VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Lecture 2: March, 10

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Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

2.1 Basic Fourier Analysis

2.1.1 Fourier transform, time frequency shifts, convolutions, derivatives

Definition 2.1 (Time frequency shifts)
For a function f the time shift T_x is defined via

$$(T_x f)(t) := f(t-x),$$

and the frequency shift M_ω is defined via

$$(M_\omega f)(t) := e^{2\pi i \omega t} \cdot f(t).$$

Facts 2.2

1. $T_x M_\omega = e^{-2\pi i x \omega} M_\omega T_x$.
2. $\|T_x M_\omega f\|_p = \|f\|_p$ *($p = 2$ energy conserving).*
3. $\mathcal{F}(T_x f) = M_{-x}\hat{f}$ *and* $\mathcal{F}(M_\omega f) = T_\omega \hat{f}$.

Proof:
2)

$$\int_{\mathbb{R}^d} |T_x M_\omega f(t)|^p dt = \int_{\mathbb{R}^d} |T_x f(t)|^p dt = \int_{\mathbb{R}^d} |f(t)|^p dt.$$

1)

$$\begin{aligned}(T_x M_\omega f)(t) &= (M_\omega f)(t-x) = f(t-x)e^{2\pi i (t-x)\omega} = e^{-2\pi i x \omega} \cdot f(t-x) e^{2\pi i t \omega} = \\ &= e^{-2\pi i x \omega} \cdot T_x f(t) e^{2\pi i t \omega} = e^{-2\pi i x \omega}(M_\omega T_x f)(t).\end{aligned}$$

3)

$$\begin{aligned}\mathcal{F}(T_x f)(\omega) &= \int_{\mathbb{R}^d} (T_x f)(t) e^{-2\pi i t \omega} dt = \int_{\mathbb{R}^d} f(t-x) e^{-2\pi i t \omega} dt = [y = t - x] = \\ &= \int_{\mathbb{R}^d} f(y) e^{-2\pi i (y+x)\omega} dy = e^{2\pi i(-x)\omega} \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \omega} dy = (M_{-x}\hat{f})(\omega).\end{aligned}$$

last part: exercise ∎

Definition 2.3 (Convolution)
For two functions $f, g \in L^1(\mathbb{R}^d)$ we define the convolution as

$$(f \star g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\ dy.$$

Facts 2.4

1. $\|f \star g\|_1 \leq \|f\|_1 \cdot \|g\|_1$.
2. $\mathcal{F}(f \star g) = \hat{f} \cdot \hat{g}$.

Proof:
1)

$$\begin{aligned}\int_{\mathbb{R}^d} |(f \star g)(x)|\ dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(y)g(x-y)\ dy \right| dx \leq \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)g(x-y)|\ dy\ dx = \int_{\mathbb{R}^d} |f(y)| \int_{\mathbb{R}^d} |g(x-y)|\ dx\ dy \leq \|f\|_1 \|g\|_1\end{aligned}$$

2)

$$\begin{aligned}\mathcal{F}(f \star g)(\omega) &= \int_{\mathbb{R}^d} (f \star g)(x) e^{-2\pi i x \omega} dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x-y)e^{-2\pi i x \omega} dy\ dx = \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x-y)e^{-2\pi i (x-y)\omega} e^{-2\pi i y \omega} dx\ dy \\ &= \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \omega} \int_{\mathbb{R}^d} g(x-y) e^{-2\pi i (x-y)\omega} dx\ dy = \hat{f}(\omega)\hat{g}(\omega).\end{aligned}$$

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Definition 2.5 (FT and derivatives)
For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$ we define

$$|\alpha| := \|\alpha\|_1 = \sum_{i=1}^d |\alpha_i|,\ \omega^\alpha := \prod_{i=1}^d \omega_i^{\alpha_i} \text{ and } D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d},$$

and we define the operator X^α via

$$X^\alpha f(t) := t^\alpha f(t).$$

(Note that $X^\alpha \hat{f}(\omega) = \omega^\alpha \hat{f}(\omega)$.)

