VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Exercise Sheet 3

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3.1 Gabor Frames - Theory

1. Show that the canonical tight frame associated to the Gabor frame $G(g,\alpha,\beta)$ is a Gabor frame with *canonical tight window* $\gamma_t = S_{g,g}^{-\frac{1}{2}} g$.

2. **Walnut's representation**: For g,γ sufficiently beautiful (e.g. in $W(\mathbb{R}^d)$) and $\alpha,\beta > 0$ the operator

$$S_{g,\gamma} f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k} M_{\beta n} g\rangle \, T_{\alpha k} M_{\beta n} \gamma \tag{3.1}$$

can be written as

$$S_{g,\gamma} f = \beta^{-d} \sum_{n\in\mathbb{Z}^d} G_n \cdot T_{\frac{n}{\beta}} f, \tag{3.2}$$

where the correlation functions $G_n(x)$ are defined as $G_n(x) = \sum_{k\in\mathbb{Z}^d} \bar{g}(x - \frac{n}{\beta} - \alpha k)\gamma(x - \alpha k)$.
Use the following steps:

   (a) Find an argument to flip the time-shifts with the frequency shifts in (3.1).

   (b) Interpret $S_{g,\gamma} f$ as sum of $\frac{1}{\beta}\mathbb{Z}^d$ periodic functions m_k, that is $S_{g,\gamma} f = \sum_{k\in\mathbb{Z}^d} m_k T_{\alpha k}\gamma$ and write down what $m_k(x)$ looks like.

   (c) Show that $\langle f, M_\omega T_x g\rangle = (\widehat{f \cdot T_x \bar{g}})(\omega)$ and apply it to m_k.

   (d) Use the generalised Poisson interpolation formula (last exercises) to rewrite m_k.

   (e) Switch summation orders and arrive at the statement.

   (f) Give some thought to convergence issues of (Fourier) series and change of summation.

3. Rationalise the existence of Gabor frames using Walnut's representation:

   (a) Identify the condition in the theorem with a condition on the correlation function G_0 for $\gamma = g$.

   (b) For g bounded with compact support and $n \neq 0$ what happens to $\|G_n\|_\infty$ as β goes to zero?

   (c) For g bounded with compact support what happens to $\sum_{n\in Z^d, n\neq 0} \|G_n\|_\infty$ as β goes to zero?

   (d) Assume that for our g $\sum_{n\in Z^d, n\neq 0} \|G_n\|_\infty$ goes to zero as β goes to zero and the condition of the theorem. Show that for β small enough we can bound the frame operator by $\langle Sf, f\rangle \geq (a - \sum_{n\neq 0} \|G_n\|_\infty)\|f\|_2^2$.

4. **Janssen's representation**: For g,γ sufficiently beautiful[1] we have:

$$S_{g,\gamma} f = (\alpha\beta)^{-d} \sum_{k,n\in\mathbb{Z}^d} \langle \gamma, T_{\frac{k}{\beta}} M_{\frac{n}{\alpha}} g\rangle \, T_{\frac{k}{\beta}} M_{\frac{n}{\alpha}} f. \tag{3.3}$$

[1] e.g. $\sum_{k,l\in\mathbb{Z}^d} |\langle \gamma, T_{k/\beta} M_{n/\alpha} g\rangle| < \infty$ or $g \in M^1$, meaning $V_\varphi g \in L^1(\mathbb{R}^{2d})$ for some $\varphi \in \mathcal{S}$.

Start with Walnut's representation. Method a), note that the correlation functions G_n are α periodic, so they can be expanded into a Fourier series on $[0, \alpha]^d$, calculate the Fourier coefficients to get the result. Method b), apply the generalised Poisson summation formula to the correlation functions G_n and use the identity from above that $\langle f, M_\omega T_x g\rangle = (\widehat{f \cdot T_x \overline{g}})(\omega)$.

5. Digest the Wexler-Raz biorthogonality relations based on Janssen's representation.

6. Use the the Wexler-Raz biorthogonality relations to show that if $G(g, \alpha, \beta)$ a frame, then $\alpha\beta \leq 1$. Note that there are two natural ways to expand g, first as itself and second using the minimal norm coefficients provided by the canonical dual window, which in particular satisfies the Wexler-Raz biorthogonality relations for $n = k = 0$.

7. Digest the lemma that for a Gabor frame $G(g, \alpha, \beta)$ the canonical dual window γ^o is in the closed linear span of $G(g, \frac{1}{\beta}, \frac{1}{\alpha})$ using Janssen's representation. Use that $S^{-1}_{g,g} = S_{\gamma^o,\gamma^o}$ and assume that γ^o is nice enough to be able to use Janssen's representation.

8. (Re)proof the minimality properties of the canonical dual window using all other results.

9. Read the paper ofdm.pdf, in the folder 'other stuff' on OLAT.

3.2 Gabor Frames - Experimental Math

1. Download the Gabor matlab mini-toolbox 'TFmini' from the folder 'other stuff' on OLAT, open all files and read their documentation. **Attention** in the toolbox vectors are row vectors, so matrices act from the right, ie. vM.

2. For $n = 39, 35, 30, 20$ create a Gaussian window in $d = 40 \cdot n$, and a Gabor system with shift parameters $a = b = n$, using `gaussnk` and `gabbasp`. Check whether the Gabor system is a frame. If yes calculate the canonical dual window and look at it using `plotc`. What happens as n decreases or equivalently the redundancy r, number of Gabor atoms/dimension $= (\frac{d}{a} \cdot \frac{d}{b})/d = d/(ab)$ increases? Compare the redundancy of the system to the shape of the dual window.

3. Create a Gaussian window g in $d = 144$ and the 3 times narrower resp. wider Gaussian windows g_n, g_w. Look at the STFTs $V_g g$, $V_g g_w$, $V_g g_n$ using `stft` and `imgc`.

4. For all possible lattice parameters a, b (all divisors of 144) create a Gabor system. Note the redundancy of the system (number of Gabor atoms/dimension) and check whether the resulting system is a frame, a Riesz basis for its span or nothing. Check the condition number of the frame. Whenever the Gabor system is a frame, have a look at the canonical dual window and its STFT. Check that it is in the span of $G(g, \frac{1}{\beta}, \frac{1}{\alpha})$. What do you observe as the redundancy increases? Which lattice parameters can you recommend for g, g_n, g_w. Create more dual windows using random vectors, box functions etc. and the projection onto the orthogonal span of $G(g, \frac{1}{\beta}, \frac{1}{\alpha})$. Look at the canonical tight windows $(S^{-\frac{1}{2}} g)$.

5. Repeat the exercise above using a rectangular window, a triangular window, a hamming, a hanning and a random window, (all normalised).

6. Choose your favourite Gabor system(s) and approximate a couple of functions of your choice with the system, using `gabcofs` and `gabsyn`. Find out what happens if one coefficient gets deleted. Try to recover the function by increasing the coefficients around the deleted one. Play around with this idea.

7. Go wild experimentally checking the last lecture!