{L}_0$ (defined initially on smooth functions) is closable in $\mathcal{H}$ and its closure generates a strongly continuous semigroup $(\mathbf{S}_0(\tau))_{\tau \geqslant 0}$ of bounded operators on $\mathcal{H}$, for which

$$\|\mathbf{S}_0(\tau)\mathbf{u}\|_{\mathcal{H}} \leqslant M e^{-\frac{3}{2}\tau} \|\mathbf{u}\|_{\mathcal{H}}$$

for all $\mathbf{u} \in \mathcal{H}$.

Proposition (Local existence of strong solutions)

The nonlinear operator $\mathbf{N}_0$ is locally Lipschitz continuous on $\mathcal{H}$. Consequently, given $R > 0$ there exists $T > 0$ such that for any Ψ_0 with $\|\Psi_0\|_{\mathcal{H}} < R$ there exists a unique $\Psi \in C([0,T], \mathcal{H})$ which solves

$$\Psi(\tau) = \mathbf{S}_0(\tau)\Psi_0 + \int_0^\tau \mathbf{S}_0(\tau - s)\mathbf{N}_0(\Psi(s))ds.$$

Furthermore, the data-to-solution map $\Psi_0 \mapsto \Psi$ is Lipschitz continuous.

Stability analysis - Two main difficulties

We consider the perturbation ansatz $\Psi(\tau) = \Psi_{\text{st}} + \Phi(\tau)$. This leads to

$$\partial_\tau \Phi(\tau) = (\mathbf{L}_0 + \mathbf{V})\Phi(\tau) + \mathbf{N}(\Phi(\tau)). \qquad (\star)$$

Stability of Ψ_{st} = small data - global existence and decay for $(\star)$.

This causes (at least) two serious hurdles in the stability analysis:

1. The underlying spectral problem is non-self-adjoint.
2. Spectral stability does not necessarily imply linear stability.

Resolution of the issues:

R1. Reduce the spectral stability problem to a connection problem for a Heun ODE. Then use difference equation theory and approximation theory results to solve the connection problem.

R2. Establish a spectral mapping theorem for the semigroup on $\mathcal{H}$, and thereby infer the linear stability principle.

Spectral mapping theorem

Theorem (G. '21)

Let $(X, \|\cdot\|)$ be a Hilbert space, and let $\mathbf{L}_0$ be a closed linear operator on X that generates a semigroup $\mathbf{S}_0(\tau)$ on X, such that

$$\|\mathbf{S}_0(\tau)\mathbf{u}\| \leqslant M e^{\omega_0 \tau} \|\mathbf{u}\|.$$

Furthermore, let $\mathbf{L}' : X \to X$ be a compact operator. Then the operator

$$\mathbf{L} := \mathbf{L}_0 + \mathbf{L}'$$

generates a semigroup $\mathbf{S}(\tau)$ for which, given $\varepsilon > 0$, the following hold.

i) The set $S_\varepsilon := \sigma(\mathbf{L}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega_0 + \varepsilon\}$ consists of finitely many eigenvalues of $\mathbf{L}$, all of which have finite algebraic multiplicity.

ii) The spectral mapping

$$\sigma(\mathbf{S}(\tau)) \setminus \mathbb{D}_{e^{\tau(\omega_0+\varepsilon)}} = \{e^{\tau\lambda} : \lambda \in S_\varepsilon\}$$

holds for all $\tau > 0$, where $\mathbb{D}_r$ stands for the open disk of radius r.

iii) Let $\mathbf{P} := \sum_{\lambda \in S_\varepsilon} \mathbf{P}_\lambda$, and let $r_\varepsilon := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{L}) \setminus S_\varepsilon\}$. Then for every $\omega > \max\{\omega_0, r_\varepsilon\}$ there exists $C \geqslant 1$ such that

$$\|\mathbf{S}(\tau)(1-\mathbf{P})\mathbf{u}\| \leqslant C e^{\omega\tau} \|(1-\mathbf{P})\mathbf{u}\|.$$

Spectral problem - Mode stability

Operator $\mathbf{L} = \mathbf{L}_0 + \mathbf{V}$ has a time-translation symmetry eigenvalue $\lambda = 1$.

If

$$\sigma_p(\mathbf{L}) \cap \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\} = \{1\},$$

and $\lambda = 1$ is simple, then based on the spectral theorem above, the nonlinear stability of u_T follows from the standard dynamical systems theory results.

If this is the case, we say that u_T is mode stable.

The equation $\mathbf{Lu} = \lambda \mathbf{u}$ reduces to an ODE for the first component of $\mathbf{u}$

$$(1-\rho^2)u''(\rho) + \left(\frac{d+1}{\rho} - 2(\lambda+3)\rho\right)u'(\rho) - (\lambda+2)(\lambda+3)u_1(\rho) + \frac{144(\rho^2-5)}{3\rho^2+5}u(\rho) = 0$$

Point λ with $\operatorname{Re}\lambda \geqslant 0$, is an eigenvalue if and only if the ODE above has a solution that is analytic at both $\rho = 0$ and $\rho = 1$ - the connection problem.

Method of proving mode stability

Outline:

1. Formulate the so called supersymmetric (SUSY) eigenvalue problem (to "remove" $\lambda = 1$).

2. Transform the SUSY problem into an isospectral equation of Heun type (Fuchsian eqn. with four regular singularities).

3. Find recurrence relation (of order two) for the coefficients $a_n(\lambda)$ of the analytic solution at zero.

4. Construct an approximate solution of the recurrence relation, and then use a perturbative argument to show that

$$\lim_{n\to\infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} = 1 \quad \text{for all} \quad \operatorname{Re}\lambda \geqslant 0.$$

5. Conclude that the radius of analyticity of the analytic solution at $\rho = 0$ equal to 1, which implies the mode stability.

Thank you!