Facts 2.6

1. $\mathcal{F}(D^\alpha f)(\omega) = (2\pi i \omega)^\alpha \hat{f}(\omega)$, *where $f \in \mathcal{C}^\infty$ with compact support.*
2. $\mathcal{F}((-2\pi i x)^\alpha f)(\omega) = (D^\alpha \hat{f})(\omega)$.

Proof:
1)

$$\mathcal{F}(D^\alpha f)(\omega) = \int_{\mathbb{R}^d} (D^\alpha f)(x) e^{-2\pi i x\omega} dx = \underbrace{\left[f(x) \cdot e^{-2\pi i x\omega}\right]_{x=-\infty}^{\infty}}_{=0, \text{ compact supp.}} + \int_{\mathbb{R}^d} f(x)(D^\alpha(e^{-2\pi i x\omega}))(-1)^{|\alpha|} dx =$$

$$= \int_{\mathbb{R}^d} f(x)(-2\pi i\omega)^{|\alpha|}(-1)^{|\alpha|} e^{-2\pi i x\omega} dx = (2\pi i\omega)^{|\alpha|} \hat{f}(\omega).$$

2)

$$D^\alpha \hat{f}(\omega) = D^\alpha \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\omega} dx = \int_{\mathbb{R}^d} f(x) D^\alpha e^{-2\pi i x\omega} dx =$$
$$= \int_{\mathbb{R}^d} f(x)(-2\pi i x)^\alpha e^{-2\pi i x\omega} dx = \mathcal{F}((-2\pi i x)^\alpha f)(\omega).$$

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2.1.2 FT and Gaussians

Definition 2.7 (Gaussian)
A Gaussian of width a is defined as

$$g_a := e^{-\frac{\pi x^2}{a}}.$$

Lemma 2.8
If g_a is a Gaussian, then

$$\hat{g}_a(\omega) = a^{\frac{d}{2}} g_{\frac{1}{a}}(\omega).$$

Proof:
d = 1: From

$$g_a'(x) = -\frac{2\pi x}{a} \cdot g_a(x) \tag{2.1}$$

we get

$$\frac{d}{d\omega}\hat{g}_a(\omega) = \mathcal{F}((-2\pi i x) g_a)(\omega) = ia \cdot \mathcal{F}(-\frac{2\pi x}{a} g_a)(\omega) \overset{2.1}{=} ia \cdot \mathcal{F}(\frac{d}{dx} g_a)(\omega) = ia \cdot 2\pi i\omega \hat{g}_a(\omega) = -2\pi a\omega \hat{g}_a(\omega),$$

hence

$$\hat{g}_a(\omega) = C \cdot e^{-\pi a\omega^2}, \text{ where}$$
$$C = \hat{g}_a(0) = \int_{\mathbb{R}} g_a(x)\, dx = \int_{\mathbb{R}} e^{-\frac{\pi x^2}{a}} dx = \sqrt{a}.$$

Higher dimensions: It is easy to see that for a function $y(t, x) = f(x)g(t)$ we have $\hat{y}(\omega, \xi) = \hat{f}(\omega)\hat{g}(\xi)$. Since for the Gaussian on $\mathbb{R}^d$ we have

$$e^{-\frac{\pi x^2}{a}} = \prod_{k=1}^{d} e^{-\frac{\pi x_k^2}{a}} \tag{2.2}$$

we get

$$\mathcal{F}(\prod_{k=1}^{d} e^{-\frac{\pi x_k^2}{a}})(\omega_1, \ldots, \omega_d) = \prod_{k=1}^{d} \mathcal{F}(e^{-\frac{\pi x_k^2}{a}})(\omega_k) = \prod_{k=1}^{d} e^{-\pi a\omega_k^2} = e^{-\pi a\omega^2}.$$

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