Lipschitz-free Banach spaces

Bachelor thesis topic

Supervisor: Christian Bargetz (christian.bargetz@uibk.ac.at)

Let (M, ρ) be a metric space with a distinguished point $0 \in M$. We consider the space

$$\mathrm{Lip}_0(M) := \{f\colon M \to \mathbb{R}\colon f \text{ Lipschitz}, f(0) = 0\}$$

equipped with the norm

$$\|f\|_{\mathrm{Lip}} := \mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)}\colon x, y \in M, x \neq y\right\}, \qquad f \in \mathrm{Lip}_0(M)$$

which makes it a Banach space. By $\delta(x)$ we denote the Dirac measure at the point $x \in M$, i.e, $\delta(x)(f) = f(x)$. The Lipschitz-free space $\mathcal{F}(M)$ is defined as the closure of the set $\mathrm{span}\{\delta(x)\colon x \in M\}$ in $\mathrm{Lip}_0(M)^*$ where $\mathrm{Lip}_0(M)^*$ stands for the dual space of $\mathrm{Lip}_0(M)$.

One reason why Lipschitz-free spaces are of interest is the following property: Given metric spaces M and N with distinguished points $0 \in M$ and $0 \in N$, and a Lipschitz mapping $f\colon M \to N$ there is a bounded linear operator $\tilde{f}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\tilde{f}} & \mathcal{F}(N) \\ {\scriptstyle\delta_M}\uparrow & & \uparrow{\scriptstyle\delta_N} \\ M & \xrightarrow{f} & N \end{array}$$

commutes and $\|\tilde{f}\| = \mathrm{Lip}(f)$.

The aim of this project is to understand and illustrate the basic properties of Lipschitz-free spaces over separable metric spaces and to consider some simple examples.

References

[1] G. Godefroy and N. J. Kalton. Lipschitz-free Banach spaces. *Studia Math.*, 159(1):121–141, 2003. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday.

[2] Gilles Godefroy. A survey on Lipschitz-free Banach spaces. *Comment. Math.*, 55(2):89–118, 2015.

[3] Nik Weaver. *Lipschitz algebras.* World Scientific Publishing Co., Inc., River Edge, NJ, 1999.