VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

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Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

10.1 Gabor Frames

10.1.1 Frame theory

Last time we have seen, that if we have got a frame $\mathcal{F} = \{\varphi_n, b_n : n \in J\}$ where φ_n, b_n are two ONB's, then we have got the following duals:

1. $\{\varphi_n : n \in J\}$
2. $\{b_n : n \in J\}$
3. $\{\frac{1}{2}\varphi_n, \frac{1}{2}b_n : n \in J\}$

Moreover we get $A \cdot I_{\mathcal{H}} \preccurlyeq S \preccurlyeq B \cdot I_{\mathcal{H}} = A \cdot I_{\mathcal{H}}$. This implies $S = A \cdot I_{\mathcal{H}}$ and $S^{-1} = A^{-1} \cdot I_{\mathcal{H}}$, so 3 is the canonical dual frame.

We look at the underdetermined system of equations

$$\Phi \cdot c = y, \qquad c \in \mathbb{R}^K, y \in \mathbb{R}^d$$

in $\mathbb{R}^d$, where Φ is a short-fat $d \times K$ matrix, $d \leq K$. It has a solution if Φ has full rank, that means for a frame, that we have a lower bound. And if $K > d$ it has infinitely many solutions. One solution is

$$c = \underbrace{\Phi^*}_{K \times d} \cdot \underbrace{(\Phi \cdot \Phi^*)^{-1}}_{d \times d} \cdot y$$

and we define the pseudo-inverse $\Phi^\dagger := \Phi^* \cdot (\Phi \cdot \Phi^*)^{-1}$. From numerics we recall that this solution is the one with minimal norm.

Proposition 10.1 (Canonical dual - minimum energy coefficient)
If $\{\varphi_j : j \in J\}$ is a frame and $f = \sum_{j \in J} c_j \cdot \varphi_j$, then

$$\sum_{j \in J} |c_j|^2 \geq \sum_{j \in J} |\langle f, S^{-1}\varphi_j \rangle|^2$$

and we have got equality only if $c_j = \langle f, S^{-1}\varphi_j \rangle$.

Proof: We define $a_j := \langle f, S^{-1}\varphi_j\rangle$, then we get $f = \sum_{j\in J} a_j\varphi_j$ and

$$\langle f, S^{-1}f\rangle = \langle \sum_{j\in J} a_j\varphi_j, S^{-1}f\rangle = \sum_{j\in J} a_j \underbrace{\langle \varphi_j, S^{-1}f\rangle}_{\bar{a}_j} = \sum_{j\in J} |a_j|^2 = \|a\|_2^2$$

$$\langle f, S^{-1}f\rangle = \langle \sum_{j\in J} c_j\varphi_j, S^{-1}f\rangle = \sum_{j\in J} c_j\bar{a}_j = \langle c, a\rangle.$$

So we get $\|a\|_2^2 = \langle c, a\rangle$ and moreover

$$\|c\|_2^2 = \|c-a+a\|_2^2 = \|a-c\|_2^2 + \|a\|_2^2 + \langle c-a, a\rangle + \langle a, c-a\rangle = \|a-c\|_2^2 + \|a\|_2^2 \geq \|a\|_2^2$$

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Definition and Theorem 10.2 (Riesz basis)
A frame $\mathcal{F} = \{\varphi_j : j \in J\}$ is called a Riesz basis if one of the following equivalent conditions holds

1. *The coefficients $c \in l^2(J)$ s.t. $f = \sum_{j\in J} c_j\varphi_j$ are unique.*
2. *The analysis operator $C : \mathcal{H} \to l^2(J)$ maps onto $l^2(J)$ (surjective).*
3. *There exist constants $A', B' > 0$ s.t.*
$$A'\|c\|_2^2 \leq \|\sum_{j\in J} c_j\varphi_j\| \leq B'\|c\|_2^2.$$
4. *$\{\varphi_j : j \in J\}$ is the image of an orthonormal basis $\{e_j : j \in J\}$ under an invertible bounded operator T.*
5. *The Gram matrix G*
$$G_{j,n} = \langle \varphi_j, \varphi_n\rangle, \qquad \forall j, n \in J$$
is a positive invertible operator on $l^2(J)$.

Proof:[not discussed in the lecture, just for the sake of completeness]
In essence $1) - 5) \Rightarrow C, D$ are bijective
frame $\Rightarrow C$ one-to-one (injective) with closed range and D is onto (surjective, range is closed and dense)

(recall a bounded operator is injective $\Leftrightarrow$ adjoint operator has dense range)
1) $\Leftrightarrow$ 2) c_j unique $\Leftrightarrow D$ injective $\Leftrightarrow D^* = C =$ onto
1) $\Rightarrow$ 3) D continuous $\Rightarrow B'$ exists and D bijective + cont. $\Rightarrow D^{-1}$ is cont. (open mapp th.) $\Rightarrow A'$ exists
3) $\Rightarrow$ 4) $\{e_j : j \in J\}$ an ONB, for

$$f = \sum_{j\in J} c_j e_j, \qquad Tf := \sum_{j\in J} c_j\varphi_j$$

$$\|Tf\| = \|\sum_{j\in J} c_j e_j\| \geq A'\|c\|_2 = A\|f\|_{\mathcal{H}}, \qquad \forall f \in \mathcal{H}$$

$$\|Tf\| = \|\sum_{j\in J} c_j e_j\| \leq B'\|c\|_2 = B\|f\|_{\mathcal{H}}, \qquad \forall f \in \mathcal{H}$$

