# Convergence radius and sample complexity of ITKM algorithms for dictionary learning

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**Abstract**

In this work we show that iterative thresholding and K means (ITKM) algorithms can recover a generating dictionary with K atoms from noisy $S$ sparse signals up to an error $\tilde{\varepsilon}$ as long as the initialisation is within a convergence radius, that is up to a $\log K$ factor inversely proportional to the dynamic range of the signals, and the sample size is proportional to $K \log K \tilde{\varepsilon}^{-2}$. The results are valid for arbitrary target errors if the sparsity level is of the order of the square of the signal dimension $d$ and for target errors down to $K^{-\ell}$ if $S$ scales as $S \leq d/(\ell \log K)$.

**Index Terms**

dictionary learning, sparse coding, sparse component analysis, sample complexity, convergence radius, alternating optimisation, thresholding, K-means

## 1 Introduction

The goal of dictionary learning is to find a dictionary that will sparsely represent a class of signals. That is given a set of $N$ training signals $y_n \in \mathbb{R}^d$, which are stored as columns in a matrix $Y = (y_1, \dots, y_N)$, one wants to find a collection of $K$ normalised vectors $\phi_k \in \mathbb{R}^d$, called atoms, which are stored as columns in the dictionary matrix $\Phi = (\phi_1, \dots, \phi_K) \in \mathbb{R}^{d \times K}$, and coefficients $x_n$, which are stored as columns in the coefficient matrix $X = (x_1, \dots, x_N)$ such that

$$Y = \Phi X \quad \text{and} \quad X \text{ sparse.} \tag{1}$$

Research into dictionary learning comes in two flavours corresponding to the two origins of the problem, the slightly older one in the independent component analysis (ICA) and blind source separation (BSS) community, where dictionary learning it is also known as sparse component analysis, and the slightly younger one in the signal processing community, where it is also known as sparse coding. The main motivation for dictionary learning in the ICA/BSS community comes from the assumption that the signals of interest are generated as sparse mixtures - random sparse mixing coefficients $X_0$ - of several sources or independent components - the dictionary $\Phi_0$ - which can be used to describe or explain a (natural) phenomenon, [14], [26], [23], [22]. For instance in the 1996 paper by Olshausen and Field, [14], which is widely regarded as the mother contribution to dictionary learning, the dictionary is learned on patches of natural images, and the resulting atoms bear a striking similarity to simple cell receptive fields in the visual cortex. A natural question in this context is, when the generating dictionary $\Phi_0$ can be identified from $Y$, that is the sources from the mixtures. Therefore the first theoretical insights into dictionary learning came from this community, [16]. Also the first dictionary recovery algorithms with global success guarantees, which are based on finding overlapping clusters in a graph derived from the signal correlation matrix $Y^\star Y$, take the ICA/BSS point of view, [5], [2].

The main motivation for dictionary learning in the signal processing community is that sparse signals are immensely practical, as they can be easily stored, denoised, or reconstructed from incomplete information, [12], [29], [27]. Thus the interest is less in the dictionary itself but in the fact that it will provide sparse

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representations $X$. Following the rule 'the sparser - the better' the obvious next step is to look for the dictionary that provides the sparsest representations. So given a budget of $K$ atoms and $S$ non-zero coefficients per signal, one way to concretise the abstract formulation of the dictionary learning problem in (1) is to formulate it as optimisation problem, such as

$$(P_{2,S}) \qquad \min \|Y - \Phi X\|_F \quad \text{s.t.} \quad \|x_n\|_0 \leq S \quad \text{and} \quad \Phi \in \mathcal{D}, \tag{2}$$

where $\|\cdot\|_0$ counts the nonzero elements of a vector or matrix and $\mathcal{D}$ is defined as $\mathcal{D} = \{\Phi = (\phi_1, \ldots, \phi_K) : \|\phi_k\|_2 = 1\}$. While $(P_{2,S})$ is for instance the starting point for the MOD or K-SVD algorithms, [13], [3], other definitions of *optimally* sparse lead to other optimisation problems and algorithms, [43], [32], [42], [28], [37], [33]. The main challenge of optimisation programmes for dictionary learning is finding the global optimum, which is hard because the constraint manifold $\mathcal{D}$ is not convex and the objective function is invariant under sign changes and permutations of the dictionary atoms with corresponding sign changes and permutations of the coefficient rows. In other words for every local optimum there are $2^K K! - 1$ equivalent local optima.

So while in the signal processing setting there is a priori no concept of a generating dictionary, it is often used as auxiliary assumption to get theoretical insights into the optimisation problem. Indeed without the assumption that the signals are sparse in some dictionary the optimisation formulation makes little or no sense. For instance if the signals are uniformly distributed on the sphere in $\mathbb{R}^d$, in asymptotics $(P_{2,S})$ becomes a covering problem and the set of optima is invariant under orthonormal transforms.

Based on a generating model on the other hand it is possible to gain several theoretical insights. For instance, how many training signals are necessary such that the sparse representation properties of a dictionary on the training samples (e.g. the optimiser) will extrapolate to the whole class, [30], [41], [31], [17]. What are the properties of a generating dictionary and the maximum sparsity level of the coefficients and signal noise such that this dictionary is a local optimiser or near a local optimiser given enough training signals, [18], [15], [35], [34], [20].

An open problem for overcomplete dictionaries with some first results for bases, [38], [39], is whether there are any spurious optimisers which are not equivalent to the generating dictionary, or if any starting point of a descent algorithm will lead to a global optimum? A related question (in case there are spurious optima) is, if the generating dictionary is the global optimiser? If yes, it would justify using one of the graph clustering algorithms for recovering the optimum, [5], [2], [4], [6]. This is important since all dictionary learning algorithms with global success guarantees are computationally very costly, while optimisation approaches are locally very efficient and robust to noise. Knowledge of the convergence properties of a descent algorithm, such as convergence radius (basin of attraction), rate or limiting precision based on the number of training signals, therefore helps to decide when it should take over from a global algorithm for fast local refinement, [1].

In this paper we will investigate the convergence properties of two iterative thresholding and K-means algorithms. The first algorithm ITKsM, which uses signed signal means, originates from the response maximisation principle introduced in [34]. There it is shown that a generating $\mu$-coherent dictionary constitutes a local maximum of the response principle as long as the sparsity level of the signals scales as $S = O(\mu^{-1})$. It further contains the first results showing that the maximiser remains close to the generator for sparsity levels up to $S = O(\mu^{-2}/\log K)$. For a target recovery error $\tilde{\varepsilon}$ the sample complexity $N$ is shown to scale as $N = O(SK^3\tilde{\varepsilon}^{-2})$ and the basin of attraction is conjectured to be of size $O(1/\sqrt{S}$.

Here we will not only improve on the conjecture by showing that the algorithm has a convergence radius of size $O(1/\sqrt{\log K})$ but also show that for the algorithm rather than the principle the sample complexity reduces to $N = O(K \log K \tilde{\varepsilon}^{-2})$ (omitting $\log \log$ factors). Again recovery to arbitrary precision holds for sparsity levels $S = O(\mu^{-1})$ and stable recovery up to an error $K^{-\ell}$ for sparsity levels $S = O(\mu^{-2}/(\ell \log K))$. We also show that the computational complexity assuming an initialisation within the convergence radius scales as $O(\log(\varepsilon^{-1})dKN)$ or omitting log factors $O(dK^2\varepsilon^{-2})$.

Motivated by the desire to reduce the sample complexity for the case of exactly sparse, noiseless signals, we then introduce a second iterative thresholding and K-means algorithms ITKrM, which uses residual instead of signal means. It has roughly the same properties as ITKsM apart from the convergence radius which reduces to $O(1/\sqrt{S}$ and the computational complexity, which is scales as $O(dN(K + S^2))$ and thus can go up to $O(d^2NK)$ for $S = O(d)$. However, if $S = O(\mu^{-1})$ and the signals follow an exactly sparse, noiseless model, we can show that the sample complexity reduces to $N = O(K\varepsilon^{-1})$ (omitting $\log\log$ factors). Our results are in the same spirit as the results for the alternating minimisation algorithm in [1] but have the advantage that they are valid for more general coefficient distributions and a lower level of sparsity (S larger) resp. higher level of coherence, that the convergence radius is larger and that the algorithms exhibit a lower computational complexity.

The rest of the paper is organised as follows. After summarising notation and conventions in the following section, in Section 3 we re-introduce the ITKsM algorithm, discuss our sparse signal model and analyse the convergence properties of ITKsM. Based on the shortcomings of ITKsM we motivate the ITKrM algorithm in Section 4, and again analyse its convergence properties. In Section 5 we finally compare our results to existing work and point out future directions of research.

## 2 Notations and Conventions

Before we join the melee, we collect some definitions and lose a few words on notations; usually subscripted letters will denote vectors with the exception of $\varepsilon, \alpha, \omega$, where they are numbers, eg. $x_n \in \mathbb{R}^K$ vs. $\varepsilon_k \in \mathbb{R}$, however, it should always be clear from the context what we are dealing with.

For a matrix $M$, we denote its (conjugate) transpose by $M^\star$ and its Moore-Penrose pseudo inverse by $M^\dagger$. We denote its operator norm by $\|M\|_{2,2} = \max_{\|x\|_2=1} \|Mx\|_2$ and its Frobenius norm by $\|M\|_F = \operatorname{tr}(M^\star M)^{1/2}$, remember that we have $\|M\|_{2,2} \leq \|M\|_F$.

We consider a **dictionary** $\Phi$ a collection of $K$ unit norm vectors $\phi_k \in \mathbb{R}^d$, $\|\phi_k\|_2 = 1$. By abuse of notation we will also refer to the $d \times K$ matrix collecting the atoms as its columns as the dictionary, i.e. $\Phi = (\phi_1, \dots \phi_K)$. The maximal absolute inner product between two different atoms is called the **coherence** $\mu$ of a dictionary, $\mu = \max_{k \neq j} |\langle \phi_k, \phi_j \rangle|$.

By $\Phi_I$ we denote the restriction of the dictionary to the atoms indexed by $I$, i.e. $\Phi_I = (\phi_{i_1}, \dots, \phi_{i_S})$, $i_j \in I$, and by $P(\Phi_I)$ the orthogonal projection onto the span of the atoms indexed by $I$, i.e. $P(\Phi_I) = \Phi_I \Phi_I^\dagger$. Note that in case the atoms indexed by $I$ are linearly independent we have $\Phi_I^\dagger = (\Phi_I^\star \Phi_I)^{-1} \Phi_I^\star$. We also define $Q(\Phi_I)$ the orthogonal projection onto the orthogonal complement of the span on $\Phi_I$, that is $Q(\Phi_I) = \mathbb{I}_d - P(\Phi_I)$, where $\mathbb{I}_d$ is the identity operator (matrix) in $\mathbb{R}^d$.

(Ab)using the language of compressed sensing we define $\delta_I(\Phi)$ as the smallest number such that all eigenvalues of $\Phi_I^\star \Phi_I$ are included in $[1 - \delta_I(\Phi), 1 + \delta_I(\Phi)]$ and the **isometry constant** $\delta_S(\Phi)$ of the dictionary as $\delta_S(\Phi) := \max_{|I| \leq S} \delta_I(\Phi)$. When clear from the context we will usually omit the reference to the dictionary. For more details on isometry constants, see for instance [10].

To keep the sub(sub)scripts under control we denote the **indicator function of a set** $\mathcal{V}$ by $\chi(\mathcal{V}, \cdot)$, that is $\chi(\mathcal{V}, v)$ is one if $v \in \mathcal{V}$ and zero else. The set of the first $S$ integers we abbreviate by $\mathbb{S} = \{1, \dots, S\}$.

We define the **distance between two dictionaries** $\Phi, \Psi$ as the maximal distance between two corresponding atoms, i.e.

$$d(\Phi, \Psi) := \max_k \|\phi_k - \psi_k\|_2. \tag{3}$$

We will make heavy use of the following decomposition of a dictionary $\Psi$ into a given dictionary $\Phi$ and a perturbation dictionary $Z$. If $d(\Psi, \Phi) = \varepsilon$ we set $\|\psi_k - \phi_k\|_2 = \varepsilon_k$, where by definition $\max_k \varepsilon_k = \varepsilon$. We can then find unit vectors $z_k$ with $\langle \phi_k, z_k \rangle = 0$ such that

$$\psi_k = \alpha_k \phi_k + \omega_k z_k, \quad \text{for,} \quad \alpha_k := 1 - \varepsilon_k^2/2 \quad \text{and} \quad \omega_k := (\varepsilon_k^2 - \varepsilon_k^4/4)^{\frac{1}{2}}. \tag{4}$$

The dictionary $Z$ collects the perturbation vectors on its columns, that is $Z = (z_1, \dots z_K)$ and we define the diagonal matrices $A_I, W_I$ implicitly via

$$\Psi_I = \Phi_I A_I + Z_I W_I, \tag{5}$$

or in MATLAB notation $A_I = \operatorname{diag}(\alpha_I)$ with $\alpha_I = (\alpha_k)_{k \in I}$ and analogue for $W_I$. Based on this decomposition we further introduce the short hand $b_k = \frac{\omega_k}{\alpha_k} z_k$ and $B_I = Z_I W_I A_I^{-1}$.

We consider a **frame** $F$ a collection of $K \geq d$ vectors $f_k \in \mathbb{R}^d$ for which there exist two positive constants $A, B$ such that for all $v \in \mathbb{R}^d$ we have

$$A\|v\|_2^2 \leq \sum_{k=1}^{K} |\langle f_k, v\rangle|^2 \leq B\|v\|_2^2. \tag{6}$$

If $B$ can be chosen equal to $A$, i.e. $B = A$, the frame is called tight and if all elements of a tight frame have unit norm we have $B = A = K/d$. The operator $FF^\star$ is called frame operator and by (6) its spectrum is bounded by $A, B$. For more details on frames, see e.g. [11].
Finally we introduce the Landau symbols $O, o$ to characterise the growth of a function. We write

$$\begin{aligned} f(t) &= O(g(t)) \quad \text{if} \quad \lim_{t\to 0/\infty} f(t)/g(t) = C < \infty \\ \text{and} \quad f(t) &= o(g(t)) \quad \text{if} \quad \lim_{t\to 0/\infty} f(\varepsilon)/g(\varepsilon) = 0. \end{aligned}$$

## 3 Dictionary Learning via ITKsM

Iterative thresholding and K signal means (ITKsM) for dictionary learning was introduced as algorithm to maximise the $S$-response criterion

$$(P_{R1}) \qquad \max_{\Psi\in\mathcal{D}} \sum_n \max_{|I|=S} \|\Psi_I^\star y_n\|_1, \tag{7}$$

which for $S = 1$ reduces to the K-means criterion, [34]. It belongs to the class of alternating optimisation algorithms for dictionary learning, which alternate between updating the sparse coefficients based on the current version of the dictionary and updating the dictionary based on the current version of the coefficients, [13], [3], [1]. As its name suggests, the update of the sparse coefficients is based on thresholding while the update of the dictionary is based on K signal means.

**Algorithm 3.1** (ITKsM one iteration). *Given an input dictionary $\Psi$ and $N$ training signals $y_n$ do:*

- *For all $n$ find* $I^t_{\Psi,n} = \arg\max_{I:|I|=S} \|\Psi_I^\star y_n\|_1$.
- *For all $k$ calculate*

$$\bar{\psi}_k = \frac{1}{N}\sum_n y_n \cdot \operatorname{sign}(\langle \psi_k, y_n\rangle) \cdot \chi(I^t_{\Psi,n}, k). \tag{8}$$

- *Output* $\bar{\Psi} = (\bar{\psi}_1/\|\bar{\psi}_1\|_2, \ldots, \bar{\psi}_K/\|\bar{\psi}_K\|_2)$.

