VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Lecture 9: April, 21

Lecturer: Karin Schnass *Scribe: Thomas Zwanowetz*

Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

9.1 Gabor frames

Until now we have only discussed continuous time frequency analysis. For practical purposes it seems useful to extend this approach. We recall the inversion formula for the short-time fourier transform

Proposition 9.1 (Inversion formula for STTF) *Suppose that $g \in L^2(\mathbb{R}^d)$ and $\langle g, \gamma \rangle \neq 0$. Then for all $f \in L^2(\mathbb{R}^d)$:*

$$f = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(x, \omega) M_\omega T_x \gamma \; d\omega \; dx$$

If the synthesis window γ has its essential time frequency support on a set $E \subset \mathbb{R}^{2d}$ in the time-frequency plane, then for two neighbouring points (x_1, ω_1) and $(x_2, \omega_2) \in \mathbb{R}^{2d}$, the supports of $M_\omega T_x h$ are in $(x_i, \omega_i) + E$. These sets largely overlap, and the coefficients $V_g f(x_i, \omega_i), i = 1, 2$, in the inversion formula carry roughly the same information about the time-frequency content of f at (x_1, ω_1). The representation of f by the inversion formula is thus highly redundant.
Consequently our goal is a discrete representation of f by countably many time-frequency shifts of γ with only minimal overlap of their supports.
The first attempt would try to replace the integral in 9.1 by a Riemann sum over a sufficiently dense lattice, writing f as

$$f = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} \gamma \rangle T_{\alpha k} M_{\beta n} g$$

for some suitable windows $g, \gamma \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$.
A second, more modest attempt leads to series expansions of the form

$$f = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} c_{kn} T_{\alpha k} M_{\beta n} g$$

with the coefficients $c_{kn} = c_{kn}(f)$ to be determined.
A second idea for the discretization comes from the interpretation of $|V_g f(x, \omega)|^2$ as the energy of f in a time-frequency cell centered at (x, ω). In order to capture the entire energy of f, we sample $V_g f$ densely enough so that the energy is preserved under the discretization. More formally we need the inequalities

$$A \|f\|_2^2\| \leq \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} |V_g f(\alpha k, \beta n)|^2 \leq B \|f\|_2^2$$

to be satisfied for all $f \in L^2(\mathbb{R}^d)$ and some constants $A, B > 0$.

9.1.1 Frame Theory

Motivated by the last inequality we make the following definition:

Definition 9.2 (Frame) *A sequence $\{\phi_j : j \in J\}$ in a (separable) Hilbert space H is called a frame if there exist positive constants $A, B > 0$ such that for all $f \in H$*

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, \phi_j\rangle|^2 \leq B\|f\|^2$$

Any two constants A, B satisfying the inequality are called frame bounds. If $A = B$ then $\{\phi_j : j \in J\}$ is called a tight frame.
Examples: An orthonormal basis is a tight frame with frame bounds $A = B = 1$. The union of any two orthonormal bases is a tight frame with frame bounds $A = B = 2$. The union of an orthonormal basis with L arbitrary unit vectors is a frame with bounds $A = 1$ and $B = L + 1$. In this lecture we will construct less trivial examples without an orthonormal basis in the background.
An equiangular tight frame is a $d \times k$ matrix, $d \leq k$ that has unit norm columns and every two columns have the same absolute inner product, for instance the mercedes star in R^2. In this case the bounds are $A = \frac{k}{d} = B$.
To understand frames and reconstruction methods better, we study some important associated operators.

Definition 9.3 (Analysis operator) *For any subset $\{\phi_j : j \in J\} \subset H$, the coefficient operator or analysis operator C is given by*

$$Cf = \{\langle f, \phi_j\rangle : j \in J\}.$$

Definition 9.4 (Synthesis operator) *The synthesis operator or reconstruction operator D is defined for a finite sequence $c = (c_j)_{j\in J}$ by*

$$Dc = \sum_{j\in J} c_j e_j \in H,$$

Definition 9.5 (frame operator) *and the frame operator is defined on H by*

$$Sf = \sum_{j\in J} \langle f, e_j\rangle e_j.$$

9.1.2 Basic frame facts

Proposition 9.6 *Suppose that $\{\phi_j : j \in J\}$ is a frame for H.*

1. *C is a bounded operator from H into $l^2(J)$ with closed range.*
2. *The operators C and D are adjoint to each other; that is $D = C^*$. Consequently D extends to a bounded operator from $l^2(J)$ into H and satisfies*