$\Rightarrow T$ well defined, invertible operator on $\mathcal{H}$
4) $\Rightarrow$ 1) $Te_j = \varphi_j, T$ invertible, e_j ONB

$$\sum_{j\in J} c_j\varphi_j = T(\sum_{j\in J} c_j e_j) = 0 \Leftrightarrow \sum_{j\in J} c_j e_j = 0 \Leftrightarrow c_j = 0, \forall j \in J$$

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Lemma 10.3

1. *$\{\varphi_j : j \in J\}$ is a tight frame with $A = B = 1$ and $||\varphi_j||_2 = 1, \forall j \in J$*
   $\Leftrightarrow \{\varphi_j : j \in J\}$ is an ONB.
2. *If $\{\varphi_j : j \in J\}$ is a frame $\Rightarrow \{S^{-\frac{1}{2}}\varphi_j : j \in J\}$ is a tight frame with $A = B = 1$.*
3. *If $\{\varphi_j : j \in J\}$ is a frame, then the inverse frame operator S^{-1} is given by*
$$S^{-1}f = \sum_{j \in J} \langle f, S^{-1}\varphi_j \rangle S^{-1}\varphi_j.$$
   S^{-1} is the frame operator of the canonical dual frame.

Proof:[not discussed in the lecture, just for the sake of completeness]
1)
$$1 \overset{A=B=1}{=} \|\varphi_k\|_2^2 = \sum_{j \in J} |\langle \varphi_k, \varphi_j \rangle|^2 = 1 + \sum_{j \neq k} |\langle \varphi_k, \varphi_j \rangle|^2$$

2) $S^{-\frac{1}{2}}$ defined via spectral theorem
$$f = S^{-\frac{1}{2}} S S^{-\frac{1}{2}} f = S^{-\frac{1}{2}} \sum_{j \in J} \langle S^{-\frac{1}{2}} f, \varphi_j \rangle \varphi_j =$$
$$= \sum_{j \in J} \langle f, S^{-\frac{1}{2}} \varphi_j \rangle S^{-\frac{1}{2}} \varphi_j$$
$$\langle f, f \rangle = \sum_{j \in J} |\langle f, S^{-\frac{1}{2}} \varphi_j \rangle|^2$$

Note $S^{-\frac{1}{2}}\varphi_j$ need not be normalized.
3)
$$S^{-1}f = S^{-1}SS^{-1}f = \sum_{j \in J} \langle f, S^{-1}\varphi_j \rangle S^{-1}\varphi_j$$

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10.1.2 Gabor Frames

Definition 10.4 (Gabor system)
Given a normalized window function $g \in \mathcal{L}^2(\mathbb{R})$ and lattice parameters $\alpha, \beta > 0$ the set of TF-shifts
$$G(g, \alpha, \beta) = \{T_{\alpha k} M_{\beta n} g : k, n \in \mathbb{Z}^d\}$$
is called a Gabor system (or Weyl Heisenberg system).
If $G(g, \alpha, \beta)$ is a frame, then it is called a Gabor frame (or Weyl Heisenberg frame).

The associated frame operator has the form
$$S_{gg}^{\alpha,\beta} = Sf = \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} g \rangle T_{\alpha k} M_{\beta n} g =$$
$$= \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g \rangle M_{\beta n} T_{\alpha k} g =$$
$$= \sum_{n \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} V_g f(\alpha k, \beta n) M_{\beta n} T_{\alpha k} g.$$

Proposition 10.5 (Dual of a Gabor frame)
If $G(g,\alpha,\beta)$ is a frame, there exists a so called dual window γ such that $G(\gamma,\alpha,\beta)$ is a dual frame.

$$\begin{aligned}\forall f \in \mathcal{L}^2(\mathbb{R}^d) : f &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}\gamma = \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{\alpha k}M_{\beta n}\gamma\rangle T_{\alpha k}M_{\beta n}g.\end{aligned}$$

(all sums converge unconditionally)

Proof: We have to show that the frame operator commutes with TF-shifts $T_{\alpha r}M_{\beta s}$, e.g. $(T_{\alpha r}M_{\beta s})^{-1}S\ T_{\alpha r}M_{\beta s}f = Sf$.

$$\begin{aligned}(T_{\alpha r}M_{\beta s})^{-1}S\ T_{\alpha r}M_{\beta s}f &= (T_{\alpha r}M_{\beta s})^{-1}\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle T_{\alpha r}M_{\beta s}f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}g = \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, M_{-\beta s}T_{-\alpha r}T_{\alpha k}M_{\beta n}g\rangle M_{-\beta s}T_{-\alpha r}T_{\alpha k}M_{\beta n} = \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{\alpha(k-r)}M_{\beta(n-s)}g\rangle T_{\alpha(k-r)}M_{\beta(n-s)}g = \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}g = \\ &= Sf.\end{aligned}$$

Now that we know that S and TF-shifts commutes, with $\pi := T_{\alpha k}M_{\beta n}$ we get

$$\pi^{-1}S^{-1}\pi f = \pi^{-1}S^{-1}\pi S S^{-1}f = \pi^{-1}S^{-1}S\pi S^{-1}f = \pi^{-1}\pi S^{-1}f = S^{-1}f.$$

So also S^{-1} commutes with TF-shifts. Finally we get

$$\{S^{-1}T_{\alpha k}M_{\beta n}g\} = \{T_{\alpha k}M_{\beta n}S^{-1}g\},$$

and we set $\gamma = S^{-1}g$, this is called the canonical dual window. Further from general frame theory it follows that $(S^{\alpha,\beta}_{g,g})^{-1} = S^{\alpha,\beta}_{\gamma,\gamma}$. ■