The algorithm can be stopped after a fixed number of iterations or once a stopping criterion, such as improvement $d(\Psi, \bar{\Psi}) \leq \theta$ for some threshold $\theta$, is reached. Its advantages over most other dictionary learning algorithms are threefold. First it has very low computational complexity. In each step the most costly operation is the calculation of the $N$ matrix vector products $\Psi^\star y_n$, that is the matrix product $\Phi^\star Y$, of order $O(dKN)$. In comparison the globally successful graph clustering algorithms need to calculate the signal correlation matrix $Y^\star Y$, cost $O(dN^2)$.
Second due to its structure only one signal has to be processed at a time. Instead of calculating $I^t_n$ for all $n$ and calculating the sum, one simply calculates $I^t_{\Psi,n}$ for the signal at hand, updates all atoms $\bar{\psi}_k$ for which $k \in I^t_{\Psi,n}$ as $\bar{\psi}_k \to \bar{\psi}_k + y_n \cdot \operatorname{sign}(\langle \psi_k, y_n\rangle)$ and turns to the next signal. Once $N$ signals have been processed one does the normalisation step and outputs $\bar{\Psi}$. Further in this online version only $(2K+1)d$ values corresponding to the input dictionary, the current version of the updated dictionary and the signal at hand, need to be stored rather than the $N \times d$ signal matrix. Parallelisation can be achieved in a similar way. Again for comparison, the graph clustering algorithms, K-SVD, [3], and the alternating minimisation algorithm in [1] need to store the whole signal resp. residual matrix as well as the dictionary.
The third advantage is that with high probability the algorithm converges locally to a generating dictionary $\Phi$ assuming that we have enough training signals and that these follow a sparse random model in $\Phi$. In order to prove the corresponding result we next introduce our sparse signal model.

### 3.1 Signal Model

We employ the same signal model, which has already been used for the analyses of the S-response and K-SVD principles, [35], [34]. Given a $d \times K$ dictionary $\Phi$, we assume that the signals are generated as,

$$y = \frac{\Phi x + r}{\sqrt{1 + \|r\|_2^2}}, \tag{9}$$

where $x$ is drawn from a sign and permutation invariant probability distribution $\nu$ on the unit sphere $S^{K-1} \subset \mathbb{R}^K$ and $r = (r(1) \dots r(d))$ is a centred random subgaussian vector with parameter $\rho$, that is $\mathbb{E}(r) = 0$ and for all vectors $v$ the marginals $\langle v, r \rangle$ are subgaussian with parameter $\rho$, meaning they satisfy $\mathbb{E}(e^{t\langle v,r \rangle}) \leq e^{t^2\rho^2/2}$ for all $t > 0$. We recall that a probability measure $\nu$ on the unit sphere is sign and permutation invariant, if for all measurable sets $\mathcal{X} \subseteq S^{K-1}$, for all sign sequences $\sigma \in \{-1, 1\}^d$ and all permutations $p$ we have

$$\nu(\sigma\mathcal{X}) = \nu(\mathcal{X}), \quad \text{where} \quad \sigma\mathcal{X} := \{(\sigma(1)x(1), \dots, \sigma(K)x(d)) : x \in \mathcal{X}\} \tag{10}$$

$$\nu(p(\mathcal{X})) = \nu(\mathcal{X}), \quad \text{where} \quad p(\mathcal{X}) := \{(x(p(1)), \dots, x(p(K))) : x \in \mathcal{X}\}. \tag{11}$$

We can get a simple example of such a measure by taking a positive, non increasing sequence $c$, that is $c(1) \geq c(2) \geq \ldots \geq c(K) \geq 0$, choosing a sign sequence $\sigma$ and a permutation $p$ uniformly at random and setting $x = x_{p,\sigma}$ with $x_{p,\sigma}(k) = \sigma(k)c(p(k))$. Conversely we can factorise any sign and permutation invariant measure into a random draw of signs and permutations and a measure on the space of non-increasing sequences.
By abuse of notation let $c$ now denote the mapping that assigns to each $x \in S^{K-1}$ the non increasing rearrangement of the absolute values of its components, i.e. $c : x \to c_x$ with $c_x(k) := |x(p(k))|$ for a permutation $p$ such that $|x(p(1))| \geq |x(p(2))| \geq \ldots \geq |x(p(K))| \geq 0$. Then the mapping $c$ together with the probability measure $\nu$ on $x \in S^{K-1}$ induces a probability measure $\nu_c$ on $c(S^{K-1}) = S^{K-1} \cap [0,1]^K$, by $\nu_c(\Omega) := \nu(c^{-1}(\Omega))$ for any measurable set $\Omega \subseteq c(S^{K-1})$.
Using this new measure we can rewrite our signal model as

$$y = \frac{\Phi x_{c,p,\sigma} + r}{\sqrt{1 + \|r\|_2^2}}, \tag{12}$$

where we define $x_{c,p,\sigma}(k) = \sigma(k)c(p(k))$ for a positive, non-increasing sequence $c$ distributed according to $\nu_c$, a sign sequence $\sigma$ and a permutation $p$ distributed uniformly at random and $r$ again a centred random subgaussian vector with parameter $\rho$. Note that we have $\mathbb{E}(\|r\|_2^2) \leq d\rho^2$, with equality for instance in the case of Gaussian noise. To incorporate sparsity into our signal model we make the following definitions.

**Definition 3.1.** *A sign and permutation invariant coefficient distribution $\nu$ on the unit sphere $S^{K-1} \subset \mathbb{R}^K$ is called $S$-sparse with absolute gap $\beta_S > 0$ and relative gap $\Delta_S > \beta_S$, if*

$$\nu\left(c_x(S) - c_x(S+1) > \beta_S\right) = 0 \qquad \textit{and} \qquad \nu\left(\frac{c_x(S) - c_x(S+1)}{c_x(1)} > \Delta_S\right) = 0, \tag{13}$$

*or equivalently*

$$\nu_c\left(c(S) - c(S+1) > \beta_S\right) = 0 \qquad \textit{and} \qquad \nu_c\left(\frac{c(S) - c(S+1)}{c(1)} > \Delta_S\right) = 0. \tag{14}$$

*The coefficient distribution is called strongly $S$-sparse if $\Delta_S \geq 2\mu S$.*

For exactly sparse signals $\beta_S$ is simply the smallest non-zero coefficient and $\Delta_S$ is the inverse dynamic range of the non-zero coefficients. We have the bounds $\beta_S \leq \frac{1}{\sqrt{S}}$ and $\Delta_S \leq 1$. Since equality holds for the 'flat' distribution generated from $c(k) = \frac{1}{\sqrt{S}}$ for $k \leq S$ and zero else, we will usually think of $\beta_S$ being of the order $O(\frac{1}{\sqrt{S}})$ and $\Delta_S$ being of the order $O(1)$. We can also see that coefficient distributions can only

be strongly $S$-sparse as long as $S$ is smaller than $\frac{\Delta_S}{2\mu}$, that is $S = O(\mu^{-1}) = O(\sqrt{d})$.
For the statement of our results we will use three other signal statistics,

$$\gamma_{1,S} := \mathbb{E}_c\left(c(1) + \ldots + c(S)\right) \qquad \gamma_{2,S} := \mathbb{E}_c\left(c^2(1) + \ldots + c^2(S)\right) \qquad C_r := \mathbb{E}_r\left(\frac{1}{\sqrt{1 + \|r\|_2^2}}\right). \tag{15}$$

The constants $\gamma_{1,S}$ and $C_r^2$ will help characterise the expected size of $\bar{\psi}_k$. We have $S\beta_s \leq \gamma_{1,S} \leq \sqrt{S}$ and

$$C_r \geq \frac{1 - e^{-d}}{\sqrt{1 + 5d\rho^2}} \tag{16}$$

compare [34]. From the above inequality we can see that $C_r$ captures the expected signal to noise ratio, that is for large $\rho$ we have

$$C_r^2 \approx \frac{1}{d\rho^2} \approx \frac{\mathbb{E}(\|\Phi x\|_2^2)}{\mathbb{E}(\|r\|_2^2)}. \tag{17}$$

Similarly the constant $\gamma_{2,S}$ can be interpreted as the expected squared energy of the signal approximation using the largest $S$ generating coefficients and the generating dictionary, or in other words $1 - \gamma_{2,S}$ is a bound for the expected squared energy of the approximation error.
For noiseless signals generated from the flat distribution described above we have $\gamma_{1,S} = \sqrt{S}$, $C_r = 1$ and $\gamma_{2,S} = 1$, so we will usually think of these constants having the orders $\gamma_{1,S} = O(\sqrt{S})$, $C_r = O(1)$ and $\gamma_{2,S} = O(1)$.
From the discussion we see that, while being relatively simply, our signal model allows us to capture both approximation error and noise. We will also see that it is straightforward to extend our results to models that include outliers or a portion of signals without gap.

### 3.2 Convergence analysis of ITKsM

To prove that, given a good enough initialisation, both the batch and the online version of ITKsM converge with high probability we first show that with high probability one iteration of ITKsM reduces the error by at least a factor $\kappa < 1$.

**Theorem 3.2.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with coefficients that are $S$-sparse with absolute gap $\beta_S$ and relative gap $\Delta_S$. Fix a target error $\tilde{\varepsilon} \geq 4\varepsilon_{\mu,\rho}$ where the*

$$\varepsilon_{\mu,\rho} := \frac{8K^2\sqrt{B+1}}{C_r\gamma_{1,S}} \exp\left(\frac{-\beta_S^2}{98\max\{\mu^2, \rho^2\}}\right). \tag{18}$$

*Given an input dictionary $\Psi$ such that*

$$d(\Psi, \Phi) \leq \frac{\Delta_S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{1060K^2(B+1)}{\Delta_S C_r \gamma_{1,S}}\right)}\right)}, \tag{19}$$

*the output dictionary $\bar{\Psi}$ of one iteration of ITKsM satisfies*

$$d(\bar{\Psi}, \Phi) \leq 0.83\max\{\tilde{\varepsilon}, d(\Psi, \Phi)\}, \tag{20}$$

*except with probability*

$$\exp\left(\frac{-C_r\gamma_{1,S}N\tilde{\varepsilon}}{120K\sqrt{B+1}}\right) + \exp\left(\frac{-C_r\gamma_{1,S}N\max\{\tilde{\varepsilon}, d(\Psi,\Phi)\}}{60K\sqrt{B+1}}\right) + 2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N\tilde{\varepsilon}^2}{200SK}\right). \tag{21}$$

Before providing the proof let us discuss the result above. We first see that one step of ITKsM will only provide an improvement if the input dictionary is within a radius $O(\Delta_S/\sqrt{\log K})$ to the generating dictionary $\Phi$. In case of exactly sparse signals this means that the convergence radius is up to a log factor

inversely proportional to the dynamic range of the coefficients. This should not be come as a big surprise, considering that the average success of thresholding for sparse recovery with a ground truth dictionary depends on the dynamic range, [36]. It also means that in the best case the convergence radius is actually of size $O(1/\sqrt{\log K})$, since for the flat distribution $\Delta_S = 1$.
Next we want to highlight the relation between the sparsity level and the minimal target error for which one iteration of ITKsM will still provide an improvement. Assume coefficients, that are drawn from the flat distribution, meaning $\beta_S = 1/\sqrt{S}$, white Gaussian noise with variance $\rho^2 = 1/d$, resulting in an expected signal to noise ratio of 1, and an incoherent dictionary with $\mu \leq 1/\sqrt{d}$. If $S \leq \frac{d}{98\ell \log K}$ for some $\ell \geq 2$ then the minimal target error can be as small as $O(K^{2-\ell})$. Again using O-notation and assuming a reasonable signal to noise ratio of 1, we get that the minimal target error scales as $O(K^{2-\ell})$ as long as $S = O(\frac{1}{\mu^2 \ell \log K})$.
Last we want to get feeling for the number of training samples we need to have a good success probability. The determining term in the failure probability bound is the third. So for $\gamma_{1,S} = O(\sqrt{S})$ we get that as soon as $N = O(K \log K \varepsilon^{-2})$ one step of ITKsM is successful with high probability.

*Proof:* The proof is based on the following ideas, compare also [34]: For most sign sequences $\sigma_n$ and therefore most signals $y_n$ thresholding with a perturbation of the original dictionary will still recover the generating support $I_n := p_n^{-1}(\mathbb{S})$, that is $I^t_{\Psi,n} = I_n$. Assuming that the generating support is recovered, for each $k$ the expected difference of the sum in (8) between using the original $\Phi$ and the perturbation $\Psi$ is small, that is smaller than $d(\Phi,\Psi) = \varepsilon$, and due to concentration of measure also the difference on a finite number of samples will be small. Finally for each $k$ the sum in (8) will again concentrate around its expectation, a scaled version of the atom $\phi_k$.
Formally we write,

$$\bar{\psi}_k = \frac{1}{N}\sum_n y_n \operatorname{sign}(\langle \psi_k, y_n\rangle)\,\chi(I^t_{\Psi,n},k) - \frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k) \tag{22}$$

$$+ \frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k) - \mathbb{E}\left(\frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k)\right) + \mathbb{E}\left(\frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k)\right). \tag{23}$$

Since $\mathbb{E}\left(\frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k)\right) = \frac{C_r\gamma_{1,S}}{K}\phi_k$, see the proof of Lemma B.5 in the appendix, using the triangle inequality and the bound $\|y_n\|_2 \leq \sqrt{B+1}$ we get,

$$\begin{aligned}\left\|\bar{\psi}_k - \frac{C_r\gamma_{1,S}}{K}\phi_k\right\|_2 &\leq \left\|\frac{1}{N}\sum_n y_n\left[\operatorname{sign}(\langle \psi_k, y_n\rangle)\,\chi(I^t_{\Psi,n},k) - \sigma_n(k)\,\chi(I_n,k)\right]\right\|_2 \\ &\quad + \left\|\frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k) - \frac{C_r\gamma_{1,S}}{K}\phi_k\right\|_2 \\ &\leq \frac{2\sqrt{B+1}}{N}\,\sharp\{n : \operatorname{sign}(\langle \psi_k, y_n\rangle)\,\chi(I^t_{\Psi,n},k) \neq \sigma_n(k)\,\chi(I_n,k)\} \\ &\quad + \left\|\frac{1}{N}\sum_n y_n\,\sigma_n(k)\,\chi(I_n,k) - \frac{C_r\gamma_{1,S}}{K}\phi_k\right\|_2 \end{aligned} \tag{24}$$

Next note that for the draw of $y_n$ the event that for a given index $k$ the signal coefficient using thresholding with $\Psi$ is different from the oracle signal is contained in the event that thresholding does not recover the entire generating support $I^t_{\Psi,n} \neq I_n$ or that on the generating support the empirical sign pattern using $\Psi$ is different from the generating pattern, $\operatorname{sign}(\langle \psi_k, y_n\rangle) \neq \sigma_n(k)$ for a $k \in I_n$,

$$\{y_n : \operatorname{sign}(\langle \psi_k, y_n\rangle)\,\chi(I^t_{\Psi,n},k) \neq \sigma_n(k)\,\chi(I_n,k)\} \subseteq \{y_n : I^t_{\Psi,n} \neq I_n\} \cup \{y_n : \operatorname{sign}(\Psi^\star_{I_n} y_n) \neq \sigma_n(I_n)\}. \tag{25}$$

From [34], e.g. proof of Proposition 7, we know that the right hand side in (25) is in turn contained in the event $\mathcal{E}_n \cup \mathcal{F}_n$, where

$$\mathcal{E}_n := \Big\{ y_n : \exists k \text{ s.t. } \Big| \sum_{j\neq k} \sigma_n(j) c_n\big(p_n(j)\big) \langle \phi_j, \phi_k \rangle \Big| \geq u_1 \text{ or } |\langle r_n, \phi_k\rangle| \geq u_2 \Big\} \tag{26}$$

$$\mathcal{F}_n := \Big\{ y_n : \exists k \text{ s.t. } \omega_k \Big| \sum_{j} \sigma_n(j) c_n\big(p_n(j)\big) \langle \phi_j, z_k \rangle \Big| \geq u_3 \text{ or } \omega_k |\langle r_n, z_k\rangle| \geq u_4 \Big\} \tag{27}$$

$$\text{for} \quad 2(u_1 + u_2 + u_3 + u_4) \leq c_n(S) \left(1 - \frac{\varepsilon^2}{2}\right) - c_n(S+1). \tag{28}$$

In particular if we choose $u_1 = u_2 = (c_n(S) - c_n(S+1))/7$, $u_3 = u_1 - \frac{\varepsilon^2 c_n(S)}{6}$ and $u_4 = u_3/2$ we get that $\mathcal{E}_n$, which contains the event that thresholding using the generating dictionary $\Phi$ fails, is independent of $\Psi$. To estimate the number of signals for which the thresholding summand is different from the oracle summand, it suffices to count how often $y_n \in \mathcal{E}_n$ or $y_n \in \mathcal{F}_n$,

$$\sharp\{n : \operatorname{sign}(\langle \psi_k, y_n\rangle)\, \chi(I^t_{\Psi,n}, k)\} \leq \sharp\{n : y_n \in \mathcal{E}_n\} + \sharp\{n : y_n \in \mathcal{F}_n\}. \tag{29}$$

