VU Time Frequency Analysis, Wavelets and Signal Processing	**SS 2016**

Lecture 5: April, 6

Lecturer: Karin Schnass — *Scribe: Hessel Tuinhof*

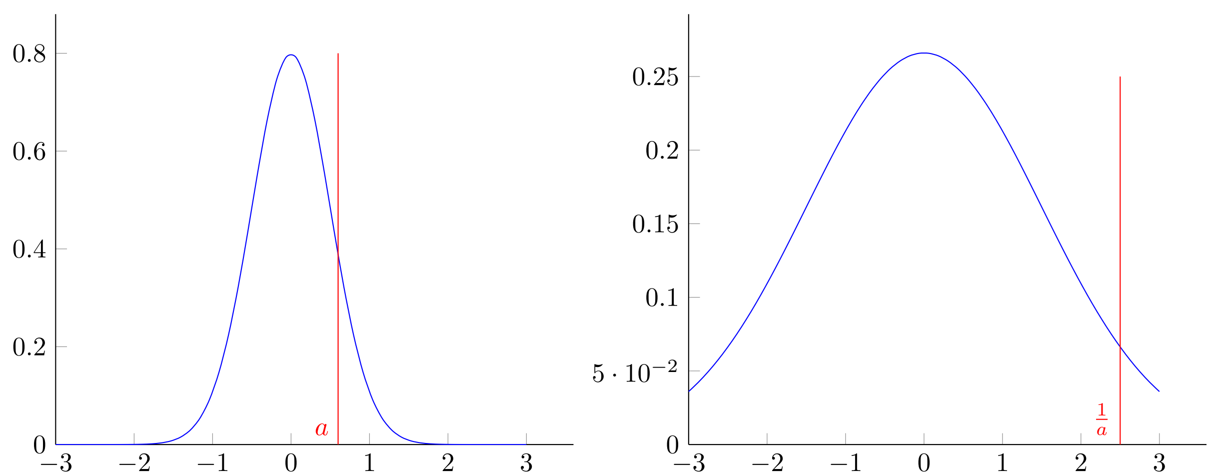
Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

5.2 Uncertainty Principles

Uncertainty principles (UP) describe relations between the position $f(t)$ and the momentum $\hat{f}(\omega)$ of a signal. In a rather sloppy formulation one could say that an uncertainty principle states the following:

> *A function cannot be well concentrated in time and in frequency.*

Example 1 (Gauss function well concentrated in time) *The relation $\hat{g}_a \approx g_{\frac{1}{a}}$ holds.*



5.2.1 Classical Uncertainty Principle

Lemma 5.1 (Abstract UP) *Let H be a Hilbert space and A, B two selfadjoint operators on H. Furthermore define the commutator operator $[\cdot\,,\cdot]$ by*

$$[A, B] = AB - BA\,.$$

Then

$$\|(A-a)f\|\|(B-b)f\| \geq \frac{1}{2}|\langle [A,B]f, f\rangle| \tag{5.1}$$

for all $a, b \in \mathbb{R}$ and for all f in the domain of AB and BA. Equality holds if and only if $(A-a)f = \mathrm{i}c(B-b)f$ for some $c \in \mathbb{R}$.

Proof:

$$\begin{aligned}\langle [A,B]f,f\rangle &= \langle \big((A-a)(B-b)-(B-b)(A-a)\big)f,f\rangle\\ &= \langle (B-b)f,(A-a)f\rangle - \langle (A-a)f,(B-b)f\rangle\\ &= 2\mathrm{i}\,\mathrm{Im}(\langle (B-b)f,(A-a)f\rangle)\end{aligned}$$

Applying Cauchy-Schwartz inequality

$$\begin{aligned}|\langle [A,B]f,f\rangle| &\le 2|\langle (B-b)f,(A-a)f\rangle| &(5.2)\\ &\le 2\|(B-b)f\|\|(A-a)f\|\,. &(5.3)\end{aligned}$$

Equality in (5.2) holds if and only if $\langle (B-b)f,(A-a)f\rangle = \mathrm{i}c$ for $c\in\mathbb{R}$ and equality in (5.3) if and only if $(A-a)f=\lambda(B-b)f, \lambda\in\mathbb{C}$. Together these imply $\lambda=\mathrm{i}c$, $C\in\mathbb{R}$. ■

Theorem 5.2 (Classical UP, Heisenberg-Pauli-Weyl inequality) *Let $f\in L^2(\mathbb{R})$ and $a,b\in\mathbb{R}$. Then*

$$\left(\int_{\mathbb{R}}(x-a)^2|f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(\omega-b)^2|\hat{f}(\omega)|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}} \ge \frac{1}{4\pi}\|f\|_2^2\,.$$

