VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Lecture 11: May, 4

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Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

11.1 Existence of Gabor frames

Definition 11.1 (Wiener space) *A function g is in the Wiener space $W(\mathbb{R}^d)$ if*

$$\|g\|_W = \sum_{n\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} |g(x+n)| < \infty.$$

The subspace of continuous functions in $W(\mathbb{R}^d)$ is denoted by $W_0(\mathbb{R}^d)$.
We have the identity $\|g\|_W = \sum_k \|g \cdot T_k \chi_{[0,1]^d}\|_\infty$.

Lemma 11.2 (Boundedness) *If $g \in W(\mathbb{R}^d)$ or $\hat{g} \in W(\mathbb{R}^d)$, then $\forall \alpha, \beta > 0:$ $D_g^{\alpha,\beta}, C_g^{\alpha,\beta}, S_{gg}^{\alpha,\beta}$ are bounded with*

$$\|C_g^{\alpha,\beta}\|_{op} \leq \left(\frac{1}{\alpha}+1\right)^{\frac{d}{2}} \left(\frac{1}{\beta}+1\right)^{\frac{d}{2}} \min\left(\|g\|_W, \|\hat{g}\|_W\right), \tag{11.1}$$

$$\|S_{gg}^{\alpha,\beta}\|_{op} = \|C_g^{*\alpha,\beta} C_g^{\alpha,\beta}\|_{op} \leq \|C_g^{\alpha,\beta}\|_{op}^2 \leq \left(\frac{1}{\alpha}+1\right)^{d} \left(\frac{1}{\beta}+1\right)^{d} \min\left(\|g\|_W, \|\hat{g}\|_W\right). \tag{11.2}$$

Theorem 11.3 *If $g \in W\left(\mathbb{R}^d\right)$ and $\alpha > 0$ s.t. $\exists a, b > 0$ with*

$$a \leq \sum_{k\in\mathbb{Z}^d} |g(x-\alpha k)|^2 \leq b \qquad \text{almost everywhere,} \tag{11.3}$$

then there exists $\beta_0(\alpha)$ s.t. $\forall \beta < \beta_0(\alpha) : G(g,\alpha,\beta)$ is a frame.

11.2 Structure of Gabor Frames

Theorem 11.4 (Ron-Shen duality principle) *Let $g \in L^2$, $\alpha, \beta > 0$, then $G(g,\alpha,\beta)$ is a frame $\Leftrightarrow$ $G(g, \frac{1}{\alpha}, \frac{1}{\beta})$ is a Riesz basis for its closed linear span.*

11.2.1 Density of Gabor Frames

Theorem 11.5 *If $G(g,\alpha,\beta)$ is a Gabor frame, then $\alpha\beta \leq 1$ and $G(g,\alpha,\beta)$ is a Riesz basis if and only if $\alpha\beta = 1$.*

Remark:

- If $\alpha\beta < 1$, then we have oversampling (hope for a frame)
- If $\alpha\beta = 1$, then we have critical sampling (hope for a Riesz-basis)
- If $\alpha\beta > 1$, then we have undersampling (hope for a Riesz-basis for its closed linear span)
- $G(\varphi, \alpha, \beta)$ with $\varphi = e^{-\pi x^2}$ is a frame if and only if $\alpha\beta < 1$

Theorem 11.6 (Balian-Low theorem) *If $G(g, \alpha, \frac{1}{\alpha})$ is a Riesz-basis, then $g \notin W(\mathbb{R}^d)$ and $\hat{g} \notin W(\mathbb{R}^d)$. In the special case $d = 1$, if $G(g, 1, 1)$ is an ONB, then $xg(x) \notin L^2(\mathbb{R})$ or $g'(x) \in L^2(\mathbb{R}^2)$*

11.2.2 Dual windows

Theorem 11.7 (Wexler-Raz biorthogonality relations) *If for $\alpha, \beta > 0$ the operators D_g, D_γ are bounded then for $S_{g,\gamma} = D_g C_\gamma$ we have $S_{g,\gamma} = S_{\gamma,g} = I$ if and only if $\langle \gamma, T_{\frac{n}{\beta}} M_{\frac{l}{\alpha}} g \rangle = \delta_{n0}\delta_{l0} \cdot (\alpha\beta)^d$.*

Let $\gamma^o = S^{-1}g$ be the canonical dual window and γ another dual window for $G(g, \alpha, \beta)$. From the theorem above we get that $\langle \gamma - \gamma^o, T_{\frac{n}{\beta}} M_{\frac{l}{\alpha}} g \rangle = 0$ for all n, l or in other words $\gamma - \gamma^o$ is orthogonal to the closed linear span of $\left\{ T_{\frac{n}{\beta}} M_{\frac{l}{\alpha}} g \right\}$.

Lemma 11.8 *Let $G(g, \alpha, \beta)$ be a Gabor frame with canonical dual winow γ^o and K the closed linear span of $G(g, \frac{1}{\beta}, \frac{1}{\alpha})$, then*

1. *$\gamma^o \in K$,*
2. *and γ is a dual window $\Leftrightarrow \gamma \in \gamma^o + K^\perp$.*

Lemma 11.9 *Let γ be a dual window for $G(g, \alpha, \beta)$, then the following statements are equivalent.*

1. *$\gamma = \gamma^o$.*
2. *$\|\gamma\|_2 < \|\tilde{\gamma}\|_2$ for all dual windows $\tilde{\gamma} \neq \gamma$.*
3. *$\| \frac{g}{\|g\|_2} - \frac{\gamma}{\|\gamma\|_2} \|_2 \leq \| \frac{g}{\|g\|_2} - \frac{\tilde{\gamma}}{\|\tilde{\gamma}\|} \|_2$ for all dual windows $\tilde{\gamma} \neq \gamma$.*