Substituting these bounds into (24) we get,

$$\begin{aligned} \left\| \bar\psi_k - \frac{C_r \gamma_{1,S}}{K} \phi_k \right\|_2 \leq \frac{2\sqrt{B+1}}{N} \sharp\{n : y_n \in \mathcal{E}_n\} + \frac{2\sqrt{B+1}}{N} \sharp\{n : y_n \in \mathcal{F}_n\} \\ + \left\| \frac{1}{N} \sum_n y_n\, \sigma_n(k)\, \chi(I_n, k) - \frac{C_r \gamma_{1,S}}{K} \phi_k \right\|_2 . \end{aligned} \tag{30}$$

If we want the error between $\bar\psi_k / \|\bar\psi\|_2$ and $\phi_k$ to be of the order $\kappa\varepsilon$, we need to ensure that the right hand side of (30) is less than $\kappa\varepsilon \cdot \frac{C_r \gamma_{1,S}}{K}$.
From Lemma B.3 in the appendix we know that

$$\mathbb{P}\left( \sharp\{n : y_n \in \mathcal{E}_n\} \geq \frac{C_r \gamma_{1,S} N}{2K\sqrt{B+1}} \cdot (\varepsilon_{\mu,\rho} + t_1) \right) \leq \exp\left( \frac{-t_1^2 C_r \gamma_{1,S} N}{2K\sqrt{B+1}\,(2\varepsilon_{\mu,\rho} + t_1)} \right). \tag{31}$$

Next Lemma B.4 tells us that

$$\mathbb{P}\left( \sharp\{n : y_n \in \mathcal{F}_n\} \geq \frac{C_r \gamma_{1,S} N}{2K\sqrt{B+1}} \cdot (\tau\varepsilon + t_2) \right) \leq \exp\left( \frac{-t_2^2 C_r \gamma_{1,S} N}{2K\sqrt{B+1}\,(2\tau\varepsilon + t_2)} \right), \tag{32}$$

whenever

$$\varepsilon \leq \frac{\Delta_S}{\sqrt{98B}\left( \frac{1}{4} + \sqrt{\log\left( \frac{106 K^2 (B+1)}{\Delta_S C_r \gamma_{1,S} \tau} \right)} \right)}. \tag{33}$$

Finally by Lemma B.5 we have

$$\mathbb{P}\left( \left\| \frac{1}{N} \sum_n \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1 + \|r_n\|_2^2}} \cdot \sigma_n(k) \cdot \chi(I_n, k) - \frac{C_r \gamma_{1,S}}{K} \phi_k \right\|_2 \geq t_3 \frac{C_r \gamma_{1,S}}{K} \right) \leq \exp\left( \frac{-t_3^2 C_r^2 \gamma_{1,S}^2 N}{8SK} + \frac{1}{4} \right), \tag{34}$$

whenever $0 \leq t_3 \leq \frac{\sqrt{S}}{\sqrt{B+2}}$. Thus with high probability we have,

$$\left\| \bar\psi_k - \frac{C_r \gamma_{1,S}}{K} \phi_k \right\|_2 \leq \frac{C_r \gamma_{1,S}}{K} \left( \varepsilon_{\mu,\rho} + t_1 + \tau\varepsilon + t_2 + t_3 \right). \tag{35}$$

To be more precise if we choose a target error $\tilde\varepsilon \geq 4\varepsilon_{\mu,\rho}$ and set $t_1 = \tilde\varepsilon/10$, $t_2 = \max\{\tilde\varepsilon, \varepsilon\}/10$, $\tau = 1/10$ and $t_3 = \tilde\varepsilon/5$, then except with probability

$$\exp\left( \frac{-C_r \gamma_{1,S} N \tilde\varepsilon}{120K\sqrt{B+1}} \right) + \exp\left( \frac{-C_r \gamma_{1,S} N \max\{\tilde\varepsilon, \varepsilon\}}{60K\sqrt{B+1}} \right) + 2K \exp\left( \frac{-C_r^2 \gamma_{1,S}^2 N \tilde\varepsilon^2}{200SK} \right) \tag{36}$$

we have

$$\max_k \left\| \bar{\psi}_k - \frac{C_r \gamma_{1,S}}{K} \phi_k \right\|_2 \leq \frac{C_r \gamma_{1,S}}{K} \cdot \frac{3}{4} \cdot \max\{\tilde{\varepsilon}, \varepsilon\}. \tag{37}$$

By Lemma B.10 this further implies that

$$d(\bar{\Psi}, \Phi) = \max_k \left\| \frac{\bar{\psi}_k}{\|\bar{\psi}_k\|_2} - \phi_k \right\|_2 \leq 0.83 \max\{\tilde{\varepsilon}, \varepsilon\}. \tag{38}$$

□

For most desired precisions repeated application of Theorem 3.4, which is valid for a quite large hypercube of input dictionaries and a wide range of sparsity levels, will actually be sufficient. However, for completeness we specialise the theorem above to the case of strongly S-sparse, noiseless signals, in which case we can get ITKsM to make improvements to input dictionaries with arbitrarily small errors, provided enough samples.

**Corollary 3.3.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with $r = 0$ and coefficients that are strongly $S$-sparse with relative gap $\Delta_S > 2\mu S$. Fix a target error $\tilde{\varepsilon} \geq 0$. If for the input dictionary $\Psi$ we have*

$$d(\Psi, \Phi) \leq \frac{\Delta_S - 2\mu S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{1060K^2B}{(\Delta_S - 2\mu S)\gamma_{1,S}}\right)}\right)}, \tag{39}$$

*then the output dictionary $\bar{\Psi}$ of one iteration of ITKsM satisfies*

$$d(\bar{\Psi}, \Phi) \leq 0.83 \max\{\tilde{\varepsilon}, d(\Psi, \Phi)\}, \tag{40}$$

*except with probability*

$$\exp\left(\frac{-\gamma_{1,S} N \max\{\tilde{\varepsilon}, d(\Psi, \Phi)\}}{60K\sqrt{B}}\right) + 2K \exp\left(\frac{-\gamma_{1,S}^2 N \tilde{\varepsilon}^2}{200SK}\right).$$

The proof is analogue to the one of Theorem 3.2 and can be found in Appendix A.1.

Let us again discuss the result. The main difference to Theorem 3.2 is that the condition $\Delta_S \geq 2\mu S$ can only hold for much lower sparsity levels, that is $S = O(\mu^{-1})$ or up to the square root of the ambient dimension $O(\sqrt(d)) \ll O(d/\log K)$ for incoherent dictionaries. It is also no surprise that once the input dictionary is up to a log factor within this radius, ITKsM can always provide an improvement given enough samples, considering that $\Delta_S \geq 2\mu S$ guarantees the success of thresholding for sparse recovery with a ground truth dictionary, [36].

We will now iterate the results above to prove our main theorem for ITKsM.

**Theorem 3.4.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with coefficients that are $S$-sparse with absolute gap $\beta_S$ and relative gap $\Delta_S$. Fix a target error $\tilde{\varepsilon} \geq 4\varepsilon_{\mu,\rho}$, compare* (18). *Given an input dictionary $\Psi$ such that*

$$d(\Psi, \Phi) \leq \frac{\Delta_S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{1060K^2(B+1)}{\Delta_S C_r \gamma_{1,S}}\right)}\right)}, \tag{41}$$

*then after $6\lceil\log(\tilde{\varepsilon}^{-1})\rceil$ iterations the output dictionary $\tilde{\Psi}$ of ITKsM in its batch resp. online version satisfies*

$$d(\tilde{\Psi}, \Phi) \leq \tilde{\varepsilon} \tag{42}$$

*except with probability*

$$3K \exp\left(\frac{-C_r^2 \gamma_{1,S}^2 N \tilde{\varepsilon}^2}{200SK}\right) \qquad \textit{resp.} \qquad 18\lceil\log(\tilde{\varepsilon}^{-1})\rceil K \exp\left(\frac{-C_r^2 \gamma_{1,S}^2 N \tilde{\varepsilon}^2}{200SK}\right).$$

The proof consists of iterative application of the results for one step, taking into account which probability estimates depend on the current iteration for the batch resp. online version, and can be found in Appendix A.2.

While the result again has the advantage of being plug-and-play, all the constants make it rather messy. To get a better feeling for its quality we will restate it in O-notation using the conventions explained in Subsection 3.1.
Assuming the number of training samples $N$ scales as $O(K \log K \tilde{\varepsilon}^{-2})$. If $\frac{d}{98S\log K} = \ell \geq 2$ then with high probability for any starting dictionary $\Psi$ within distance $\varepsilon \leq O(1/\sqrt{\log K})$ to the generating dictionary after $O(\log(\tilde{\varepsilon}^{-1})$ iterations of ITKsM the distance of the output dictionary $\bar{\Psi}$ to the generating dictionary will be smaller than

$$\max\left\{\tilde{\varepsilon}, O\left(K^{2-\ell}\right)\right\}. \tag{43}$$

Again we specialise our result to noiseless signals.

**Corollary 3.5.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in (12) with $r = 0$ and coefficients that are strongly $S$-sparse with relative gap $\Delta_S > 2\mu S$. Fix a target error $\tilde{\varepsilon} \geq 0$. If for the input dictionary $\Psi$ we have*

$$d(\Psi, \Phi) \leq \frac{\Delta_S - 2\mu S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{1060K^2B}{(\Delta_S - 2\mu S)\gamma_{1,S}}\right)}\right)}, \tag{44}$$

*then after $6\lceil\log(\tilde{\varepsilon}^{-1})\rceil$ iterations the output dictionary $\tilde{\Psi}$ of ITKsM in its batch resp. online version satisfies*

$$d(\tilde{\Psi}, \Phi) \leq \tilde{\varepsilon} \tag{45}$$

*except with probability*

$$3K \exp\left(\frac{-\gamma_{1,S}^2 N\tilde{\varepsilon}^2}{200SK}\right) \qquad \textit{resp.} \qquad 18\log(\tilde{\varepsilon}^{-1})K\exp\left(\frac{-\gamma_{1,S}^2 N\tilde{\varepsilon}^2}{200SK}\right).$$

To turn the corollary into something less technical and more interesting we have to combine it with the corresponding theorem. If the coefficients are strongly $S$-sparse the minimally achievable error using Theorem 3.4 will be smaller than the error we need for Corollary 3.5 to take over and we get the following O notation result. Assuming the number of noiseless exactly S-sparse training samples $N$ scales as $O(K \log K \tilde{\varepsilon}^{-2})$. If $S \leq O(\sqrt{d})$ then with high probability for any starting dictionary $\Psi$ within distance $\varepsilon \leq O(1/\sqrt{\log K})$ to the generating dictionary after $O(\log(\tilde{\varepsilon}^{-1})$ iterations of ITKsM the distance of the output dictionary $\bar{\Psi}$ to the generating dictionary will be smaller than $\tilde{\varepsilon}$.
While for ITKsM a convergence radius of around $1/\sqrt{\log K}$, admissible sparsity levels up to $d/\log K$ and a dependence on the sample complexity of only $K \log K$ is very positive, the dependence of the sample complexity on the squared inverse target error $\tilde{\varepsilon}^{-2}$ is somewhat disappointing. Looking at the proof of Theorem 3.2 we see that the reason for this factor is the slow concentration of the sums $\frac{1}{N}\sum_n y_n\, \sigma_n(k)\, \chi(I_n, k)$ around the atom $\phi_k$, which can in turn be explained by the fact that we have to cancel out the equally sized contribution of all other atoms. Actively trying to cancel out these contributions before the summation, that is summing residuals instead of signals, should therefore accelerate the concentration, and lead to a lower sample complexity. We will concretise these ideas in the next section.

## 4 Dictionary Learning via ITKrM

There are several ways to remove the contribution of all atoms in the current support $I^t_{\Psi,n}$ except for $\psi_k$. The maybe most obvious way is to consider $P(\Psi_{I^t_{\Psi,n}/k})y_n$. Unfortunately this residual has several disadvantages, the most severe being that it is not clear whether for the oracle residuals $P(\Phi_{I_n/k})y_n$ and oracle signs the corresponding sum concentrates around a multiple of the $\phi_k$,

$$\mathbb{E}\left(\frac{1}{N}\sum_n P(\Phi_{I_n/k})y_n \cdot \sigma_n(k)\cdot \chi(I_n,k)\right) \propto \mathbb{E}_{I:k\in I}\left(P(\Phi_{I/k})\right)\,\phi_k \overset{?}{\propto} \gamma\phi_k. \tag{46}$$

We suspect that equality can only hold for tight dictionaries and that an additional straight such as minimally incoherent is needed. We therefore choose a perhaps less obvious but more stable residual $r_{n,k}(\Psi) = y_n - P(\Psi_{I^t_{\Psi,n}})y_n + P(\psi_k)y_n$, which captures the contribution of the current atom $\phi_k$ as well as the approximation error in $\Psi$, that is $y_n - P(\Psi_{I^t_{\Psi,n}})y_n$. Replacing the signal means in ITKsM with residual means we arrive at the new algorithm, iterative thresholding and K residual means (ITKrM).

**Algorithm 4.1** (ITKrM one iteration)**.** *Given an input dictionary $\Psi$ and $N$ training signals $y_n$ do:*

- *For all $n$ find $I^t_{\Psi,n} = \arg\max_{I:|I|=S}\|\Psi_I^\star y_n\|_1$.*
- *For all $k$ calculate*

$$\bar{\psi}_k = \frac{1}{N}\sum_n \left[y_n - P(\Psi_{I^t_{\Psi,n}})y_n + P(\psi_k)y_n\right]\cdot \operatorname{sign}(\langle \psi_k, y_n\rangle)\cdot \chi(I^t_{\Psi,n},k). \tag{47}$$

- *Output $\bar{\Psi} = (\bar{\psi}_1/\|\bar{\psi}_1\|_2,\dots,\bar{\psi}_K/\|\bar{\psi}_K\|_2)$.*

Again ITKrM inherits most computational properties of ITKsM. As such it can again be stopped after a fixed number of iterations or once a stopping criterion, such as improvement below some threshold, is reached. Only one signal has to be processed at a time, making it suitable for an online version and parallelisation. Its computational complexity is slightly larger than for ITKsM because of the projections $P(\Psi_{I^t_{\Psi,n}})y_n$, which have an overall cost of $O(S^2dN)$ and so for $S \geq \sqrt{d}$ become the determining factor - $S$ can again be of the order $O(\mu^{-2}/\log K) \approx O(d/\log K)$. In the next subsection we will analyse which of the convergence properties of ITKsM translate to ITKrM.

### 4.1 Convergence Analysis of ITKrM

As for ITKsM in order to prove convergence of ITKrM we start by showing that with high probability one iteration of ITKrM will reduce the error by at least a factor $\kappa < 1$. We focus on the more realistic case of non exactly S-sparse and/or relatively noisy signals. For comparison to other work we will later specialise our results to exactly S-sparse, noiseless signals and moreover the case where $S \leq O(\mu^{-1})$.

**Theorem 4.2.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with coefficients that are $S$-sparse with absolute gap $\beta_S$ and relative gap $\Delta_S$. Assume further that $S \leq \frac{K}{98B}$ and $\varepsilon_\delta := K\exp\left(-\frac{1}{4741\mu^2 S}\right) \leq \frac{1}{24(B+1)}$. Fix a target error $\tilde{\varepsilon} \geq 8\varepsilon_{\mu,\rho}$, with $\varepsilon_{\mu,\rho}$ as defined in* (18)*, and assume that $\tilde{\varepsilon} \leq 1-\gamma_{2,S} + d\rho^2$, ie. the target error is smaller than the expected noisepower and approximation error*[1]*. If for the input dictionary $\Psi$ we have $d(\Psi,\Phi) \leq \frac{1}{32\sqrt{S}}$ and*

$$d(\Psi,\Phi) \leq \frac{\Delta_S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{2544K^2(B+1)}{\Delta_S C_r \gamma_{1,S}}\right)}\right)} \tag{48}$$

*then the output dictionary $\bar{\Psi}$ of one iteration of ITKsM satisfies*

$$d(\bar{\Psi},\Phi) \leq 0.92\max\{\tilde{\varepsilon}, d(\Psi,\Phi)\}, \tag{49}$$

1. In case the reversed inequality holds we get the same result but with better probability estimates. To get an idea of these improved probability estimates see the Corollary 4.3 below.

