Iterative Reconstruction Techniques in CT

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1 Background

Inversion of the Radon transform forms the mathematical basis of computerized tomography (CT). The Radon transform is named after the Austrian mathematician Johann Radon [7], who studied this transform in 1917 long before the development of CT.

The basic principle of CT is as follows. Denote by $f\colon \mathbb{R}^2 \to \mathbb{R}$ the spatially varying x-ray absorption coefficient in some slice of a patient. Suppose, further, that an x-ray beam origins at some position x_0, propagates along the line L, and is recorded at another point x_1; see Figure 1.1. Further write $L = \{s\theta + t\theta^\perp \colon t \in \mathbb{R}\}$, where $\theta \in S^1$ is a normal vector, $s \in \mathbb{R}$ is the oriented distance of the line from the origin, and $\theta^\perp \in S^1$ denotes a unit vector orthogonal to θ. One then easily shows, that the intensity values measured at x_0 an x_1 provide the line integral

$$(\mathcal{R}f)(\theta, s) := \int_L f(x)\mathrm{d}s(x)\,. \tag{1.1}$$

By varying the positions of the x-ray sources and detectors, respectively, one collects integrals of f over various lines. The mathematical task of CT is to recover the function f from these line integrals. The function $\mathcal{R}f\colon S^1 \times \mathbb{R} \to \mathbb{R}\colon (\theta, s) \mapsto (\mathcal{R}f)(\theta, s)$ is the Radon transform of f; hence CT yields to the problem of reconstructing the unknown function f from certain values of its Radon transform.

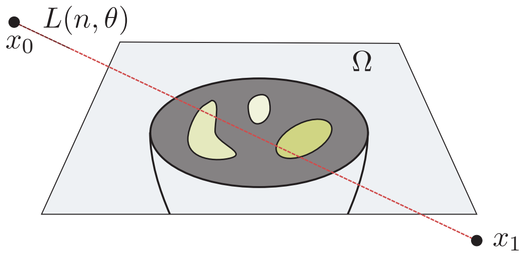

Figure 1.1: Illustration of CT. Suppose an x-ray is emitted at x_0 outside of Ω, propagates along the line $L(n, \theta)$ through Ω, and is finally recorded at another location x_1 outside of Ω. The intensity values at x_0 and x_1 then yield the line integral $(\mathcal{R}f)(\theta, s)$.

2 Iterative Reconstruction Techniques

The most common way to invert the Radon transform is by filtered backprojection, which may be seen as a numerical implementation of an explicit inversion formula [6]. In some applications, however it is preferable to use iterative techniques for the inversion of $\mathcal{R}$. For that purposes one approximates the continuous Radon transform $\mathcal{R}$ by some finite dimensional analogon that is inverted numerically. Such an approach yields to the discrete inverse problem

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{z}\,, \tag{2.1}$$

where $\mathbf{A} \in \mathbb{R}^{d \times n}$ represents a discretized version of the Radon transform, $\mathbf{x} \in \mathbb{R}^n$ is the vector to be estimated, $\mathbf{y} \in \mathbb{R}^d$ are the given data, and $\mathbf{z} \in \mathbb{R}^d$ is the noise (or error) in the data.

In CT, the matrix $\mathbf{A}$ is ill-conditioned, which prohibits direct solution of (2.2). As a remedy, regularization methods have to applied. Here we consider iterative regularization methods, which apply certain iterative schemes for solving the noise free equation $\mathbf{y} = \mathbf{A}\mathbf{x}$ in combination with a stopping to avoid over-fitting of the noisy data. Popular iterative reconstruction techniques in CT are the algebraic reconstruction technique (ART) [4], the simultaneous iterative reconstruction technique (SIRT) [3], the simultaneous algebraic reconstruction technique (SART) [1], and the diagonally relaxed orthogonal projection (DROP) method [2]. All these iterative schemes can be written in the form

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k+1)} + \lambda_k \mathbf{V}_k \mathbf{A}^\mathsf{T} \mathbf{W}_k \left(\mathbf{y} - \mathbf{A}\mathbf{x}^{(k)} \right) \qquad \text{for } k \in \mathbb{N}\,, \tag{2.2}$$

for some particular choices of the matrices $\mathbf{V}_k$ and $\mathbf{W}_k$, and certain numbers λ_k.

3 Aims of the Bachelor thesis

The aim of this Bachelor thesis is to efficiently implement the ART, SIRT, SART and that DROP iterative schemes in Matlab. Further, numerical studies should be performed and summarized, that compare the performance of ART, SIRT, SART and DROP when applied for the inversion of the Radon transform. For deriving the matrix $\mathbf{A}$ from the Radon transform $\mathcal{R}$, the function f should be assumed to be constant on square pixels and $(\mathcal{R}f)(\theta_k, s_\ell)$ be evaluated at uniform samples θ_k, s_ℓ as described in [5, Chapter 4].

References

[1] A. H. Andersen and A. C. Kak. Simultaneous algebraic reconstruction technique (sart): a superior implementation of the art algorithm. *Ultrason. Imag.*, 6:81–94, 1984.

[2] Y. Censor, T. Elfving, G. T. Herman, and T. Nikazad. On diagonally relaxed orthogonal projection methods. *SIAM J. Sci. Comput.*, 30(1):473–504, 2007/08.

[3] P. Gilbert. Iterative methods for the three-dimensional reconstruction of an object from projections. *J. Theor. Biol.*, 36:105–17, 1972.

[4] R. Gordon, R. Bender, and G. T. Herman. Algebraic reconstruction techniques (art) for three-dimensional electron microscopy and x-ray photography. *J. Theor. Biol.*, 29(3):471–481, 1970.

[5] G. T. Herman. *Fundamentals of computerized tomography*. Advances in Pattern Recognition. Springer, Dordrecht, 2009.

[6] F. Natterer. *The Mathematics of Computerized Tomography*, volume 32 of *Classics in Applied Mathematics*. SIAM, Philadelphia, 2001.

[7] J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verh. Kön. Sächs. Ges. Wiss. Leipzig Math. Phys. Kl.*, 69:262–277, 1917.