$$\left\| \sum_{j\in J} c_j \phi_j \right\| \leq B^{1/2} \|c\|_2 \tag{9.1}$$

3. *The frame operator $S = CC^* = DD^*$ maps H onto H and is a positive invertible operator satisfying $AI_H \leq S \leq BI_H$ and $B^{-1}I_H \leq S^{-1} \leq A^{-1}I_H$.*
   In particular $\{\phi_j : j \in J\}$ is a tight frame if and only if $S = AI_H$.
4. *The optimal frame bounds are $B_{opt} = \|S\|_{op}$ and $A_{opt} = \|S^{-1}\|_{op}^{-1}$ where $\|\cdot\|_{op}$ is the operator norm of S.*

To understand the convergence properties of the non-orthogonal series $\sum_j c_j\phi_j$ better, we exploit the proposition further.

Corollary 9.7 (Unconditional convergence) *Let $\{\phi_j : j \in J\}$ be a frame for H. If $f = \sum_{j\in J} c_j\phi_j$ for some $c \in l^2(J)$, then for every $\epsilon > 0$ there exists a finite subset $F_\epsilon \subseteq J$ such that*

$$\|f - \sum_{j\in F} c_j\phi_j\| < \epsilon \tag{9.2}$$

for all finite subsets $F \supseteq F_\epsilon$.
We say that the series $\sum_{j\in J} c_j\phi_j$ converges unconditionally to $f \in H$.

Proof: Choose $F_\epsilon \subseteq J$ such that $\sum_{j\neq F} |c_j|^2 < e/B^{1/2}$ for $F \supset F$. Let $c_F = c \cdot \chi_F \in l^2(J)$ be the finite sequence with terms $c_{F,j} = c_j$ if $j \in F$ and $c_{F,j} = 0$ if $j \notin F$. Then $\sum_{j\in F} c_j\phi_j = Dc_F$ and

$$\left\|f - \sum_{j\in F} c_j\phi_j\right\| = \|Dc - Dc_F\| = \|D(c - c_F)\| \leq B^{1/2}\|c - c_F\|_2 < \epsilon$$

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Unconditional convergence is the most important notion of convergence for non-orthogonal series over general, unstructured index sets.
As another consequence of 9.6 we obtain a first reconstruction formula for f from the frame coefficients $\langle f, \phi_j\rangle$.

Corollary 9.8 (Dual frame) *If $\{\phi_j : j\ inJ\}$ is a frame with bounds $A, B > 0$, then $\{S^{-1}\phi_j : j \in J\}$ is a frame with bounds $B^{-1}, A^{-1} > 0$, the so called (canonical) dual frame. Every $f \in H$ has non orthogonal expansions*

$$f = \sum_{j\in J} \langle f, S^{-1}\phi_j\rangle\phi_j \tag{9.3}$$

and

$$f = \sum_{j\in J} \langle f, \phi_j\rangle S^{-1}\phi_j \tag{9.4}$$

where both sums converge unconditionally in H.

Proof: First we observe that

$$\sum_{j\in J} |\langle f, S^{-1}\phi_j\rangle|^2 = \sum_{j\in J} |\langle \underbrace{S^{-1}f}_{g}, \phi_j\rangle|^2 = \langle Sg, g\rangle = \langle S(S^{-1}f), S^{-1}f\rangle = \langle S^{-1}f, f\rangle. \tag{9.5}$$

Therefore Proposition 9.1.2 implies that

$$B^{-1}\|f\|^2 \leq \langle S^{-1}f, f\rangle = \sum_{j\in J} |f, S^{-1}\phi_j|^2 \leq A^{-1}\|f\|^2 \tag{9.6}$$

and

$$f = S^{-1}Sf = \sum_{j \in J} \langle f, \phi_j \rangle S^{-1}\phi_j. \tag{9.7}$$

Because both $\{\langle f, \phi_j \rangle\}$ and $\{\langle f, S^{-1}\phi_j \rangle\}$ are in $l^2(J)$, both series convergence unconditionally by corollary 9.8. ∎

Remark: In general for a given frame there are many different dual frames, for example the frame $\{\phi_j, b_j : j \in J\}$ has among its dual frames: $\{\frac{1}{2}b_j, \frac{1}{2}\phi_j\}$, $\{b_j, 0\}$ and $\{\phi_j, 0\}$.