*except with probability*

$$\exp\left(\frac{-C_r\gamma_{1,S}N\tilde{\varepsilon}}{336\,K\sqrt{B+1}}\right)+\exp\left(\frac{-C_r\gamma_{1,S}N\max\{\tilde{\varepsilon},\varepsilon\}}{144\,K\sqrt{B+1}}\right)+K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N}{K(5103+34\,C_r\gamma_{1,S}\sqrt{B+1})}\right)$$
$$+2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N\tilde{\varepsilon}^2}{512K\max\{S,B+1\}\,(1-\gamma_{2,S}+d\rho^2)}\right)+2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N\max\{\tilde{\varepsilon},\varepsilon\}^2}{576K\max\{S,B+1\}\,(\varepsilon+1-\gamma_{2,S}+d\rho^2)}\right).$$

*In case $\tilde{\varepsilon}\geq\varepsilon_\delta$, e.g. because $\beta_S\leq\frac{1}{7\sqrt{S}}$, the last term in the sum above reduces to*

$$2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N\max\{\tilde{\varepsilon},\varepsilon\}}{576K\max\{S,B+1\}\,(2-\gamma_{2,S}+d\rho^2)}\right). \tag{50}$$

*Proof:* The proof follows the same idea as the proof of Theorem (3.2). First we check how often thresholding with $\Psi$ fails. Assuming thresholding recovers the generating support we show that the difference of the residuals using $\Phi$ or $\Psi$ concentrates around its expectation, which is small. Finally we show that the sum of residuals using $\Phi$ converges to a scaled version of $\phi_k$. To keep the flow of the paper we do not give the full proof here but in Appendix A.3 □

The perhaps most disappointing fact about ITKrM compared to ITKsM is that the convergence radius decreases to $O(1/\sqrt{S})$. The reason for this is revealed in the proof of Lemma B.8, where we see that the expected difference between $R^o(\Psi,y_n,k)$ and $R^o(\Phi,y_n,k)$ depends on the operator norms of the rescaled perturbation matrices $\|B_I\|_{2,2}$. If the perturbation dictionary is quasi constant, that is before normalisation $z_k=v-P(\phi_k)v$ for some $v\neq 0$, then $\|B_I\|_{2,2}\approx\sqrt{S}\varepsilon$ for all $I$, so we need $\varepsilon\leq 1\sqrt{S}$.
The advantage over ITKrM is that for exactly sparse signals we save a factor $1/d\rho^2$ in the exponents of the failure bound containing $\varepsilon^{-2}$. From this we can already guess that for exactly sparse, noiseless signals we can actually reduce the $\varepsilon^{-2}$ in these exponents to $\varepsilon^{-1}$. We have the following corollary.

**Corollary 4.3.** *Let $\Phi$ be a unit norm frame with frame constants $A\leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with $r=0$ and coefficients that are exactly and strongly $S$-sparse with relative gap $\Delta_S>2\mu S$. Fix a target precision $\tilde{\varepsilon}>0$. If for the input dictionary $\Psi$ we have $d(\Psi,\Phi)\leq\frac{1}{32\sqrt{S}}$ and*

$$d(\Psi,\Phi)\leq\frac{\Delta_S-2\mu S}{\sqrt{12}\left(\frac{1}{4}+\sqrt{\log\left(\frac{23K^2\sqrt{B}}{(\Delta_S-2\mu S)\gamma_{1,S}}\right)}\right)}, \tag{51}$$

*then the output dictionary $\bar{\Psi}$ of one iteration of ITKrM satisfies*

$$d(\bar{\Psi},\Phi)\leq 0.92\max\{\tilde{\varepsilon},d(\Psi,\Phi)\}, \tag{52}$$

*except with probability*

$$\exp\left(\frac{-\gamma_{1,S}N\max\{\tilde{\varepsilon},\varepsilon\}}{96\,K\sqrt{B}}\right)+K\exp\left(\frac{-\gamma_{1,S}N}{2830K\sqrt{B})}\right)+2K\exp\left(\frac{-\gamma_{1,S}^2N}{8192K\max\{S,B\}}\right). \tag{53}$$

The proof sketch can be found in the Appendix A.4.

The above corollary clearly reveals the influence of the underlying signal model on dictionary learning results. So assuming that the signals are noiseless and exactly sparse and that $S$ is only of the order $O(\mu^{-1})=O(\sqrt{d})$, we get that one iteration of ITKrM will reduce the error as long as the number of samples scales as $O(K\varepsilon^{-1})$, meaning the influence of the target error is reduced by a factor $\varepsilon^{-1}$!
Unlike before for ITKrM the size of the hyper-cube of dictionaries for which we can expect improvement is not strongly influenced by the presence of noise since the constraint $1/\sqrt{S}$ will usually be stronger than those in (54)/(58).
We again iterate both results to arrive at the corresponding convergence results for ITKrM.

**Theorem 4.4.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with coefficients that are $S$-sparse with absolute gap $\beta_S$ and relative gap $\Delta_S$. Assume further that $S \leq \frac{K}{98B}$ and $\varepsilon_\delta := K \exp\left(-\frac{1}{4741\mu^2 S}\right) \leq \frac{1}{24(B+1)}$. Fix a target error $\tilde{\varepsilon} \geq 8\varepsilon_{\mu,\rho}$, with $\varepsilon_{\mu,\rho}$ as defined in* (18)*, and assume that $\tilde{\varepsilon} \leq 1 - \gamma_{2,S} + d\rho^2$. If for the input dictionary $\Psi$ we have $d(\Psi, \Phi) \leq \frac{1}{32\sqrt{S}}$ and*

$$d(\Psi, \Phi) \leq \frac{\Delta_S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{2544K^2(B+1)}{\Delta_S C_r \gamma_{1,S}}\right)}\right)} \tag{54}$$

*then after $12\lceil \log(\tilde{\varepsilon}^{-1})\rceil$ iterations the output dictionary $\tilde{\Psi}$ of ITKrM both in its batch and online version satisfies*

$$d(\bar{\Psi}, \Phi) \leq \tilde{\varepsilon} \tag{55}$$

*except with probability*

$$60\lceil \log(\tilde{\varepsilon}^{-1})\rceil K \exp\left(\frac{-C_r^2 \gamma_{1,S}^2 N \tilde{\varepsilon}^2}{576K \max\{S, B+1\}\left(\tilde{\varepsilon} + 1 - \gamma_{2,S} + d\rho^2\right)}\right). \tag{56}$$

Because of the reduced convergence radius we need to combine the result above with the results for ITKsM to arrive at a useful result. That is we first need to exploit the large convergence radius of ITKsM and run ITKsM to arrive at an error $O(1/\sqrt{S}$. Then we exploit the lower sample complexity of ITKrM to arrive at the target precision.
Assuming the number of training samples $N$ scales as $O(K \log K \tilde{\varepsilon}^{-2})$. If $\frac{d}{98S \log K} = \ell \geq 2$ then with high probability for any starting dictionary $\Psi$ within distance $\varepsilon \leq O(1/\sqrt{\log K})$ to the generating dictionary after $O(\log(S))$ iterations of ITKsM and $O(\log(\tilde{\varepsilon}^{-1}))$ iterations of ITKrM the distance of the output dictionary $\bar{\Psi}$ to the generating dictionary will be smaller than

$$\max\left\{\tilde{\varepsilon}, O\left(K^{2-\ell}\right)\right\}. \tag{57}$$

Unfortunately the slightly lower sample complexity of ITKrM in the case of noisy signals disappears in the O notation and we cannot really see the improvement over ITKrM. We therefore specialise again to noiseless exact sparse signals.

**Corollary 4.5.** *Let $\Phi$ be a unit norm frame with frame constants $A \leq B$ and coherence $\mu$ and assume that the $N$ training signals $y_n$ are generated according to the signal model in* (12) *with $r = 0$ and coefficients that are exactly and strongly $S$-sparse with relative gap $\Delta_S > 2\mu S$. Fix a target precision $\tilde{\varepsilon} > 0$. If for the input dictionary $\Psi$ we have $d(\Psi, \Phi) \leq \frac{1}{32\sqrt{S}}$ and*

$$d(\Psi, \Phi) \leq \frac{\Delta_S - 2\mu S}{\sqrt{12}\left(\frac{1}{4} + \sqrt{\log\left(\frac{23K^2\sqrt{B}}{(\Delta_S - 2\mu S)\gamma_{1,S}}\right)}\right)}, \tag{58}$$

*then after $12\lceil \log(\tilde{\varepsilon}^{-1})\rceil$ iterations the output dictionary $\tilde{\Psi}$ of ITKrM both in its batch resp. online version satisfies*

$$d(\bar{\Psi}, \Phi) \leq \tilde{\varepsilon}$$

*except with probability*

$$36K\lceil \log(\tilde{\varepsilon}^{-1})\rceil \exp\left(\frac{-\gamma_{1,S} N \tilde{\varepsilon}}{96\, K\sqrt{B}}\right) \quad \textit{resp.} \quad 36\lceil \log(\tilde{\varepsilon}^{-1})\rceil \exp\left(\frac{-\gamma_{1,S} N \tilde{\varepsilon}}{96\, K\sqrt{B}}\right) \quad \textit{in case} \quad \tilde{\varepsilon} \leq \frac{1}{86S \log K}.$$

Again combining with ITKsM we get the following quantitative results. Assuming the number of noiseless exactly S-sparse training samples $N$ scales as $O(K \log K \tilde{\varepsilon}^{-1})$. If $S \leq O(\sqrt{d})$ then with high probability for any starting dictionary $\Psi$ within distance $\varepsilon \leq O(1/\sqrt{\log K})$ to the generating dictionary after $O(\log(S))$ iterations of ITKsM and $O(\log(\tilde{\varepsilon}^{-1}))$ iterations of ITKrM the distance of the output dictionary $\bar{\Psi}$ to the generating dictionary will be smaller than $\tilde{\varepsilon}$.

We now turn to a final discussion of our results.

## 5 Discussion

We have shown that iterative thresholding and K-means is a very attractive local dictionary learning method, since it has low computational complexity $O(dKN)$ omitting log factors, can be use in parallel or online, has a convergence radius $O(1/\sqrt{logK})$ and a sample complexity $O(K\log K\varepsilon^{-2}$ for a target error $\varepsilon$ and reduces to $O(K\log K\varepsilon^{-1}$ in the case of noiseless exactly sparse signals. Further to the best of our knowledge it is the only algorithm for learning overcomplete dictionaries, that is proven to be (locally) stable for sparsity ranges up to a log factor of the ambient dimension - that is recovery up to a target error $K^{-\ell}$ for sparsity levels $S = O(\mu^{-2}/(\ell\log K)) = O(d/(\ell\log K)$. As such it improves on the most closely related results in [1] in terms of computational efficiency, convergence radius and admissible sparsity level. In the case of noiseless signals, which is the only valid regime for [1], the sample complexity increases by a factor $\varepsilon^{-1}$. However, note that in case of noisy signals the dependence of the sample complexity on the inverse squared target error $\varepsilon^{-2}$ is optimal, [21].

One disappointment of the results is that the computationally more involved version using residuals ITKrM has only a convergence radius of the order $O(1/\sqrt{S}$. One possibility to increase this radius to $O(1/\log K)$ is to make an additional assumption on the perturbation dictionary, that is the normalised difference between the input and the generating dictionary, such as a flat spectrum, and show that most perturbations satisfy this additional assumption.

Another weak point, hidden in the O notation is, that both the convergence radius and implicitly also the limiting precision decrease with the dynamic range of the coefficients. This seems unavoidable since the success of thresholding depends on the dynamic range. So while we could improve our results to depend on an average dynamic range instead of the worst dynamic range by assuming a probability distribution on the dynamic range in our proofs, this average dynamic range will remain a limitation. To remove the dependence on the dynamic range we would have to replace thresholding by another sparse approximation method such as (Orthogonal) Matching Pursuit or Basis Pursuit, which is used in [1]. However, the only method that is known to be on average stable for sparsity levels $S \geq \sqrt{d}$ is Basis Pursuit, [40], and it will need some work to extend the corresponding results to perturbed dictionaries, noise and approximation error. An alternative, maybe less daunting strategy we are interested in is to extend the stability results for thresholding to iterative (hard) thresholding, [8], [9].

Another important question is how to find an initialisation within the convergence radius. The results for the strategy in [5] are only valid for sparsity levels $S \leq O(\mu^{-1})$ and it is an open question whether they extend to $S = O(\mu^{-2}/\log K)$. A complementary approach we are actively pursuing is based on the earlier mentioned additional assumptions. If the perturbation dictionary not only has a flat spectrum but is incoherent to the generating dictionary we expect one step of ITKrM to reduce the perturbation sizes but to keep the perturbation directions roughly the same. Estimating the volume of 'good' perturbations we could then calculate the probability that a random initialisation is successful. Finally we also want to investigate wether the convergence of ITKM can be accelerated using weighted instead of signed signal/residual means.

## Acknowledgements

This work was supported by the Austrian Science Fund (FWF) under Grant no. J3335.

Thanks also go to the Computer Vision Laboratory of the University of Sassari, Italy, which provided the surroundings, where the inspirational part of the presented work was done.

# Appendix A
# Proof Sketches

## A.1 Proof of Theorem 3.3

The proof is analogue to the one of Theorem 3.2. We only need to take into account that without noise $C_r = 1$ and in all estimates the constant $B+1$ can be replaced by B. Further since the coefficients are strongly $S$-sparse, thresholding using the generating dictionary $\Phi$ will always (almost surely) recover the generating support with a margin $u_s \geq (\Delta - 2\mu S)c_n(1)$, that is $\min_{k\in I_n} |\langle \phi_k, y_n\rangle| \geq \max_{k\notin I_n} |\langle \phi_k, y_n\rangle| + u_s$, compare [34]. Therefore the event that thresholding using $\Psi$ fails or that the empirical signs differ from the generating ones is contained in

$$\mathcal{F}_n^s := \Big\{ y_n : \exists k \text{ s.t. } \omega_k \Big| \sum_j \sigma_n(j) c_n\big(p_n(j)\big) \langle \phi_j, z_k\rangle \Big| \geq \frac{u_s - \frac{\varepsilon^2}{2\sqrt{S}}}{2} \Big\} \tag{59}$$

and we get

$$\Big\| \bar\psi_k - \frac{\gamma_{1,S}}{K}\phi_k \Big\|_2 \leq \frac{2\sqrt{B}}{N} \sharp\{n : y_n \in \mathcal{F}_n^s\} + \Big\| \frac{1}{N}\sum_n y_n\, \sigma_n(k)\, \chi(I_n,k) - \frac{\gamma_{1,S}}{K}\phi_k \Big\|_2, \tag{60}$$

which can be estimated as before.

## A.2 Proof of Theorem 3.4

In each iteration the error will either be decreased by at least a factor $0.83$ or if its already below $\tilde\varepsilon$ will stay below $\tilde\varepsilon$. So after $i$ iterations $d(\tilde\Psi, \Phi) \leq \max\{\tilde\varepsilon, 0.83^i d(\Psi,\Phi)\} \leq \max\{\tilde\varepsilon, 0.83^i\}$ and for $i = 6\lceil \log(\tilde\varepsilon^{-1})\rceil$ we have $0.83^i \leq \tilde\varepsilon$. In the batch version of ITKsM the only probability estimate that depends on the current version of the dictionary $\bar\Psi$ is

$$\mathbb{P}\left( \sharp\{n : y_n \in \mathcal{F}_n\} \geq \frac{C_r\gamma_{1,S}N}{2K\sqrt{B+1}} \cdot \frac{\varepsilon + \max\{\tilde\varepsilon,\varepsilon\}}{10} \right) \leq \exp\left( \frac{-C_r\gamma_{1,S}N\max\{\tilde\varepsilon,\varepsilon\}}{60K\sqrt{B+1}} \right) \leq \exp\left( \frac{-C_r\gamma_{1,S}N\tilde\varepsilon}{60K\sqrt{B+1}} \right).$$

Taking a union bound over all $i$ dictionaries $\bar\Psi$ we can bound the failure probability of the batch version as

$$\exp\left( \frac{-C_r\gamma_{1,S}N\tilde\varepsilon}{120K\sqrt{B+1}} \right) + 6\lceil\log(\tilde\varepsilon^{-1})\rceil \exp\left( \frac{-C_r\gamma_{1,S}N\tilde\varepsilon}{60K\sqrt{B+1}} \right) + 2K\exp\left( \frac{-C_r^2\gamma_{1,S}^2N\tilde\varepsilon^2}{200SK} \right). \tag{61}$$

Since the last term in the sum above dominates the other two we get the final bound.
In the online version of ITKsM in each of the $i$ iterations we have a new batch of $N$ signals, so all probability estimates depend on the current batch. Taking a union bound over the number of iterations and taking into account that the failure probability of one iteration is bounded by $3K\exp\left(\frac{-C_r^2\gamma_{1,S}^2N\tilde\varepsilon^2}{200SK}\right)$ leads to the final estimate.

