Compressed Dictionary Learning

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Low complexity models of high-dimensional data lie at heart of many efficient solutions in modern signal processing. One such model is that of sparsity in dictionary, where every signal in the data class at hand has a sparse expansion in a predefined basis or frame. In mathematical terms this means that there is a set of K unit-norm vectors $\boldsymbol{\phi}_k \in \mathbb{R}^d$ (also referred to as atoms) collected as columns in the dictionary matrix $\boldsymbol{\Phi} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_K)$, and that every data point $\boldsymbol{y} \in \mathbb{R}^d$ can be approximately represented as

$$\boldsymbol{y} \approx \boldsymbol{\Phi}_{\mathcal{I}} \boldsymbol{x}_{\mathcal{I}} = \sum_{i \in \mathcal{I}} \boldsymbol{x}(i) \boldsymbol{\phi}_i,$$

for an index set $\mathcal{I}$ of cardinality S with $S \ll d$.

One fundamental question associated with the sparse model is how to find a suitable dictionary providing sparse representations. When taking a learning rather than a design approach this problem is known as *dictionary learning* or *sparse component analysis*. In its most general form the dictionary learning can be seen as matrix factorisation problem. Given a set of N data points in $\mathbb{R}^d$, represented by the $d \times N$ matrix $\boldsymbol{Y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)$, decompose into a $d \times K$ dictionary $\boldsymbol{\Phi}$ and sparse coefficients, $Y = \boldsymbol{\Phi} X$, where X is sparse. Since the seminal paper by Olshausen and Field, [1], a myriad of dictionary learning algorithms have been developed and recently also theory on the problem has started to emerge. For an overview over dictionary learning algorithms see [2], while pointers to the main theoretical results can be found in [3].

However one of the remaining open problems is that so far there exist no efficient algorithms with global recovery guarantees and that even the algorithms that are not supported by theoretical results become computationally intractable as the signal dimension increases. In this paper we take a step towards increasing computational efficiency of dictionary learning and thus making it applicable to high-dimensional data.

As starting point for our development we use the residual version of the Iterative Thresholding and K-Means (ITKM) algorithm presented in [4], which is supported not only by experimental validation but also by local convergence results. Given an initialization dictionary $\boldsymbol{\Psi}$ the algorithm iteratively performs two operations: (1) finding the sparse support $\mathcal{I}_n^t$ of each point in the data set $\boldsymbol{Y}$ by using thresholding as

$$\mathcal{I}_n^t = \arg \max_{|\mathcal{I}|=S} \|\boldsymbol{\Psi}_{\mathcal{I}}^* \boldsymbol{y}_n\|_1, \tag{1}$$

and (2) updating the dictionary via K residual means. For most admissible sparsity levels which still allow for stable dictionary recovery, the computationally most expensive operation of ITKM is finding the sparse support $\mathcal{I}_n^t$. This entails the calculation of the matrix product $\boldsymbol{\Psi}^* \boldsymbol{Y}$ of cost $O(dKN)$ at each iteration. Although this cost is quite light compared to other dictionary learning algorithms such as the popular K-SVD algorithm, [5], which additionally in each iteration requires the calculation of K leading singular vectors, learning dictionaries for high-dimensional data can still be prohibitively expensive.

We therefore introduce the Iterative Compressed-Thresholding and K-Means (IcTKM) algorithm for fast dictionary learning, which has significantly reduced computational cost and can efficiently process large data sets. The key modification of the ITKM algorithm is based on a fundamental dimensionality-reduction result due to Johnson and Lindenstrauss [6]. It states that for any set $\mathcal{X}$ in $\mathbb{R}^d$ with $|\mathcal{X}| = N$, there exists a map $f : \mathbb{R}^d \to \mathbb{R}^m$ with $m = O\left(\delta^{-2} \log N\right)$ and $\delta \in (0, \frac{1}{2})$, such that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}$

$$(1-\delta) \|\boldsymbol{u} - \boldsymbol{v}\|_2^2 \leq \|f(\boldsymbol{u}) - f(\boldsymbol{v})\|_2^2 \leq (1+\delta) \|\boldsymbol{u} - \boldsymbol{v}\|_2^2 . \tag{2}$$

Moreover probabilistic matrix constructions can efficiently realize the low-distortion embedding $f : \mathbb{R}^d \to \mathbb{R}^m$ in (2). Recent developments have focused on improving the computational costs associated with the embedding and providing tighter bounds on the required embedding dimension, [7], [8], [9]. In particular, matrices with the Restricted Isometry Property (RIP), as introduced by Candès and Tao in [10], can realize the embedding $f : \mathbb{R}^d \to \mathbb{R}^m$ in (2) with high probability when their column signs are randomized [9]. In the specific case where the RIP matrix is taken to be the partial Fourier matrix, formed by choosing at random a subset of m rows from the $d \times d$ discrete Fourier matrix, the computational cost associated with embedding the data is $O(d \log d)$ and the required embedding dimension assumes the near-optimal bound $m = O(\delta^{-2} \log N \log^4 d)$.

In the proposed IcTKM algorithm, we can reduce the computational costs associated with finding the sparse support $\mathcal{I}_n^t$ in (1) by embedding the entire data set $\boldsymbol{Y}$ and the initialization dictionary $\boldsymbol{\Psi}$ with a $m \times K$ partial Fourier matrix with randomized column signs. Let $\boldsymbol{\Gamma}$ denote such a matrix; we replace the thresholding operation in (1) with the *compressed-thresholding* operation, which we define as

$$\mathcal{I}_n^{ct} := \arg \max_{|\mathcal{I}|=S} \|\boldsymbol{\Psi}_{\mathcal{I}}^* \boldsymbol{\Gamma}^* \boldsymbol{\Gamma} \boldsymbol{y}_n\|_1. \tag{3}$$

We proceed by showing that the computational cost associated with compressed thresholding reduces to $O\left(\delta^{-2} \log N \log^4(d) KN\right)$ as opposed to $O(dKN)$ of regular thresholding. Thus the embedding distortion δ controls the performance improvement of IcTKM over ITKM. It is then shown that the number of data points N (sample complexity) required for IcTKM to locally identify a dictionary with high probability is essentially the same as that of ITKM, while the embedding distortion δ increases the best achievable error $\tilde{\varepsilon}$ and reduces the convergence radius of IcTKM; However increasing the minimally achievable error is largely negligible for high-dimensional data, since the realistically achievable error is determined by the sample size. The reduction of convergence radius is somewhat more disappointing, but as we will show in our numerical experiments on very-large data sets, in practice this does not affect the good global convergence behaviour of IcTKM, [11].

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