VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Lecture 1: March, 9

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Warning: *These notes have not been subjected to the usual scrutiny reserved for formal publications. So enjoy with caution.*

1.1 Basic Fourier Analysis

Here we summarise and prove some basic facts about

- Fourier transform (FT),
- FT & TF-shifts, convolution, derivatives,
- FT & Gaussians,
- Plancherel's theorem (PT)
- Exploit PT, inversion, smoothness, Schwartz space,
- Poisson summation formula, Nyquist-Shannon sampling theorem.

1.1.1 Setting (What we already know)

- $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $f\colon \mathbb{R}^d \to \mathbb{C}$, e. g.

$$\left.\begin{array}{l} d = 1 \dots \text{ music with } x_1 = t \\ d = 2 \dots \text{ grayscale image} \end{array}\right\} \text{Engineers say (ES) "signal"}$$

- $\|f\|_p^p = \int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x$; $p = 2$ $\|f\|_2^2$ ES "signal energy"
- $L_p(\mathbb{R}^d),\ L_p,\ L_p(\mathbb{T}^d)$
- $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g}(x)\,\mathrm{d}x$
- $x \cdot \omega = \sum_{i=1}^d x_i\omega_i$, where $x,\ \omega \in \mathbb{R}^d$
- $|x|^2 = \|x\|^2 = \|x\|_2^2 = \sum x_i^2$
- Fourier series: Let f on $\mathbb{R}^d$ be $\mathbb{Z}^d$-periodic, i. e. $\forall\, k \in \mathbb{Z}^d\colon f(x+k) = f(x)$, then we define f on the torus $f(\mathbb{T}^d),\ f([0,1]^d)$.
- $\mathrm{e}^{2\pi i n x},\ n \in \mathbb{Z}^d$
  - ONS in $L^2(\mathbb{T}^d)$
  - dense in $L^2(\mathbb{T}^d)$

Hence the family $\{e^{2\pi i n x} \mid n \in \mathbb{Z}^d\}$ is a basis.

- $C^\infty(\mathbb{R}^d)$ with compact support is dense in L^p.

Theorem 1.1 (Riesz–Fischer, Parseval's Identity) *Let $f \in L^2(\mathbb{R}^d)$ and*

$$\begin{aligned}\widehat{f}(n) &= \int_{[0,1]^d} f(x) e^{-2\pi i n x} \, dx \,, \\ f(x) &= \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{2\pi i n x} \,.\end{aligned}$$

The partial sums ($|n| \le N$) converge in L^2 sense and

$$\|f\|^2_{L^2(\mathbb{T}^d)} = \int_{[0,1]^d} |f(x)|^2 \, dx = \sum_{n \in \mathbb{Z}^d} |\widehat{f}(n)|^2 \,.$$

Proof: BSc & handout. ■
(If $f \in C^1(\mathbb{T}^d)$ and $\sum |\widehat{f}(n)| < \infty$ then we have pointwise convergence, more on convergence of Fourier series: `https://en.wikipedia.org/wiki/Convergence_of_Fourier_series`.)

1.1.2 Fourier transform

Definition 1.2 $f \in L^1(\mathbb{R}^d)$

$$\begin{aligned}\widehat{f}(\omega) &= \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \omega} \, dx \\ \|\widehat{f}\|_\infty &= \sup_\omega |\widehat{f}(\omega)| \le \left| \int_{\mathbb{R}^d} \dots dx \right| \le \int |f(x)| \, dx = \|f\|_1 \,.\end{aligned}$$

Lemma 1.3 (Riemann–Lebesgue) *If $f \in L^1$, $\widehat{f}$ is uniformly continuous and* $\lim_{\|\omega\| \to \infty} |\widehat{f}(\omega)| = 0$.

Proof:

1. It holds that

$$|\widehat{f}(\omega + h) - \widehat{f}(\omega)| = \left| \int_{\mathbb{R}^d} f(x) (e^{-2\pi i x(\omega + h)} - e^{-2\pi i x \omega}) \, dx \right| \le \int_{\mathbb{R}^d} |f(x)| |e^{-2\pi i x h} - 1| \, dx \,.$$

We can find an $R > 0$ such that $\int_{\mathbb{R}^d} |f(x)| \, dx < \varepsilon$ as well as some $\delta > 0$ fulfilling

$$f \|h\| < \delta \, \forall \, \|x\| < R \colon |e^{-2\pi i x h} - 1| < \varepsilon \,.$$

Hence we can estimate the upper integral in the following way

$$\int_{\|x\| \le R} |f(x)| |e^{-2\pi i x h} - 1| \, dx + \int_{\|x\| > R} |f(x)| |e^{-2\pi i x h} - 1| \, dx \le \varepsilon \|f\|_1 + 2\varepsilon = \varepsilon (\|f\|_1 + 2)$$

2. Find $g \in C^1$ with compact support K such that $\|f-g\|_1 \leq \varepsilon$.

$$\begin{aligned} \|\widehat{f}(\omega)\| &\leq |\widehat{f}(\omega) - \widehat{g}(\omega)| + |\widehat{g}(\omega)| \\ &\leq \|f-g\|_1 + |\widehat{g}(\omega)| \end{aligned}$$

   Partial integration yields

$$\left|\int g(x) \mathrm{e}^{-2\pi \mathrm{i} \omega x} \,\mathrm{d}x\right| = \left|-\int \frac{\partial g}{\partial x_k}(x) \frac{1}{2\pi \mathrm{i} \omega_k} \mathrm{e}^{-2\pi \mathrm{i} \omega x} \,\mathrm{d}x\right| \leq \frac{1}{2\pi \omega_k} \int \left|\frac{\partial g}{\partial x_k}(x)\right| \mathrm{d}x \,.$$

   Thus we can estimate for all $k = 1, \ldots, d$

$$|\widehat{g}(\omega)| \leq \frac{1}{2\pi \|\omega\|_\infty} \max_k \int \left|\frac{\partial g}{\partial x_k}(x)\right| \mathrm{d}x \implies \lim_{\|\omega\|_2 \to \infty} |\widehat{g}(\omega)| = 0\,,$$

   We conclude that for all $\varepsilon > 0$

$$\lim_{\|\omega\|_2 \to \infty} |\widehat{f}(\omega)| < \varepsilon + \lim_{\|\omega\|_2 \to \infty} |\widehat{g}(\omega)| = \varepsilon \implies \lim_{\|\omega\|_2 \to \infty} |\widehat{f}(\omega)| = 0\,.$$

■

The real goal of this section is Plancherel's theorem

$$f \in L_1 \cap L_2 \colon \|\widehat{f}\|_2 = \|f\|_2 \,.$$