## A.3 Proof of Theorem 4.2

As already mentioned the proof follows the same ideas as the proof of Theorem (3.2). First we check how often thresholding with $\Psi$ fails. Assuming thresholding recovers the generating support we show that the difference of the residuals using $\Phi$ or $\Psi$ concentrates around its expectation, which is small. Finally we show that the sum of residuals using $\Phi$ converges to a scaled version of $\phi_k$. To make the ideas precise we first define the thresholding residual based on $\Psi$

$$R^t(\Psi, y_n, k) := \big[y_n - P(\Psi_{I^t_{\Psi,n}})y_n + P(\psi_k)y_n\big] \cdot \operatorname{sign}(\langle\psi_k, y_n\rangle) \cdot \chi(I^t_{\Psi,n}, k) \tag{62}$$

and the oracle residual based on the generating support $I_n = p_n^{-1}(\mathbb{S})$ and $\Psi$.

$$R^o(\Psi, y_n, k) := \big[y_n - P(\Psi_{I_n})y_n + P(\psi_k)y_n\big] \cdot \sigma_n(k) \cdot \chi(I_n, k). \tag{63}$$

We can now write,

$$
\begin{aligned}
\bar{\psi}_k &= \frac{1}{N}\sum_n \left[R^t(\Psi, y_n, k) - R^o(\Psi, y_n, k)\right] + \frac{1}{N}\sum_n \left[R^o(\Psi, y_n, k) - R^o(\Phi, y_n, k)\right] + \frac{1}{N}\sum_n R^o(\Phi, y_n, k) \\
&= \frac{1}{N}\sum_n \left[R^t(\Psi, y_n, k) - R^o(\Psi, y_n, k)\right] + \frac{1}{N}\sum_n \left[R^o(\Psi, y_n, k) - R^o(\Phi, y_n, k)\right] \\
&\qquad + \frac{1}{N}\sum_n \left[y_n - P(\Phi_{I_n})y_n\right]\cdot \sigma_n(k)\cdot\chi(I_n, k) + \left(\frac{1}{N}\sum_n \langle y_n, \phi_k\rangle \cdot \sigma_n(k)\cdot \chi(I_n,k)\right)\phi_k \qquad (64)
\end{aligned}
$$

Abbreviating $s_k = \frac{1}{N}\sum_n \langle y_n, \phi_k\rangle \cdot \sigma_n(k)\cdot\chi(I_n,k)$ we get

$$
\begin{aligned}
\|\bar{\psi}_k - s_k\phi_k\|_2 \leq \frac{1}{N}\Big\| \sum_n \left[R^t(\Psi, y_n, k) - R^o(\Psi, y_n, k)\right]\Big\|_2 \\
+ \frac{1}{N}\Big\|\sum_n \left[R^o(\Psi, y_n, k) - R^o(\Phi, y_n, k)\right]\Big\|_2 \\
+ \frac{1}{N}\Big\|\sum_n \left[y_n - P(\Phi_{I_n})y_n\right]\cdot\sigma_n(k)\cdot\chi(I_n,k)\Big\|_2. \qquad (65)
\end{aligned}
$$

We first estimate the norm of the first sum using the fact that the operator $\mathbb{I}_d - P(\Psi_{I_n}) + P(\psi_k)$ is an orthogonal projection and that $\|y_n\|_2 \leq \sqrt{B+1}$,

$$
\frac{1}{N}\Big\|\sum_n \left[R^t(\Psi, y_n, k) - R^o(\Psi, y_n, k)\right]\Big\|_2 \leq \frac{2\sqrt{B+1}}{N}\cdot \sharp\{n : R^t(\Psi, y_n, k) \neq R^o(\Psi, y_n, k)\}. \qquad (66)
$$

Next note that on the draw of $y_n$ the event that the thresholding residual using $\Psi$ is different from the oracle residual using $\Psi$, $\{y_n : R^t(\Psi, y_n, k) \neq R^o(\Psi, y_n, k)\}$ for any $k$ is again contained in the events $\mathcal{E}_n \cup \mathcal{F}_n$ as defined in (26)/(27),

$$
\{y_n : R^t(\Psi, y_n, k) \neq R^o(\Psi, y_n, k)\} \subseteq \{y_n : I^t_{\Psi,n} \neq I_n\} \cup \{y_n : \operatorname{sign}(\Psi^\star_{I_n} y_n) \neq \sigma_n(I_n)\} \subseteq \mathcal{E}_n \cup \mathcal{F}_n. \qquad (67)
$$

Substituting the corresponding bounds into (65) we get,

$$
\begin{aligned}
\|\bar{\psi}_k - s_k\phi_k\|_2 \leq \frac{2\sqrt{B+1}}{N}\cdot\sharp\{n : y_n \in \mathcal{E}_n\} + \frac{2\sqrt{B+1}}{N}\cdot\sharp\{n : y_n\in\mathcal{F}_n\} \\
+ \frac{1}{N}\Big\|\sum_n \left[R^o(\Psi, y_n, k) - R^o(\Phi, y_n, k)\right]\Big\|_2 + \frac{1}{N}\Big\|\sum_n \left[y_n - P(\Phi_{I_n})y_n\right]\cdot\sigma_n(k)\cdot\chi(I_n,k)\Big\|_2. \qquad (68)
\end{aligned}
$$

For the first two terms on the right hand side we use the same estimates as in the proof of Theorem 3.2. To estimate the remaining two terms on the right hand side as well as $s_k$ we use the corresponding lemmata in the appendix. From Lemma B.6 we know that

$$
\mathbb{P}\left(\left|\frac{1}{N}\sum_n \chi(I_n,k)\sigma_n(k)\langle y_n, \phi_k\rangle\right| \leq (1-t_0)\frac{C_r\gamma_{1,S}}{K}\right) \leq \exp\left(-\frac{Nt_0^2 C_r^2\gamma_{1,S}^2}{2K(1 + \frac{SB}{K} + S\rho^2 + t_0 C_r\gamma_{1,S}\sqrt{B+1}/3)}\right). \qquad (69)
$$

From Lemma B.8 we get that if $S \leq \min\{\frac{K}{98B}, \frac{1}{98\rho^2}\}$, $\varepsilon \leq \frac{1}{32\sqrt{S}}$ and $\varepsilon_\delta \leq \frac{1}{24(B+1)}$ then

$$
\begin{aligned}
\mathbb{P}\left(\frac{1}{N}\left\|\sum_n \left[R^o(\Psi, y_n, k) - R^o(\Phi, y_n, k)\right]\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}(0.38\varepsilon + t_3)\right) \\
\leq \exp\left(-\frac{t_3 C_r^2\gamma_{1,S}^2 N}{40K\max\{S, B+1\}}\min\left\{\frac{t_3}{\varepsilon^2 + \varepsilon_\delta\left(1 - \gamma_{2,S} + d\rho^2\right)/160}, \frac{5}{3}\right\} + \frac{1}{4}\right). \qquad (70)
\end{aligned}
$$

Finally from Lemma B.7 we know that for $0 \leq t_4 \leq 1 - \gamma_{2,S} + d\rho^2$, we have

$$\mathbb{P}\left(\left\|\frac{1}{N}\sum_n \left[y_n - P(\Phi_{I_n})y_n\right] \cdot \sigma_n(k) \cdot \chi(I_n,k)\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}\, t_4\right) \leq \exp\left(-\frac{t_4^2 C_r^2 \gamma_{1,S}^2 N}{8K\max\{S, B+1\}\,(1-\gamma_{2,S}+d\rho^2)} + \frac{1}{4}\right). \tag{71}$$

Thus with high probability we have

$$\|\bar{\psi}_k - s_k\phi_k\|_2 \leq \frac{C_r\gamma_{1,S}}{K}\left(\varepsilon_{\mu,\rho} + t_1 + \tau\varepsilon + t_2 + 0.38\varepsilon + t_3 + t_4\right) \quad \text{and} \quad s_k \geq (1-t_0)\frac{C_r\gamma_{1,S}}{K}. \tag{72}$$

To be more precise, if we choose a target precision $\tilde{\varepsilon} \geq 8\varepsilon_{\mu,\rho}$ and set $t_1 = \tilde{\varepsilon}/24$, $t_2 = t_3 = \max\{\tilde{\varepsilon}, \varepsilon\}/24$, $\tau = 1/24$, $t_4 = \tilde{\varepsilon}/8$ and $t_0 = 1/50$ we get

$$\max_k \left\|\bar{\psi}_k - \frac{C_r\gamma_{1,S}}{K}\phi_k\right\|_2 \leq 0.8 \cdot \frac{C_r\gamma_{1,S}}{K}\max\{\tilde{\varepsilon},\varepsilon\} \quad \text{and} \quad \min_k s_k \geq 0.98 \cdot \frac{C_r\gamma_{1,S}}{K}. \tag{73}$$

except with probability

$$\exp\left(\frac{-C_r\gamma_{1,S}N\tilde{\varepsilon}}{336\,K\sqrt{B+1}}\right) + \exp\left(\frac{-C_r\gamma_{1,S}N\max\{\tilde{\varepsilon},\varepsilon\}}{144\,K\sqrt{B+1}}\right) + K\exp\left(\frac{-C_r^2\gamma_{1,S}^2 N}{K(5103 + 34\,C_r\gamma_{1,S}\sqrt{B+1})}\right)$$
$$+2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2 N\tilde{\varepsilon}^2}{512K\max\{S,B+1\}\,(1-\gamma_{2,S}+d\rho^2)}\right) + 2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2 N\max\{\tilde{\varepsilon},\varepsilon\}^2}{576K\max\{S,B+1\}\,(\varepsilon+1-\gamma_{2,S}+d\rho^2)}\right).$$

Note that in case the target precision $\tilde{\varepsilon}$ is larger than $\varepsilon_\delta$, as happens for instance as soon as $\beta_S \leq \frac{1}{7\sqrt{S}}$ and therefore $\varepsilon_{\mu,\rho} \geq \varepsilon_\delta$, the last term in the sum above reduces to

$$2K\exp\left(\frac{-C_r^2\gamma_{1,S}^2 N\max\{\tilde{\varepsilon},\varepsilon\}}{576K\max\{S,B+1\}\,(2-\gamma_{2,S}+d\rho^2)}\right). \tag{74}$$

Lemma B.10 then again implies that

$$d(\bar{\Psi},\Phi) = \max_k \left\|\frac{\bar{\psi}_k}{\|\bar{\psi}_k\|_2} - \phi_k\right\|_2 \leq 0.92\max\{\tilde{\varepsilon},\varepsilon\}. \tag{75}$$

### A.4 Proof of Theorem 4.3

In case of exactly $S$-sparse, noiseless signals the bound (65) reduces to

$$\|\bar{\psi}_k - s_k\phi_k\|_2 \leq \frac{2\sqrt{B}}{N}\cdot\sharp\{n : y_n \in \mathcal{F}_n^s\} + \frac{1}{N}\left\|\sum_n \left[R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)\right]\right\|_2. \tag{76}$$

Since the relative gap $\Delta > 2\mu S$ we get $\delta_S \leq \mu S \leq \frac{1}{2}$ and by Lemma B.4

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{F}_n^s\} \geq \frac{\gamma_{1,S}N}{2K\sqrt{B}}\cdot(\tau\varepsilon + t_2)\right) \leq \exp\left(\frac{-t^2\gamma_{1,S}N}{2K\sqrt{B}\,(2\tau\varepsilon + t_2)}\right), \tag{77}$$

whenever

$$\varepsilon \leq \frac{\Delta - 2\mu S}{\sqrt{12}\left(\frac{1}{4} + \sqrt{\log\left(\frac{23K^2\sqrt{B}}{(\Delta-2\mu S)\gamma_{1,S}\tau}\right)}\right)}. \tag{78}$$

Further by Lemma B.8

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n \left[R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)\right]\right\|_2 \geq \frac{\gamma_{1,S}}{K}(0.61\varepsilon + t_3)\right) \leq \exp\left(-\frac{t_3\gamma_{1,S}^2 N}{32\varepsilon K\max\{S,B\}}\min\left\{\frac{t_3}{\varepsilon},1\right\} + \frac{1}{4}\right),$$

and again by B.6

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{n}\chi(I_n,k)\sigma_n(k)\langle y_n,\phi_k\rangle\right| \le (1-t_0)\frac{C_r\gamma_{1,S}}{K}\right) \le \exp\left(-\frac{Nt_0^2\gamma_{1,S}^2}{2K(1+\mu^2(S-1)+t_0\gamma_{1,S}\sqrt{B}/3)}\right). \quad (79)$$

Thus with high probability we have

$$\left\|\bar{\psi}_k - s_k\phi_k\right\|_2 \le \frac{\gamma_{1,S}}{K}\left(\tau\varepsilon + t_2 + 0.61\varepsilon + t_3\right) \quad \text{and} \quad s_k \ge (1-t_0)\frac{\gamma_{1,S}}{K}. \quad (80)$$

The final result follows as before from setting $t_0 = 1/50$, $\tau = 1/24$ and $t_2 = t_3 = \max\{\tilde{\varepsilon},\varepsilon\}/24$.

## Appendix B
## Probability Estimates & Technicalities

**Theorem B.1** (Vector Bernstein, [24], [19], [25]). *Let $(v_n)_n \in \mathbb{R}^d$ be a finite sequence of independent random vectors. If $\|v_n\|_2 \le M$ almost surely, $\|\mathbb{E}(v_n)\|_2 \le m_1$ and $\sum_n \mathbb{E}(\|v_n\|_2^2) \le m_2$, then for all $0 \le t \le m_2/(M+m_1)$*

$$\mathbb{P}\left(\left\|\sum_n v_n - \sum_n \mathbb{E}(v_n)\right\|_2 \ge t\right) \le \exp\left(-\frac{t^2}{8m_2} + \frac{1}{4}\right), \quad (81)$$

*and in general*

$$\mathbb{P}\left(\left\|\sum_n v_n - \sum_n \mathbb{E}(v_n)\right\|_2 \ge t\right) \le \exp\left(-\frac{t}{8}\cdot\min\left\{\frac{t}{m_2},\frac{1}{M+m_1}\right\} + \frac{1}{4}\right). \quad (82)$$

Note that the general statement is simply a consequence of the first part, since for $t \ge m_2/(M+m_1)$ we can choose $m_2 = t(M+m_1)$.

For the simple case of random variables we also state a scalar version of Bernstein's inequality leading to better constants.