Proof: Define the multiplication and momentum operators $Xf(t)=tf(t)$ and $Pf(t)=\frac{1}{2\pi\mathrm{i}}f'(t)$ for $f\in\mathcal{S}(\mathbb{R})$. Then by the latter lemma we get

$$\begin{aligned}[X,P]f(t) &= (XP)f(t)-(PX)f(t)\\ &= \frac{t}{2\pi\mathrm{i}}f'(t)-\frac{f(t)}{2\pi\mathrm{i}}-\frac{tf'(t)}{2\pi\mathrm{i}}\\ &= -\frac{f(t)}{2\pi\mathrm{i}}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}|\langle [X,P]f,f\rangle| = \frac{1}{2}|\langle -\frac{1}{2\pi\mathrm{i}}f,f\rangle| &= \frac{1}{4\pi}\|f\|_2^2\\ &\le \|(X-a)f\|_2\|(P-b)f\|_2 = \|(X-a)f\|_2\|\widehat{(P-b)f}\|_2\\ &= \|(X-a)f\|_2\|(X-b)\hat{f}\|_2\,.\end{aligned}$$

In the former equation equality holds if and only if $(P-b)f=\mathrm{i}c(X-a)f$ for some $c\in\mathbb{R}$. This is the differential equation

$$f'-2\pi\mathrm{i}bf=-2\pi c(x-a)f\,.$$

with the only solutions being time-frequency shifted Gaussian functions $T_aM_bg_c$, $c>0$ (see Exercise). ■

Now let $f\in L^2(\mathbb{R})$ with $\|f\|_2=1$. Then

$$\Delta_t^2 f=\min_a\int_{\mathbb{R}}(t-a)^2|f(t)|^2\,\mathrm{d}t$$

and

$$\Delta_\omega^2 f=\min_b\int_{\mathbb{R}}(\omega-b)^2|\hat{f}(\omega)|^2\,\mathrm{d}\omega\,.$$

In ES $\Delta_t^2 f$ is a measure for the signal duration and $\Delta_\omega^2 f$ measures the essential bandwidth of the signal centered around the average frequency $\overline{\omega}$.

Physics in a nutshell

- Classical mechanics:
  - bunch of particles is described by a point in phase space $\mathbb{R}^d$, i.e. position and momentum of particles
  - observables are functions $\sigma(x,\omega)$
- Quantum mechanics:
  - bunch of particles are described by a function f in a Hilbert space H (state space)
  - observables are selfadjoint operators on H
    * expected value $\langle tf, f\rangle$
    * standard realization $H = L^2(\mathbb{R}^d)$
    * position operator $X_j f(t) = t_j f(t)$
    * momentum operator $P_j f(t) = \frac{1}{2\pi \mathrm{i}} \frac{\partial f}{\partial t_j}(t)$
    * expected position $q_j = \langle X_j f, f\rangle = \int_{\mathbb{R}^d} x_j |f(x)|^2 \,\mathrm{d}x$
    * expected momentum $p_j = \langle P_j f, f\rangle = \int_{\mathbb{R}^d} \omega_j |\hat{f}(\omega)|^2 \,\mathrm{d}\omega$
    * position uncertainty/variance $\Delta^2_{x_j} f = \langle (X_j - q_j)^2 f, f\rangle = \int_{\mathbb{R}^d} (x_j - q_j)^2 |f(x)|^2 \,\mathrm{d}x$
    * momentum uncertainty/variance $\Delta^2_{\omega_j} f = \langle (P_j - p_j)^2 f, f\rangle = \int_{\mathbb{R}^d} (\omega_j - p_j)^2 |\hat{f}(\omega)|^2 \,\mathrm{d}\omega$
- $[X_j, P_k] = -\frac{1}{2\pi\mathrm{i}}\delta_{jk} I \Rightarrow$ (abstract UP)$\Delta^2_{x_j} f \Delta^2_{\omega_j} f \geq \frac{1}{4\pi}$

Proposition 5.3 *Let $f \in L^2(\mathbb{R})$. Then*

$$\|Xf\|_2^2 + \|Pf\|_2^2 \geq \frac{1}{2\pi}\|f\|_2^2 .$$

Equality holds if and only if $f(x) = c\mathrm{e}^{-\pi x^2}$, i. e. f is a Gaussian function.

Proof: Follows from the classical UP with $a = b = 0$, $\alpha = \|Xf\|_2$ and $\beta = \|Pf\|_2$ by applying the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$. For equality use $\alpha = \beta$ and the equality in the UP. ■

5.2.2 Support Uncertainty Principle

Definition 5.4 (ε-concentration) *We call a function $f \in L^2(\mathbb{R}^d)$* ε-concentrated *on a measurable set $U \subseteq \mathbb{R}^d$, in*

$$\left(\int_U |f(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \geq (1-\varepsilon)\|f\|_2$$
$$\left(\int_{U^c} |f(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \leq \varepsilon\|f\|_2$$

Theorem 5.5 (Weak support Uncertainty Principle) *Let $\in L^2(\mathbb{R}^d)$, $f \neq 0$, be ε_T-concentrated on $T \subseteq \mathbb{R}^d$ and $\hat{f}$ be ε_Ω-concentrated on $\Omega \subseteq \mathbb{R}^d$. Then*

$$|T||\Omega| \geq (1 - \varepsilon_T - \varepsilon_\Omega)^2 .$$

Proof: See Gröchenig (2001). ∎

Theorem 5.6 (Strong support Uncertainty Principle) *Let $f \in L^1(\mathbb{R}^d)$ with $\operatorname{supp} f \subseteq T$ and $\operatorname{supp} \hat{f} \subseteq \Omega$. If $|T||\Omega| < \infty$, then $f = 0$.*

Proof: See Gröchenig (2001). ∎