VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Exercise Sheet 2

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2.1 Uncertainty Principle

1. Show that $\min_a \left(\int_{\mathbb{R}}(x-a)^2|f(x)|^2dx\right)^{1/2}$ takes its minimum at $\bar{x} = \int x|f(x)|^2dx$, compare your findings with the quantum mechanic view of the uncertainty principle.

2.2 STFT

1. Check that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x \bar{g}})(\omega) \tag{2.1}$$
$$= \langle f, M_\omega T_x g\rangle \tag{2.2}$$
$$= \langle \hat{f}, T_\omega M_{-x}\hat{g}\rangle \tag{2.3}$$
$$= e^{-2\pi i x\omega}(\widehat{\hat{f} \cdot T_\omega \bar{\hat{g}}})(-x) = V_{\hat{g}}\hat{f}(\omega,-x). \tag{2.4}$$

2. Show that the STFT of an L^2-function with an L^2-window is uniformly continuous.

3. Make the density argument of the proof of the orthogonality relations for the STFT explicit. (Cauchy sequences!).

4. Implement a discrete version of the STFT in Matlab (use a wrap-around for the timeshifts). Compare it to the version you found at `http://ltfat.sourceforge.net/`

5. Take three Gaussians $g_1, g_a, g_{1/a}$, for some suitable value $a > 0$, two centered box functions b_1, b_2 of different size (remember the negative values are at the end of the vector, infinity is in the middle), the convolution of the box functions with themselves $b_i * b_i$, convolution of the box functions with the Gaussians $b_i * g_j$ and their Fourier transforms $\widehat{b_i * b_i}$, $\widehat{b_i * g_j}$. Decide how many samples you want to use. Calculate all possible STFTs $V_g f$ and look at them (`imagesc`).

6. Analyse the function

$$f(t) = \begin{cases} \sin(2\pi t) + \sin(16\pi t) & t \leq 0 \\ \sin(2\pi t^2) + \sin(2\pi t^3) & t > 0 \end{cases}$$

with the windows from the last exercise.

7. Implement/download the inverse STFT in Matlab. Reconstruct the function from the last exercise from its STFT using various windows. Try to filter out the different elements (the 2 pure frequencies and the 2 chirps), and resynthesize the signal. Plot the original and the filtered version.

2.3 Quadratic TF representations

1. Show that for the cross-ambiguity function we have $A(f,g)(x,\omega) = e^{\pi i x\omega} V_g f(x,\omega)$.
2. Show that for the ambiguity function we have $\|Af\|_2 = \|f\|_2^2$.
3. Prove the radar uncertainty principle:
   If $\|f\|_2 = 1$ and $U \subseteq \mathbb{R}^{2d}$ and $\varepsilon > 0$ such that
   $$\int\int_U |Af(x,\omega)|^2 dx d\omega \geq 1-\varepsilon, \quad \text{then} \quad |U| \geq 1-\varepsilon.$$
4. Derive an inversion formula for the ambiguity function $Af(x,w)$ of a (Schwartz) function with $f(0) = 0$.
5. Show that for the cross-Wigner distribution we have
   $$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{\mathrm{flip}(g)} f(2x, 2\omega),$$
   where $\mathrm{flip}(g)(x) = g(-x)$.
6. For $f, g \in L^2(\mathbb{R}^d)$ show the following properties of the (cross)-Wigner distribution,
   - $W(f,g)$ is uniformly continuous on $\mathbb{R}^{2d}$, and $\|W(f,g)\|_\infty \leq 2^d \|f\|_2 \|g\|_2$.
   - $W(f,g) = \overline{W(g,f)}$, conclude that Wf is real valued.
   - $W(T_u M_\eta f)(x,\omega) = Wf(x-u, \omega-\eta)$, (covariance).
   - $W(\hat{f},\hat{g})(x,\omega) = W(f,g)(-\omega, x)$.
   - Moyal's formula: For $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$, we have $\langle W(f_1,g_1), W(f_2,g_2)\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}$.
7. Show the support property of the Wigner distribution for $f \in L^2(\mathbb{R})$.
   If $\mathrm{supp}\, f \subseteq [a,b]$ then $Wf(x,\omega) = 0$ for $x \notin [a,b]$.
   If $\mathrm{supp}\, \hat{f} \subseteq [\alpha,\beta]$ then $Wf(x,\omega) = 0$ for $\omega \notin [\alpha,\beta]$.
8. Show the following properties of the marginal densities of the Wigner distribution.
   If $f, \hat{f} \in L^1 \cap L^2$ then $\int_{\mathbb{R}^d} Wf(x,\omega) d\omega = |f(x)|^2$ and $\int_{\mathbb{R}^d} Wf(x,\omega) dx = |\hat{f}(\omega)|^2$.
   What can you conclude for $\int\int_{\mathbb{R}^{2d}} Wf(x,\omega) dx d\omega$?
9. Show the weak uncertainty principle for the Wigner distribution:
   If $\varepsilon > 0$ and $U \subseteq \mathbb{R}^{2d}$ such that
   $$\int\int_U Wf(x,\omega) dx d\omega \geq (1-\varepsilon)\|f\|_2^2, \quad \text{then} \quad |U| \geq (1-\varepsilon) 2^{-d}.$$
10. Show that for an odd real-valued function, ie. $f(-x) = -f(x)$, the Wigner distribution is negative at zero, $Wf(0,0) = -2^d \|f\|_2^2$.