**Theorem B.2** (Scalar Bernstein, [7]). *Let $v_n \in \mathbb{R}$, $n = 1\ldots N$ be a finite sequence of independent random variables. If $\mathbb{E}(v_n^2) \le m$ and $\mathbb{E}(|v_n|^k) \le \frac{1}{2}k!\,mM^{k-2}$ for all $k > 2$ then for all $t > 0$*

$$\mathbb{P}\left(\left|\sum_n v_n - \sum_n \mathbb{E}(v_n)\right| \ge t\right) \le \exp\left(-\frac{t^2}{2(Nm+Mt)}\right).$$

**Lemma B.3.** *For $y_n$ following model* (12) *with coefficients that have an absolute gap $\beta_S$and $\varepsilon_{\mu,\rho} = \frac{8K^2\sqrt{B+1}}{C_r\gamma_{1,S}}\exp\left(\frac{-\beta_S^2}{98\max\{\mu^2,\rho^2\}}\right)$ we have,*

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{E}_n\} \ge \frac{C_r\gamma_{1,S}N}{2K\sqrt{B+1}}\cdot(\varepsilon_{\mu,\rho}+t)\right) \le \exp\left(\frac{-t^2C_r\gamma_{1,S}N}{2K\sqrt{B+1}\,(2\varepsilon_{\mu,\rho}+t)}\right). \quad (83)$$

*Proof:* We apply Theorem B.2 to the sum of indicator functions $\mathbf{1}_{\mathcal{E}_n}$ to get

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{E}_n\} \ge \sum_n \mathbb{P}(\mathcal{E}_n) + tN\right) \le \exp\left(\frac{-t^2N^2}{2\sum_n \mathbb{P}(\mathcal{E}_n) + tN}\right). \quad (84)$$

To estimate $\mathbb{P}(\mathcal{E}_n)$ we applying Hoeffding's inequality to (26) resp. use the subgaussian property of $r_n$. Omitting subscripts for simplicity and abbreviating $u = c(S) - c(S+1)$ we get,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\le \sum_k \mathbb{P}\left(\Big|\sum_{j\neq k}\sigma(j)c\big(p(j)\big)\langle\phi_j,\phi_k\rangle\Big| \ge \frac{u}{7}\right) + \sum_k \mathbb{P}\left(|\langle r,\phi_k\rangle| \ge \frac{u}{7}\right) && (85)\\
&\le \sum_k 2\exp\left(\frac{u^2}{98\sum_{j\neq k} c\big(p(j)\big)^2|\langle\phi_j,\phi_k\rangle|^2}\right) + 2K\exp\left(\frac{-u^2}{98\rho^2}\right) && (86)\\
&\le 2K\exp\left(\frac{-\beta_S^2}{98\mu^2}\right) + 2K\exp\left(\frac{-\beta_S^2}{98\rho^2}\right) && (87)\\
&\le 4K\exp\left(\frac{-\beta_S^2}{98\max\{\mu^2,\rho^2\}}\right) = \frac{C_r\gamma_{1,S}}{2K\sqrt{B+1}}\cdot\varepsilon_{\mu,\rho}. && (88)
\end{aligned}$$

The result follows from the substitution $t \to \frac{C_r \gamma_{1,S}}{2K\sqrt{B+1}}\, t$. □

**Lemma B.4.** *(a) For $y_n$ following model* (12) *with coefficients that have a relative gap $\Delta_S$ we have,*

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{F}_n\} \geq \frac{C_r \gamma_{1,S} N}{2K\sqrt{B+1}} \cdot (\tau\varepsilon + t)\right) \leq \exp\left(\frac{-t^2 C_r \gamma_{1,S} N}{2K\sqrt{B+1}\,(2\tau\varepsilon + t)}\right), \tag{89}$$

*whenever*

$$\varepsilon \leq \frac{\Delta_S}{\sqrt{98B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{106K^2(B+1)}{\Delta_S C_r \gamma_{1,S}\tau}\right)}\right)} \tag{90}$$

*(b) For $y_n$ following model* (12) *with coefficients that have a relative gap $\Delta_S \geq 2\mu S$ we have,*

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{F}_n^s\} \geq \frac{\gamma_{1,S} N}{2K\sqrt{B}} \cdot (\tau\varepsilon + t)\right) \leq \exp\left(\frac{-t^2 \gamma_{1,S} N}{2K\sqrt{B}\,(2\tau\varepsilon + t)}\right), \tag{91}$$

*whenever*

$$\varepsilon \leq \frac{\Delta_S - 2\mu S}{\sqrt{8B}\left(\frac{1}{4} + \sqrt{\log\left(\frac{19K^2 B}{(\Delta_S - 2\mu S)\gamma_{1,S}\tau}\right)}\right)} \tag{92}$$

*Proof:* We apply Theorem B.2 to the sum of indicator functions $\mathbf{1}_{\mathcal{F}_\mathbf{n}^{(\mathbf{s})}}$ to get

$$\mathbb{P}\left(\sharp\{n : y_n \in \mathcal{F}_n^{(s)}\} \geq \sum_n \mathbb{P}(\mathcal{F}_n^{(s)}) + tN\right) \leq \exp\left(\frac{-t^2 N^2}{2\sum_n \mathbb{P}(\mathcal{F}_n^{(s)}) + tN}\right) \tag{93}$$

To estimate $\mathbb{P}(\mathcal{F}_n^{(s)})$ we again apply Hoeffding's inequality this time to (27)/(59) resp. use the subgaussian property of $r_n$. Omitting subscripts and using the short hand $u = c(S) - c(S+1)$ and $u_s = (\Delta_S - 2\mu S)c(1)$ we get,

$$\mathbb{P}(\mathcal{F}) \leq \sum_k \mathbb{P}\left(\omega_k \Big|\sum_{j\neq k} \sigma(j) c\big(p(j)\big)\langle \phi_j, z_k\rangle\Big| \geq \frac{u}{7} - \frac{\varepsilon^2 c(S)}{6}\right) + \sum_k \mathbb{P}\left(\omega_k |\langle r, z_k\rangle| \geq \frac{u}{14} - \frac{\varepsilon^2 c(S)}{12}\right) \tag{94}$$

$$\leq \sum_k 2\exp\left(\frac{-\left(u - \frac{7\varepsilon^2 c(S)}{6}\right)^2}{98\omega_k^2 \sum_{j\neq k} c\big(p(j)\big)^2 |\langle \phi_j, z_k\rangle|^2}\right) + 2K\exp\left(\frac{-\left(u - \frac{7\varepsilon^2 c(S)}{6}\right)^2}{4\cdot 98\rho^2}\right) \tag{95}$$

$$\leq 2K\exp\left(\frac{-\left(u - \frac{7\varepsilon^2 c(S)}{6}\right)^2}{98\varepsilon^2 \min\{c(1)^2 B, 1\}}\right) + 2K\exp\left(\frac{-\left(u - \frac{7\varepsilon^2 c(S)}{6}\right)^2}{4\cdot 98\varepsilon^2\rho^2}\right) \tag{96}$$

$$\leq 5K\exp\left(\frac{-(c(S) - c(S+1))^2}{98\varepsilon^2 c(1)^2 B}\right) \leq 5K\exp\left(\frac{-\Delta_S^2}{98\varepsilon^2 B}\right) \tag{97}$$

From Lemma A.3 in [35] we further know that condition (90) implies

$$5K\exp\left(\frac{-\Delta_S^2}{98\varepsilon^2 B}\right) \leq \frac{C_r \gamma_{1,S}}{2K\sqrt{B+1}} \cdot \tau\varepsilon, \tag{98}$$

and the result in (a) follows again from the substitution $t \to \frac{C_r \gamma_{1,S}}{2K\sqrt{B+1}}\, t$.
Similarly we get

$$\mathbb{P}(\mathcal{F}^s) \leq \sum_k \mathbb{P}\left(\omega_k \Big|\sum_{j\neq k} \sigma(j) c\big(p(j)\big)\langle \phi_j, z_k\rangle\Big| \geq \frac{u_s}{2} - \frac{\varepsilon^2 c(S)}{4}\right) \tag{99}$$

$$\leq 2K\exp\left(\frac{-\left((\Delta_S - 2\mu S)c(1) - \frac{\varepsilon^2 c(S)}{2}\right)^2}{8\varepsilon^2 \min\{c(1)^2 B, 1\}}\right) \leq 3K\exp\left(\frac{-(\Delta_S - 2\mu S)^2}{8\varepsilon^2 B}\right) \leq \frac{\gamma_{1,S}}{2K\sqrt{B}} \cdot \tau\varepsilon, \tag{100}$$

whenever (92) holds and the result follows from the substitution $t \to \frac{\gamma_{1,S}}{2K\sqrt{B}}\, t$.

Finally note that other (messier) ways to bound $\sum_{j\neq k} c\big(p(j)\big)^2 |\langle \phi_j, z_k\rangle|^2$ are

$$\sum_{j\neq k} c\big(p(j)\big)^2 |\langle \phi_j, z_k\rangle|^2 \leq \min\{c(1)^2\|\Phi_I\|_{2,2}^2 + 1 - \gamma_{2,S}, c(1)^2\|\Phi_I\|_{2,2}^2 + c(S+1)^2 B\} \tag{101}$$

However, in the case of exactly S-sparse signals these can lead to better (and again clean) estimates, such as $c(1)^2(1+\mu S)$ or $c(1)^2(1+\delta_S)$ if $\Phi$ has isometry constant $\delta_S < 1$. □

**Lemma B.5.** *For $y_n = \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}}$ as in model* (12) *and $0 \leq t \leq \frac{\sqrt{S}}{\sqrt{B}+2}$ we have*

$$\mathbb{P}\left(\left\| \frac{1}{N}\sum_n \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}} \cdot \sigma_n(k)\cdot \chi(I_n,k) - \frac{C_r\gamma_{1,S}}{K}\phi_k \right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}\, t\right) \leq \exp\left(-\frac{t^2 C_r^2 \gamma_{1,S}^2 N}{8SK} + \frac{1}{4}\right). \tag{102}$$

*Proof:* We apply Theorem B.1 to $v_n = \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}} \cdot \sigma_n(k)\cdot\chi(I_n,k)$. Since the $v_n$ are identically distributed we drop the index $n$ for our estimates. Remembering that $I = p^{-1}(\mathbb{S})$ we get,

$$\begin{aligned}
\mathbb{E}(v) &= \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{\sqrt{1+\|r\|_2^2}}\left(\sum_j \phi_j c\big(p(j)\big)\sigma(j)\cdot\sigma(k) + r\cdot\sigma(k)\right)\right)\\
&= \mathbb{E}_{c,p,r}\left(\frac{\chi(\mathbb{S},p(k))\cdot c\big(p(k)\big)}{\sqrt{1+\|r\|_2^2}}\,\phi_k\right)\\
&= \mathbb{E}_r\left(\frac{1}{\sqrt{1+\|r\|_2^2}}\right)\mathbb{E}_c\left(\frac{c(1)+\ldots+c(S)}{K}\right)\phi_k = \frac{C_r\gamma_{1,S}}{K}\,\phi_k,
\end{aligned} \tag{103}$$

and $\|\mathbb{E}(v)\|_2 \leq \sqrt{S}/K$. Together with the estimates,

$$\mathbb{E}\left(\|v\|_2^2\right) = \mathbb{E}\left(\frac{\chi(I,k)}{1+\|r\|_2^2}\cdot\left(\|\Phi x_{c,p,\sigma}\|_2^2 + \langle\Phi x_{c,p,\sigma}, r\rangle + \|r\|_2^2\right)\right) = \mathbb{E}\left(\chi(I,k)\right) = \frac{S}{K}$$

$$\text{and} \qquad \|v\|_2 \leq \frac{\|\Phi x_{c,p,\sigma} + r\|_2}{\sqrt{1+\|r\|_2^2}} \leq \frac{\sqrt{B}+\|r\|_2}{\sqrt{1+\|r\|_2^2}} \leq \sqrt{B+1},$$

this leads to

$$\mathbb{P}\left(\left\| \frac{1}{N}\sum_n \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}} \cdot \sigma_n(k)\cdot \chi(I_n,k) - \frac{C_r\gamma_{1,S}}{K}\phi_k \right\|_2 \geq t\right) \leq \exp\left(-\frac{t^2 KN}{8S} + \frac{1}{4}\right), \tag{104}$$

for $0 \leq t \leq \frac{S}{K(\sqrt{B+1}+\frac{S}{K})}$. The final statements follows from the substitution $t \to \frac{C_r\gamma_{1,S}}{K}\, t$ and simplifications. □

**Lemma B.6.** *For $y_n = \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}}$ as in model* (12) *we have*

$$\mathbb{P}\left(\left| \frac{1}{N}\sum_n \chi(I_n,k)\sigma_n(k)\langle y_n,\phi_k\rangle \right| \leq (1-t)\frac{C_r\gamma_{1,S}}{K}\right) \leq \exp\left(-\frac{N t^2 C_r^2 \gamma_{1,S}^2}{2K(1+\frac{SB}{K} + S\rho^2 + tC_r\gamma_{1,S}\sqrt{B+1}/3)}\right). \tag{105}$$

*Proof:* We apply Theorem B.2 to $v_n = \chi(I_n,k)\sigma_n(k)\langle y_n,\phi_k\rangle$, as usual dropping the index $n$ in the estimates for conciseness. For the expectation we get

$$\mathbb{E}(v) = \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{\sqrt{1+\|r\|_2^2}}\left(\sum_j c\big(p(j)\big)\sigma(j)\langle\phi_j,\phi_k\rangle\cdot\sigma(k) + \langle r,\phi_k\rangle\cdot\sigma(k)\right)\right) \tag{106}$$

$$= \mathbb{E}_{c,p,r}\left(\frac{\chi(\mathbb{S},p(k))\cdot c\big(p(k)\big)}{\sqrt{1+\|r\|_2^2}}\right) = \frac{C_r\gamma_{1,S}}{K}. \tag{107}$$

We further estimate the second moment $m$ as

$$\begin{aligned}\mathbb{E}\left(v^2\right) &= \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{1+\|r\|_2^2}\Big(\sum_j c\big(p(j)\big)\sigma(j)\langle\phi_j,\phi_k\rangle + \langle r,\phi_k\rangle\Big)^2\right)\\
&\le \mathbb{E}_{c,p}\left(\chi(I,k)\cdot\left(\sum_j c\big(p(j)\big)^2|\langle\phi_j,\phi_k\rangle|^2 + \mathbb{E}_r\left(|\langle r,\phi_k\rangle|^2\right)\right)\right)\\
&\le \mathbb{E}_{c,p}\left(\chi(I,k)\cdot\left(\frac{\gamma_{2,S}}{S} + \frac{1-\frac{\gamma_{2,S}}{S}}{K-1}\sum_{j\in I, j\neq k}|\langle\phi_j,\phi_k\rangle|^2 + \rho^2\right)\right) \le \frac{S}{K}\cdot\left(\frac{\gamma_{2,S}}{S} + \frac{B}{K} + \rho^2\right). \qquad (108)\end{aligned}$$

In the case of exactly $S$-sparse signals, where $\gamma_{2,S}=1$ we get the alternative bound, $\mathbb{E}\left(v^2\right) \le \frac{1}{K}(1+(S-1)\mu^2 + S\rho^2)$. Since $|v| \le |\langle y,\phi_k\rangle| \le \|y\|_2 \le \sqrt{B+1}$ we can choose $M = \frac{\sqrt{B+1}}{3}$. □

**Lemma B.7.** *For $y_n = \frac{\Phi x_{c_n,p_n,\sigma_n} + r_n}{\sqrt{1+\|r_n\|_2^2}}$ as in model* (12)

$$\begin{aligned}\mathbb{P}\left(\left\|\frac{1}{N}\sum_n \left(y_n - P(\Phi_{I_n})y_n\right)\cdot\sigma_n(k)\cdot\chi(I_n,k)\right\|_2 \ge \frac{C_r\gamma_{1,S}}{K}\,t\right)&\\
\le \exp\left(-\frac{tC_r^2\gamma_{1,S}^2N}{8K\max\{S,B+1\}}\max\left\{\frac{t}{1-\gamma_{2,S}+d\rho^2},1\right\}+\frac{1}{4}\right).& \qquad (109)\end{aligned}$$

