VU Time Frequency Analysis, Wavelets and Signal Processing **SS 2016**

Exercise Sheet 1

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Don't be scared, they are mostly one-liners!

1.1 Useful Identities

Show the following:

1. $\widehat{(M_\omega f)} = T_\omega \hat{f}$ or $\mathcal{F} M_\omega f = T_\omega \mathcal{F} f$.
2. $\hat{\hat{f}}(x) = f(-x)$ or $(\mathcal{F}\mathcal{F}f)(x) = f(-x)$.
3. M_ω and T_x are linear and continuous operators on $L^2(\mathbb{R}^d)$.
4. The adjoint operator of the time-shift T_x is the timeshift T_{-x}, ie. $T_x^\star = T_{-x}$.
5. The adjoint operator of the frequency-shift M_ω is the frequency-shift $M_{-\omega}$, ie. $M_\omega^\star = M_{-\omega}$.
6. M_ω and T_x are unitary operators on $L^2(\mathbb{R}^d)$.
7. Let $L(L^2(\mathbb{R}^d))$ be the space of linear operators on $L^2(\mathbb{R}^d)$ equipped with the strong operator topology, meaning a sequence of operators T_n converges to an operator T if $T_n f$ converges to Tf for all $f \in L^2(\mathbb{R}^d)$.
   (a) The mapping $A : \mathbb{R}^d \to L^2(\mathbb{R}^d)$ with $A(x) = T_x$ is continuous.
   Hint: For continuous functions with compact support things are usually easy!
   (b) The mapping $B : \mathbb{R}^d \to L^2(\mathbb{R}^d)$ with $B(\omega) = M_\omega$ is continuous.
8. Define the operators $X, P : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by $Xf(x) = x \cdot f(x)$ and $Pf(x) = \frac{1}{2\pi i} f'(x)$.
   We have $\frac{d}{d\omega} M_\omega f\big|_{\omega=0} = 2\pi i X f$ and $\frac{d}{dx} T_x f\big|_{x=0} = -2\pi i P f$.
9. Define the dilation operator D_a for $a \in \mathbb{R}/\{0\}$ by $D_a f(x) = |a|^{-d/2} f(x/a)$.
   D_a is unitary on $L^2(\mathbb{R}^d)$ and $\widehat{(D_a f)} = D_{1/a} \hat{f}$.
10. The Fourier transform of an even function ($f(x) = f(-x)$) is real, the Fourier transform of an odd function ($f(x) = -f(-x)$) is purely imaginary.

1.2 Reprocessing the Lecture

1. Prove that from $\langle T_x M_\omega g_1, T_y M_\eta g_1 \rangle = \langle \mathcal{F}(T_x M_\omega g_1), \mathcal{F}(T_y M_\eta g_1) \rangle$ and the fact that the linear span of $X = \{T_{x_i} M_{\omega_i} g_1\}$ is dense in L^2 or you can conclude that the Fouriertransform is an isometry on $L^2 \cap L^1$.
2. Show that the extension of the Fourier transform to L^2 is well defined.
3. Check other handwaving issues from the lecture.

1.3 More or Less Brainless Fourier Transforms

1. Calculate the Fouriertransform of the box function $b = \chi_{[-c,c]} \in L^2(\mathbb{R})$.
   Remember $\chi_\Omega(x) = 1$ for $x \in \Omega$ and zero else.
2. Calculate the Fouriertransform of the d-dimensional box function $b = \chi_{[-c,c]^d} \in L^2(\mathbb{R}^d)$.
3. Calculate the Fouriertransform of the triangular function $\Delta \in L^2(\mathbb{R})$.

$$\Delta(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \tag{1.1}$$

4. Calculate the Fouriertransform of a pure frequency $f_\omega(x) = e^{2\pi i x\omega}$ (in the distributional sense).
5. Calculate the Fouriertransform of the sine and the cosine $f_\omega(x) = \sin(2\pi x\omega)$, $f_\omega(x) = \cos(2\pi x\omega)$ (in the distributional sense).
6. Calculate the Fouriertransform of the absolute value function $f(x) = |x|$ and the sign function $f(x) = \mathrm{sign}(x)$ for $x \in \mathbb{R}$ (in the distributional sense).

1.4 Poisson Summation & Shannon-Nyquist Sampling Theorem

1. Generalize the Poisson summation formula, ie. for $\alpha > 0$

$$\sum_{n\in\mathbb{Z}^d} f(x + \alpha n) = \alpha^{-d} \sum_{n\in\mathbb{Z}^d} \hat{f}\left(\frac{n}{\alpha}\right) e^{2\pi i n x/\alpha}. \tag{1.2}$$

2. Show the 'reverse' Poisson summation formula, for $\alpha > 0$

$$\sum_{n\in\mathbb{Z}^d} \hat{f}(x + \alpha n) = \alpha^{-d} \sum_{n\in\mathbb{Z}^d} f\left(\frac{n}{\alpha}\right) e^{-2\pi i n x/\alpha}. \tag{1.3}$$

3. **Shannon-Nyquist Sampling Theorem:**
   If f is a bandlimited function, i.e. $\operatorname{supp} \hat{f} \subseteq [-B, B]$, then f is completely determined by equidistant sampling points with distance $T = \frac{1}{2B}$, in particular

$$f(x) = \sum_{n\in\mathbb{Z}} f(Tn)\,\mathrm{sinc}(x/T - n) \quad \text{where} \quad \mathrm{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases} \tag{1.4}$$

   Proof the theorem. (Hints: A bandlimited function can be recovered from its 2B-periodisation by multiplying it with the box-function $\chi_{[-B,B]}$. Then use the reverse Poisson summation formula. Apply the inverse Fourier transform on both sides remembering all the useful identities and the Fourier transform of the box function.)

1.5 Practical stuff

1. Approximate the Fouriertransform of a function f with a Riemann integral on the interval $[-\sqrt{N}/2, \sqrt{N}/2]$ using N equidistant sampling points (Stützstellen) $x_k = -\sqrt{N}/2 + k/\sqrt{N}$. What approximation do you get for the Fouriertransform on the same N equidistant sampling points $\hat{f}(-\sqrt{N}/2 + j/\sqrt{N})$?

2. Check the Matlab documentation of the fft. Convince yourself that doing a Fouriertransform is more or less the same as doing an fft. Find out where the high frequencies are and where the low frequencies by taking the inverse Fouriertransform ifft of the identity matrix and plotting the rows (columns).

3. Take your favourite black and white picture, easier if square.

```
I=imread('mypic.jpg')
im = 0.2989*im(:,:,1)+0.5870*im(:,:,2)+0.1140*im(:,:,3);
im=double(im);
```

   Downsample it, by taking only every second, third pixel. Look at the resulting downsampled image. Now take the twodimensional Fouriertransform (fft2) of the picture. Trash the high frequencies (set to zero if $\|\omega\|_2 \geq c_1, c_2 \ldots$), remembering where in the matrix the high frequencies are. Now go back to the time domain and downsample the image again, ie. take every second, third pixel. Compare the two downsampled versions of your image. Find an explanation for what happened.