Drawing of graphs and the Hanani–Tutte theorem

Bachelor thesis topic

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Given a graph $G = (V, E)$ how can one determine whether it is *planar*, i.e., that it can be drawn in the plane without any crossings? In 1934 Hanani [1] showed that if one can draw G in the plane in such a way that every two non-adjacent edges cross an even number of times (including 0 times), then G must be planar. This statement, which has been later rediscovered several times in various forms, has become known as the (strong) Hanani–Tutte theorem.

At first sight it is not clear how can the Hanani–Tutte theorem help with deciding whether a given graph is planar or not. However, it turns out that the existence of the drawing in which every two non-adjacent edges cross an even number of times can be rephrased as an existence of a solution to a system of linear equations over $\mathbb{Z}_2$. Since it is easy to come up with such a system of linear equations and also it is easy to solve it, this provides an efficient algorithm for deciding planarity of graphs. This is certainly *not* the fastest algorithm to decide planarity of graphs known up to date, however, it is very simple. It is also quite robust, since it adapts to several modifications of the notion of planarity for which the more specialised approaches fail.

The Hanani–Tutte theorem is also of interest from the theoretical point of view. For instance, one can ask to what extend is the analogous statement true for other surfaces than the plane? It is known that the weak version of the theorem (not stated here) holds true for all surfaces [4]. The strong version is known to be true for the projective plane [5, 2] and there are counter-examples for orientable surfaces of genus four and higher [3]. The remaining cases are open.

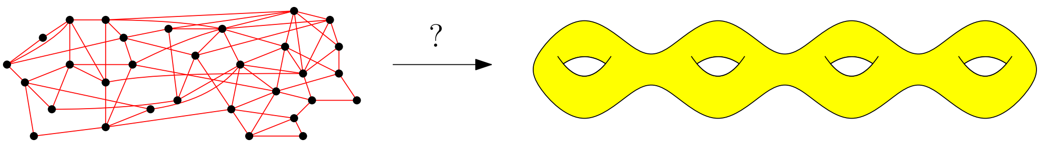

Figure 1: Can this graph be drawn on the orientable surfaces of genus 4 without crossings?

The presented area is rather large, so the specific topic for a bachelor project can be adjusted upon one's interest. The possible options include looking at some of the variants of planarity and presenting known proofs and counter-examples, as in [6]. Or one can look at the algorithmic side of the problems, describe the algorithms and try to implement them. Another possibility would be to take a look at other surfaces than the plane and survey what is known there, with particular emphasis on the recent counter-example to the strong Hanani–Tutte theorem [3]. Several parts of the area are accessible with little to no prior knowledge. On the other hand, the topic also offers several possibilities to learn more advanced results and techniques.

There are also many open problems which are easy to state and which can be attacked without any prior knowledge. Optionally, an interested student can try to work on one of these.

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