*Proof:* We apply Theorem B.1 to $v_n = (y_n - P(\Phi_{I_n})y_n)\cdot\sigma_n(k)\cdot\chi(I_n,k)$. For brevity we again drop the index $n$ in the estimates and define the orthogonal projection $Q(\Phi_I) = \mathbb{I}_d - P(\Phi_I)$. For the expectation we get

$$\begin{aligned}\mathbb{E}(v) &= \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{\sqrt{1+\|r\|_2^2}}\,Q(\Phi_I)\left(\sum_j \phi_j c\big(p(j)\big)\sigma(j)\cdot\sigma(k) + r\cdot\sigma(k)\right)\right)\\
&= \mathbb{E}_{c,p,r}\left(\frac{\chi(I,k)}{\sqrt{1+\|r\|_2^2}}\,c\big(p(k)\big)Q(\Phi_I)\phi_k\right) = 0, \qquad (110)\end{aligned}$$

and for the second moment

$$\begin{aligned}\mathbb{E}\left(\|v\|_2^2\right) &= \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{1+\|r\|_2^2}\cdot\left(\|Q(\Phi_I)\Phi x_{c,p,\sigma}\|_2^2 + \langle Q(\Phi_I)\Phi x_{c,p,\sigma}, Q(\Phi_I)r\rangle + \|Q(\Phi_I)r\|_2^2\right)\right)\\
&\le \mathbb{E}_{c,p}\left(\chi(I,k)\cdot\left(\sum_j c\big(p(j)\big)^2\|Q(\Phi_I)\phi_j\|_2^2 + \mathbb{E}_r\left(\frac{\|Q(\Phi_I)r\|_2^2}{1+\|r\|_2^2}\right)\right)\right)\\
&\le \mathbb{E}_{c,p}\left(\chi(I,k)\cdot\left(\sum_{j\notin I} c\big(p(j)\big)^2 + \min\{1,(d-S)\rho^2\}\right)\right) \le \frac{S}{K}\cdot\left(1-\gamma_{2,S}+d\rho^2\right). \qquad (111)\end{aligned}$$

Since $v$ is bounded,

$$\|v\|_2 \le \frac{\|Q(\Phi_I)(\Phi x_{c,p,\sigma}+r)\|_2}{\sqrt{1+\|r\|_2^2}} \le \frac{\sqrt{B(1-\gamma_{2,S,\min})}+\|r\|_2}{\sqrt{1+\|r\|_2^2}} \le \sqrt{B(1-\gamma_{2,S,\min})+1} \le \sqrt{B+1}, \qquad (112)$$

we get for $t \to \frac{C_r\gamma_{1,S}}{K}\, t$

$$\mathbb{P}\left(\left\|\frac{1}{N}\sum_n (y_n - P(\Phi_{I_n})y_n)\cdot\sigma_n(k)\cdot\chi(I_n,k)\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}\, t\right)$$
$$\leq \exp\left(-\frac{tC_r\gamma_{1,S}N}{8K}\max\left\{\frac{tC_r\gamma_{1,S}}{S(1-\gamma_{2,S}+d\rho^2)}, \frac{1}{\sqrt{B+1}}\right\}+\frac{1}{4}\right)$$
$$\leq \exp\left(-\frac{tC_r^2\gamma_{1,S}^2N}{8K}\max\left\{\frac{t}{S(1-\gamma_{2,S}+d\rho^2)}, \frac{1}{C_r\gamma_{1,S}\sqrt{B+1}}\right\}+\frac{1}{4}\right). \tag{113}$$

The final bound follows from the fact that $C_r < 1$ and $\gamma_{1,S} \leq \sqrt{S}$. □

**Lemma B.8.** *Assume that* $y_n = \frac{\Phi x_{c_n,p_n,\sigma_n}+r_n}{\sqrt{1+\|r_n\|_2^2}}$ *follows the random model in* (12)*. Assume* $S \leq \min\{\frac{K}{98B}, \frac{1}{98\rho^2}\}$ *and* $d(\Phi,\Psi) = \varepsilon \leq \frac{1}{32\sqrt{S}}$.

*(a) If* $\varepsilon_\delta := K\exp\left(-\frac{1}{4741\mu^2 S}\right) \leq \frac{1}{24(B+1)}$ *we have*

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n [R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)]\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}(0.38\varepsilon + t)\right)$$
$$\leq \exp\left(-\frac{tC_r\gamma_{1,S}N}{8K}\min\left\{\frac{tC_r\gamma_{1,S}}{S\left[5\varepsilon^2+\varepsilon_\delta\left(1-\gamma_{2,S}+d\rho^2\right)/32\right]}, \frac{1}{3\sqrt{B+1}}\right\}+\frac{1}{4}\right). \tag{114}$$

*(b) If* $\gamma_{2,S} = 1, \rho = 0$ *together with* $\varepsilon_\delta \leq \frac{1}{24(B+1)}$ *or* $\delta_S(\Phi) \leq 1/4$ *this reduces to*

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n [R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)]\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}(0.38\varepsilon + t)\right)$$
$$\leq \exp\left(-\frac{t\gamma_{1,S}^2N}{32\varepsilon K\max\{S,B\}}\min\left\{\frac{t}{\varepsilon},1\right\}+\frac{1}{4}\right). \tag{115}$$

*(c) If* $\gamma_{2,S} = 1, \rho = 0$ *and* $\delta_S(\Phi) \leq 1/2$ *we have*

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n [R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)]\right\|_2 \geq \frac{\gamma_{1,S}}{K}(0.61\varepsilon + t)\right)$$
$$\leq \exp\left(-\frac{t\gamma_{1,S}^2N}{32\varepsilon K\max\{S,B\}}\min\left\{\frac{t}{\varepsilon},1\right\}+\frac{1}{4}\right). \tag{116}$$

*Proof:* We apply Theorem B.1 to $v_n = R^o(\Psi,y_n,k) - R^o(\Phi,y_n,k)$. Again we drop the index $n$ in the estimates. Remembering the definition of $R^o(\Psi,y_n,k)$ in (63) we first expand $v$ as

$$v = \big(y_n - P(\Psi_{I_n})y_n + P(\psi_k)y_n\big)\cdot\sigma_n(k)\cdot\chi(I_n,k) - \big(y_n - P(\Phi_{I_n})y_n + P(\phi_k)y_n\big)\cdot\sigma_n(k)\cdot\chi(I_n,k)$$
$$= [P(\Phi_I) - P(\Psi_I) - P(\phi_k) + P(\psi_k)]\, y\cdot\sigma(k)\cdot\chi(I,k). \tag{117}$$

Abbreviate $T(I,k) := P(\Phi_I) - P(\Psi_I) - P(\phi_k) + P(\psi_k)$. Taking the expectation we get

$$\mathbb{E}(v) = \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{\sqrt{1+\|r\|_2^2}}\, T(I,k)\left(\sum_j \phi_j c\big(p(j)\big)\sigma(j)\cdot\sigma(k) + r\cdot\sigma(k)\right)\right)$$
$$= \mathbb{E}_{c,p,r}\left(\frac{\chi(I,k)\cdot c\big(p(k)\big)}{\sqrt{1+\|r\|_2^2}}\left[P(\Phi_I) - P(\Psi_I) - P(\phi_k) + P(\psi_k)\right]\phi_k\right)$$
$$= \frac{C_r\gamma_{1,S}}{K}\binom{K-1}{S-1}^{-1}\sum_{|I|=S,k\in I}\left[P(\psi_k) - P(\Psi_I)\right]\phi_k. \tag{118}$$

We next split the sum above into a sum over the well-conditioned subsets, where $\delta_I(\Phi) \leq \delta_0$, and the ill-conditioned subsets, $\delta_I(\Phi) > \delta_0$,

$$\mathbb{E}(v) = \frac{C_r\gamma_{1,S}}{K}\binom{K-1}{S-1}^{-1}\left(\sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}\big[P(\psi_k)-P(\Psi_I)\big]\phi_k + \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)>\delta_0}}\big[P(\psi_k)-P(\Psi_I)\big]\phi_k\right). \tag{119}$$

We further expand the sum over the well-conditioned sets using Sublemma B.9,

$$\begin{aligned}
\sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}\big[P(\psi_k)-P(\Psi_I)\big]\phi_k &= \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}(P(\Phi_I)b_k+\eta_{I,k})\\
&= \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}(\Phi_I\Phi_I^\star b_k + [P(\Phi_I)-\Phi_I\Phi_I^\star]\,b_k+\eta_{I,k})\\
&= \sum_{|I|=S,k\in I}\Phi_I\Phi_I^\star b_k - \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)>\delta_0}}\Phi_I\Phi_I^\star b_k + \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}([P(\Phi_I)-\Phi_I\Phi_I^\star]\,b_k+\eta_{I,k})\\
&= \binom{K-2}{S-2}\Phi\Phi^\star b_k - \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)>\delta_0}}\Phi_I\Phi_I^\star b_k + \sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}([P(\Phi_I)-\Phi_I\Phi_I^\star]\,b_k+\eta_{I,k})\,,
\end{aligned} \tag{120}$$

where for the last equality have used that $\langle b_k,\phi_k\rangle = 0$. Substituting the last expression into (119) we get,

$$\begin{aligned}
\mathbb{E}(v) = \frac{C_r\gamma_{1,S}}{K}\Bigg[&\frac{S-1}{K-1}\Phi\Phi^\star b_k + \binom{K-1}{S-1}^{-1}\sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)\leq\delta_0}}([P(\Phi_I)-\Phi_I\Phi_I^\star]\,b_k+\eta_{I,k})\\
&+\binom{K-1}{S-1}^{-1}\sum_{\substack{|I|=S,k\in I\\ \delta(\Phi_I)>\delta_0}}\big(\big[P(\psi_k)-P(\Psi_I)\big]\phi_k - \Phi_I\Phi_I^\star b_k\big)\Bigg].
\end{aligned} \tag{121}$$

Substituting the bound $\|P(\Phi_I)-\Phi_I\Phi_I^\star\|_{2,2} \leq \delta(\Phi_I) \leq \delta_0$ as well as the bound for $\|\eta_{I,k}\|_2$ from Sublemma B.9 for the well-conditioned subsets and the bound

$$\big\|\big[P(\psi_k)-P(\Psi_I)\big]\phi_k\big\|_2 = \|P(\Psi_I)Q(\psi_k)\phi_k\|_2 \leq \|Q(\psi_k)\phi_k\|_2 = \sqrt{1-|\langle\psi_k,\phi_k\rangle|^2} \leq \varepsilon_k \tag{122}$$

for the ill-conditioned subsets finally leads to

$$\begin{aligned}
\|\mathbb{E}(v)\|_2 \leq \frac{C_r\gamma_{1,S}}{K}\Bigg[&\frac{S-1}{K-1}B\|b_k\|_2 + \delta_0\|b_k\|_2 + \frac{2\varepsilon\sqrt{S}}{\sqrt{(1-\delta_0)(1-\frac{\varepsilon^2}{2})-2\varepsilon\sqrt{S}}}\cdot\|b_k\|_2\\
&+\mathbb{P}(\delta(\Phi_I)>\delta_0 : |I|=S, k\in I)\cdot(\varepsilon_k + B\|b_k\|_2)\Bigg],\\
\leq \frac{C_r\gamma_{1,S}}{K}\Bigg[&\frac{SB}{K}+\delta_0+\frac{2\varepsilon\sqrt{S}}{\sqrt{(1-\delta_0)(1-\frac{\varepsilon^2}{2})-2\varepsilon\sqrt{S}}} + \mathbb{P}(\delta(\Phi_I)>\delta_0 : |I|=S,k\in I)\cdot(B+1)\Bigg]\|b_k\|_2.
\end{aligned} \tag{123}$$

If $\delta_S \leq \frac{1}{2}$, we choose $\delta_0 = \delta_S$, which for $S \leq \frac{K}{98B}$ and $\varepsilon \leq \frac{1}{32\sqrt{S}}$ leads to

$$\|\mathbb{E}(v)\|_2 \leq 0.61\varepsilon\cdot\frac{C_r\gamma_{1,S}}{K}. \tag{124}$$

In the non-trivial case, where the $\Phi$ does not have a uniform isometry constant $\delta_S \leq \frac{1}{2}$, we can estimate (123) using J. Tropp's results on the conditioning of random subdictionaries. Reformulating Theorem 12 in [40] for our purposes we get that

$$\mathbb{P}(\delta(\Phi_I) > \delta_0 : |I| = S) \leq e^{-s} \qquad \text{for} \qquad s = \frac{\left(e^{-1/4}\delta_0 - \frac{2SB}{K}\right)^2}{144\mu^2 S}, \tag{125}$$

whenever $e^{-1/4}\delta_0 \geq \frac{2SB}{K}$, $s \geq \log(S/2+1)$ and $S \geq 4$. Together with the union bound,

$$\begin{aligned}\mathbb{P}(\delta(\Phi_I) > \delta_0 : |I| = S, k \in I) &= \binom{K-1}{S-1}^{-1} \sharp\{I : \delta(\Phi_I) > \delta_0, |I| = S, k \in I\} \\ &\leq \binom{K-1}{S-1}^{-1} \sharp\{I : \delta(\Phi_I) > \delta_0, |I| = S\} = \frac{K}{S} \cdot \mathbb{P}(\delta(\Phi_I) > \delta_0 : |I| = S), \end{aligned} \tag{126}$$

this leads to

$$\mathbb{P}(\delta(\Phi_I) > \delta_0 : |I| = S) \leq \max\left\{S, \frac{K}{S}\right\} \exp\left(-\frac{\left(e^{-1/4}\delta_0 - \frac{2SB}{K}\right)^2}{144\mu^2 S}\right), \tag{127}$$

whenever $e^{-1/4}\delta_0 \geq \frac{2SB}{K}$ - in case one of the other original conditions is violated the statement is trivially true. Using the assumption $S \leq \frac{K}{98B}$, which does not represent a hard additional constraint, considering that in order to have $\varepsilon_{\mu,\rho} < 1$ we need $S \leq \frac{1}{98\mu^2}$ and that $\mu^2 \geq \frac{B-1}{K-1} \approx \frac{B}{K}$, we get for $\delta_0 = \frac{1}{4}$,

$$\mathbb{P}\left(\delta(\Phi_I) > \frac{1}{4} : |I| = S\right) \leq K \exp\left(-\frac{1}{4741\mu^2 S}\right) := \varepsilon_\delta, \tag{128}$$

Substituting this bound for the choice $\delta_0 = \frac{1}{4}$ into (123) and using that $\varepsilon \leq \frac{1}{32\sqrt{S}}$ and $\varepsilon_\delta \leq \frac{1}{24(B+1)}$ we get

$$\|\mathbb{E}(v)\|_2 \leq 0.38\varepsilon \cdot \frac{C_r \gamma_{1,S}}{K}. \tag{129}$$

The second quantity we need to bound is the expected squared energy of $v = T(I,k)y \cdot \sigma(k) \cdot \chi(I,k)$,

$$\begin{aligned}\mathbb{E}(\|v\|_2^2) &= \mathbb{E}_{c,p,\sigma,r}\left(\frac{\chi(I,k)}{1+\|r\|_2^2} \cdot \left\|T(I,k)\Big(\sum_j \phi_j c\big(p(j)\big)\sigma(j) + r\Big)\right\|_2^2\right) \\ &= \mathbb{E}_{c,p,r}\left(\frac{\chi(I,k)}{1+\|r\|_2^2}\left(\sum_j c\big(p(j)\big)^2 \|T(I,k)\phi_j\|_2^2 + \|T(I,k)r\|_2^2\right)\right) \\ &= \mathbb{E}_{p,r}\left(\frac{\chi(I,k)}{1+\|r\|_2^2}\left(\frac{\gamma_{2,S}}{S}\sum_{j\in I}\|T(I,k)\phi_j\|_2^2 + \frac{1-\gamma_{2,S}}{K-S}\sum_{j\notin I}\|T(I,k)\phi_j\|_2^2 + \|T(I,k)r\|_2^2\right)\right), \\ &\leq \mathbb{E}_p\left(\chi(I,k)\left(\frac{\gamma_{2,S}}{S}\sum_{j\in I}\|T(I,k)\phi_j\|_2^2 + \frac{1-\gamma_{2,S}}{K-S}\sum_{j\notin I}\|T(I,k)\phi_j\|_2^2 + \mathbb{E}_r\left(\|T(I,k)r\|_2^2\right)\right)\right). \end{aligned} \tag{130}$$

We first estimate the two sums above given that $k \in I$. Note that we always have $\|P(\phi_k) - P(\psi_k)\|_{2,2} \leq \varepsilon_k$ and $\|P(\phi_k) - P(\psi_k)\|_F \leq \sqrt{2}\varepsilon_k$. Thus we get for the sum over $I$,

$$\begin{aligned}\sum_{j\in I}\|T(I,k)\phi_j\|_2^2 &\leq \sum_{j\in I}\left(\|[P(\Phi_I) - P(\Psi_I)]\phi_j\|_2 + \|[P(\phi_k) - P(\psi_k)]\phi_j\|_2\right)^2 \\ &= \sum_{j\in I}\left(\|Q(\Psi_I)]\phi_j\|_2 + \|[P(\phi_k) - P(\psi_k)]\phi_j\|_2\right)^2 \\ &\leq \sum_{j\in I}\left(\|Q(\psi_j)]\phi_j\|_2 + \|P(\phi_k) - P(\psi_k)\|_{2,2}\right)^2 \leq \sum_{j\in I}(\varepsilon_j + \varepsilon_k)^2 \leq 4S\varepsilon^2, \end{aligned} \tag{131}$$

and for the sum over the complement $I^c$,

$$\sum_{j\notin I}\|T(I,k)\phi_j\|_2^2=\|T(I,k)\Phi_{I^c}\|_F^2\le\|T(I,k)\|_F^2\|\Phi_{I^c}\|_{2,2}^2\le B\|T(I,k)\|_F^2. \tag{132}$$

