BACHELOR PROJECT:
THE LÉVY-KHINTCHINE FORMULA FOR LÉVY PROCESSES

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Field: Stochastic Analysis; Stochastic processes

Lévy processes are types of stochastic processes that play a major role in stochastic analysis. This is due to their importance as generalization for Brownian motion as well as their connection to the theory of infinitely divisible distributions. Their mathematically beautiful structure makes them versatilely applicable in a large class of mathematical challenges such as stochastic differential equations (SDEs), exit-time problems, ruin models, partial integro-differential equations (PIDEs), etc. Such questions arise for example in models from mathematical physics or finance and insurance mathematics.

A *Lévy process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family of $\mathbb{R}^d$-valued random variables $(X_t)_{t\geq 0}$ which satisfies the following properties:

(1) $X_0 = 0$ a.s.
(2) *X has independent increments:*
For all $n \geq 1$ and time points $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_0}, X_{t_n} - X_{t_{n-1}}$ are independent.
(3) *X is time homogeneous* or *X possesses stationary increments:*
For all $t, s \geq 0$: $X_{t+s} - X_s$ and X_t are identically distributed.
(4) *X is stochastically continuous:*
For all $t \geq 0$ and all $\varepsilon > 0$: $\lim_{s\to t} \mathbb{P}\left(|X_s - X_t| > \varepsilon\right) = 0$.
(5) *X is almost sure càdlàg:*
There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, the trajectory $t \mapsto X_t(\omega)$ is right continuous in $t \geq 0$ and has left limits in $t > 0$.

The strong one-to-one connection to infinitely divisible distributions leads to a en explicit formula for the characteristic function of X_t. This formula is called *Lévy-Khintchine-formula* and is the key to a classification of Lévy processes by three deterministic parameters (the *characteristic triplet*).

The main goals of this project are:

- Show the connection between Lévy processes and infinitely divisible distributions.
- Ensure the existence of a Lévy process (in law) associated to such a distribution.
- Give a rigorous proof of the Lévy-Khintchine formula for Lévy processes (in contrast to the more general concept of additive processes).
- Show how a Lévy processes can be represented by a characteristic triplet.

References

[1] Sato, K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 2005.