To estimate the noise term in (130) we use the singular value decomposition of $T(I,k)=UDV^\star$,

$$\mathbb{E}\left(\|T(I,k)r\|_2^2\right)=\mathbb{E}\left(\|DV^\star r\|_2^2\right)=\mathbb{E}\left(\sum_i d_i^2|\langle v_i,r\rangle|^2\right)\le\sum_i d_i^2\rho^2=\rho^2\|T(I,k)\|_F^2, \tag{133}$$

where for the inequality we have used that for a subgaussian vector $r$ with parameter $\rho$, the marginal $\langle v_i,r\rangle$ is subgaussian with parameter $\rho$. Substituting these estimates together with the bound $\|T(I,k)\|_F\le\|P(\Phi_I)-P(\Psi_I)\|_F+\sqrt{2}\varepsilon_k$ into (130) we get,

$$\mathbb{E}(\|v\|_2^2)\le\mathbb{E}_p\left(\chi(I,k)\left[4\gamma_{2,S}\varepsilon^2+\left(\frac{B(1-\gamma_{2,S})}{K-S}+\rho^2\right)\left(\|P(\Phi_I)-P(\Psi_I)\|_F+\sqrt{2}\varepsilon_k\right)^2\right]\right). \tag{134}$$

As for the estimation of $\mathbb{E}(v)$ we now split the expectation over $p$ into the well and the ill-conditioned subsets $I=p^{-1}(\mathbb{S})$. By Lemma A.2 in [35], whenever $\delta(\Phi_I)\le\delta_0$, we have

$$\|P(\Phi_I)-P(\Psi_I)\|_F^2\le\frac{2\|Q(\Phi_I)B_I\|_F^2}{\sqrt{1-\delta_0}\left(\sqrt{1-\delta_0}-2\|B_I\|_F\right)} \tag{135}$$

which for $\varepsilon\le\frac{1}{32\sqrt{S}}$ and $\delta_0=1/4$ (resp. $\delta_S\le 1/2$) simplifies to $\|P(\Phi_I)-P(\Psi_I)\|_F^2\le 5S\varepsilon^2$. Together with the general estimate $\|P(\Phi_I)-P(\Psi_I)\|_F\le\sqrt{2S}$, this leads to

$$\begin{aligned}\mathbb{E}(\|v\|_2^2)\le\frac{S}{K}&\left[4\gamma_{2,S}\varepsilon^2+\left(\frac{B(1-\gamma_{2,S})}{K-S}+\rho^2\right)\left(\sqrt{5S}\varepsilon+\sqrt{2}\varepsilon_k\right)^2\right.\\&\left.\quad+\mathbb{P}\left(\delta(\Phi_I)>\frac{1}{4}:|I|=S,k\in I\right)\left(\frac{B(1-\gamma_{2,S})}{K-S}+\rho^2\right)\left(2S+2\varepsilon_k\sqrt{S}\right)\right]\\\le\frac{S}{K}&\left[4\gamma_{2,S}\varepsilon^2+15\varepsilon^2\left(\frac{SB}{K-S}(1-\gamma_{2,S})+S\rho^2\right)\right.\\&\left.\quad+\mathbb{P}\left(\delta(\Phi_I)>\frac{1}{4}:|I|=S,k\in I\right)\left(1-\gamma_{2,S}+d\rho^2\right)\frac{2B(S+1)}{K-S}\right].\end{aligned}$$

Substituting the probability bound from (128) and assuming again that $S\le\frac{K}{98B}$ as well as that $S\le\frac{1}{98\rho^2}$ leads to the final estimate

$$\mathbb{E}(\|v\|_2^2)\le\frac{S}{K}\left[5\varepsilon^2+\frac{\varepsilon_\delta}{32}\left(1-\gamma_{2,S}+d\rho^2\right)\right]. \tag{136}$$

Last we bound the energy of $v$ in general as

$$\|v\|_2=\|[P(\Phi_I)-P(\Psi_I)-P(\phi_k)+P(\psi_k)]y\|_2\le 2\|y\|_2\le 2\sqrt{B+1}. \tag{137}$$

In case $\gamma_{2,S}=1,\rho=0$ and therefore $y=\Phi_I x_I$ this reduces to

$$\begin{aligned}\|v\|_2&\le\|[\Phi_I-P(\Psi_I)\Phi_I\|_F\|x_I\|_2+\|P(\phi_k)-P(\psi_k)\|_{2,2}\|\Phi_I x_I\|_2\\&\le\left(\sum_{i\in I}\|\phi_i-P(\Psi_I)\phi_i\|_2^2\right)^{\frac{1}{2}}+\varepsilon\sqrt{B}\le\varepsilon\left(\sqrt{S}+\sqrt{B}\right),\end{aligned} \tag{138}$$

and in case of uniform isometry constant $\delta_S(\Phi)\le 1/4$ and $\varepsilon\le\frac{1}{32\sqrt{S}}$ to

$$\|v\|_2\le\|[P(\Phi_I)-P(\Psi_I)\|_F\|y\|_2+\|P(\phi_k)-P(\psi_k)\|_{2,2}\|y\|_2\le\varepsilon\sqrt{B+1}\left(\sqrt{3S}+1\right). \tag{139}$$

Putting all the pieces together we get that under the assumptions in (a),

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n \left[R^o(\Psi,y_n,k)-R^o(\Phi,y_n,k)\right]\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}(0.38\varepsilon+t)\right)$$
$$\leq \exp\left(-\frac{tC_r\gamma_{1,S}N}{8K}\min\left\{\frac{tC_r\gamma_{1,S}}{S\left[5\varepsilon^2+\varepsilon_\delta\left(1-\gamma_{2,S}+d\rho^2\right)/32\right]},\frac{1}{3\sqrt{B+1}}\right\}+\frac{1}{4}\right)$$
$$\leq \exp\left(-\frac{tC_r^2\gamma_{1,S}^2N}{40K\max\{S,B+1\}}\min\left\{\frac{t}{\varepsilon^2+\varepsilon_\delta\left(1-\gamma_{2,S}+d\rho^2\right)/160},\frac{3}{5}\right\}+\frac{1}{4}\right),$$

under the assumptions in (b),

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n \left[R^o(\Psi,y_n,k)-R^o(\Phi,y_n,k)\right]\right\|_2 \geq \frac{C_r\gamma_{1,S}}{K}(0.38\varepsilon+t)\right)$$
$$\leq \exp\left(-\frac{tC_r\gamma_{1,S}N}{8K}\min\left\{\frac{tC_r\gamma_{1,S}}{4\varepsilon^2S},\frac{1}{3\varepsilon\sqrt{S(B+1)}}\right\}+\frac{1}{4}\right)$$
$$\leq \exp\left(-\frac{tC_r^2\gamma_{1,S}^2N}{32\varepsilon K\max\{S,B+1\}}\min\left\{\frac{t}{\varepsilon},1\right\}+\frac{1}{4}\right),$$

and under the assumptions in (c),

$$\mathbb{P}\left(\frac{1}{N}\left\|\sum_n \left[R^o(\Psi,y_n,k)-R^o(\Phi,y_n,k)\right]\right\|_2 \geq \frac{\gamma_{1,S}}{K}(0.61\varepsilon+t)\right)$$
$$\leq \exp\left(-\frac{t\gamma_{1,S}^2N}{40\varepsilon K\max\{S,B+1\}}\min\left\{\frac{t}{\varepsilon},1\right\}+\frac{1}{4}\right).$$

□

**Sublemma B.9.** *Let $\Phi_I$ be a subdictionary of $\Phi$ with $\delta(\Phi_I)\leq\delta_0$ and $\Psi_I$ the corresponding subdictionary of an $\varepsilon$-perturbation of $\Psi$, that is $d(\Phi,\Psi)=\varepsilon$. If $k\in I$ then*

$$\big[P(\psi_k)-P(\Psi_I)\big]\phi_k = P(\Phi_I)b_k+\eta_{I,k} \quad \textit{with} \quad \|\eta_{I,k}\|_2 \leq \frac{2\varepsilon\sqrt{S}}{\sqrt{(1-\delta_0)(1-\frac{\varepsilon^2}{2})}-2\varepsilon\sqrt{S}}\cdot\|b_k\|_2. \tag{140}$$

*Proof:* If $\delta(\Phi_I)\leq\delta_0$ we can use the expression for $P(\Psi_I)$ developed in Lemma A.2 of [35],

$$P(\Psi_I) = \big(\Phi_I+Q(\Phi_I)B_IM_I\big)(\Phi_I^\star\Phi_I)^{-1}\left(\mathbb{I}_S+\sum_{i=1}^\infty(-R_I)^i\right)\big(\Phi_I+Q(\Phi_I)B_IM_I\big)^\star,$$

$$\text{with} \qquad M_I=\mathbb{I}_S+\sum_{i=1}^\infty(-\Phi_I^\dagger B_I)^i \qquad \text{and} \qquad R_I=M_I^\star B_I^\star Q(\Phi_I)B_IM_I(\Phi_I^\star\Phi_I)^{-1} \tag{141}$$

to get $P(\psi_k)\phi_k=\phi_k+b_k$ and

$$\begin{aligned}
P(\Psi_I)\phi_k &= \big(\Phi_I+Q(\Phi_I)B_IM_I\big)(\Phi_I^\star\Phi_I)^{-1}\left(\mathbb{I}_S+\sum_{i=1}^\infty(-R_I)^i\right)\Phi_I^\star\phi_k\\
&= \phi_k+Q(\Phi_I)B_IM_I(\Phi_I^\star\Phi_I)^{-1}\Phi_I^\star\phi_k+\big(\Phi_I+Q(\Phi_I)B_IM_I\big)(\Phi_I^\star\Phi_I)^{-1}\sum_{i=1}^\infty(-R_I)^i\Phi_I^\star\phi_k\\
&= \phi_k+Q(\Phi_I)B_I\left(\mathbb{I}_S+\sum_{i=1}^\infty(-\Phi_I^\dagger B_I)^i\right)e_{k|I}+\big(\Phi_I+Q(\Phi_I)B_IM_I\big)(\Phi_I^\star\Phi_I)^{-1}\sum_{i=1}^\infty(-R_I)^i\Phi_I^\star\phi_k\\
&= \phi_k+b_k-P(\Phi_I)b_k+Q(\Phi_I)B_I\sum_{i=1}^\infty(-\Phi_I^\dagger B_I)^ie_{k|I}+(\Phi_I+Q(\Phi_I)B_IM_I)\,(\Phi_I^\star\Phi_I)^{-1}\sum_{i=1}^\infty(-R_I)^i\Phi_I^\star\phi_k.
\end{aligned}$$

Subtracting the projections we see that all that remains to do is to estimate the size of

$$\eta_{I,k} := Q(\Phi_I)B_I M_I(\Phi_I^\dagger B_I)e_{k|I} - \big((\Phi_I^\dagger)^\star + Q(\Phi_I)B_I M_I(\Phi_I^\star\Phi_I)^{-1}\big)\sum_{i=1}^\infty(-R_I)^i\Phi_I^\star\phi_k. \tag{142}$$

Using standard bounds for matrix vector products and the identity $\|(\Phi_I^\star\Phi_I)^{-1}\|_{2,2} = \|\Phi_I^\dagger\|_{2,2}^2$ we get

$$\begin{aligned}\|\eta_{I,k}\|_2 &\leq \|B_I M_I\|_{2,2}\|\Phi_I^\dagger b_k\|_2 + \Big(\|\Phi_I^\dagger\|_{2,2} + \|B_I M_I\|_{2,2}\|\Phi_I^\dagger\|_{2,2}^2\Big)\sum_{i=0}^\infty \|R_I\|_{2,2}^i\|R_I\Phi_I^\star\phi_k\|_2\\ &\leq \|B_I M_I\|_{2,2}\|\Phi_I^\dagger\|_{2,2}\|b_k\|_2 + \Big(\|\Phi_I^\dagger\|_{2,2} + \|B_I M_I\|_{2,2}\|\Phi_I^\dagger\|_{2,2}^2\Big)\sum_{i=0}^\infty\Big(\|\Phi_I^\dagger\|_{2,2}^2\|B_I M_I\|_{2,2}^2\Big)^i\|R_I\Phi_I^\star\phi_k\|_2.\end{aligned}$$

We next expand $R_I\Phi_I^\star\phi_k$ remembering the definition of $R_I$ and $M_I$ as

$$\begin{aligned}R_I\Phi_I^\star\phi_k &= M_I^\star B_I^\star Q(\Phi_I)B_I\left(\mathbb{I}_S + \sum_{i=1}^\infty(-\Phi_I^\dagger B_I)^i\right)(\Phi_I^\star\Phi_I)^{-1}\Phi_I^\star\phi_k\\ &= M_I^\star B_I^\star Q(\Phi_I)\left(\mathbb{I}_d + \sum_{i=1}^\infty(-B_I\Phi_I^\dagger)^i\right)B_I e_{k|I} = M_I^\star B_I^\star Q(\Phi_I)\left(\mathbb{I}_d + \sum_{i=1}^\infty(-B_I\Phi_I^\dagger)^i\right)b_k\end{aligned}$$

to get

$$\|R_I\Phi_I^\star\phi_k\|_2 \leq \|B_I M_I\|_{2,2}\Big(1 - \|B_I\|_{2,2}\|\Phi_I^\dagger\|_{2,2}\Big)^{-1}\|b_k\|_2.$$

Substituting this estimate together with the bound $\|M_I\|_{2,2} \leq \Big(1 - \|B_I\|_{2,2}\|\Phi_I^\dagger\|_{2,2}\Big)^{-1}$ into the above bound for $\|\eta_{I,k}\|_2$ and resolving the sums and fractions leads to,

$$\|\eta_{I,k}\|_2 \leq \frac{2\|B_I\|_{2,2}}{\|\Phi_I^\dagger\|_{2,2}^{-1} - 2\|B_I\|_{2,2}}\cdot\|b_k\|_2$$

To get to the final statement we use the bounds $\|B_I\|_{2,2}^2 \leq \|B_I\|_F^2 \leq S\varepsilon^2/(1-\varepsilon^2/2)$ and $\|\Phi_I^\dagger\|_{2,2}^{-1} \geq \sqrt{1-\delta(\Phi_I)} \geq \sqrt{1-\delta_0}$. □

**Lemma B.10.** *If for two vectors $\psi,\phi$, where $\|\phi\|_2 = 1$, and two scalars $0 < t < s$ we have, $\|\psi - s\phi\|_2^2 \leq t^2$ then*

$$\left\|\frac{\psi}{\|\psi\|_2} - \phi\right\|_2^2 \leq 2 - 2\sqrt{1 - \frac{t^2}{s^2}}. \tag{143}$$

*Proof:* Writing $\psi = \alpha\phi + \omega z$ for some unit norm vector $z$ with $\langle z,\phi\rangle = 0$ we can reformulate the initial constraint $\|\psi - s\phi\|_2^2 \leq t^2$ to $(\alpha - s)^2 + \omega^2 \leq t^2$, while the quantity whose maximal size we have to estimate becomes

$$\left\|\frac{\psi}{\|\psi\|_2} - \phi\right\|_2^2 = 2 - 2\frac{\alpha}{\sqrt{\alpha^2+\omega^2}}. \tag{144}$$

Solving the resulting maximisation problem we get that the maximum is attained at $\alpha = \frac{s^2-t^2}{s}$ and $\omega = \frac{t}{s}\sqrt{s^2-t^2}$ and that therefore

$$\left\|\frac{\psi}{\|\psi\|_2} - \phi\right\|_2^2 \leq 2 - 2\sqrt{1 - \frac{t^2}{s^2}}. \tag{145}$$

□

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