Sequential Learning of Analysis Operators

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I. Introduction

Many tasks in high dimension signal processing, such as denoising or reconstruction from incomplete information, can be efficiently solved if the data at hand is known to have intrinsic low dimension. One popular model with intrinsic low dimension is the union of subspace model, where every signal is assumed to lie in one of the low dimensional linear subspaces. However, as the number of subspaces increases, the model becomes more and more cumbersome to use unless the subspaces can be parametrised. Two examples of large unions of parametrised subspaces, that have been successfully employed, are sparsity in a dictionary and cosparsity in an analysis operator. In the sparse model the subspaces correspond to the linear span of just a few normalised columns from a $K \times d$ dictionary matrix, $\Phi = (\phi_1 \dots \phi_K)$ with $\|\phi_k\|_2 = 1$, meaning any data point y can be approximately represented as superposition of $S \ll d$ dictionary elements, $y \approx \sum_{j=1}^{S} \phi_{i_j} x_{i_j}$. In the cosparse model the subspaces correspond to the orthogonal complement of the span of some normalised rows from a $d \times K$ analysis operator $\Omega = (\omega_1^\star \dots \omega_K^\star)^\star$ with $\|\omega_k\|_2 = 1$, meaning any data point y is orthogonal to ℓ analysers, meaning the vector Ωy has ℓ zero entries.

However, before being able to exploit these models for a given data class it is necessary to identify the parametrising dictionary or analysis operator. This can be done either via a theoretical analysis or a learning approach. While dictionary learning is by now an established field, see [1] for an introductory survey, results in analysis operator learning are still relatively scarse, [2], [3], [4], [5], [6], [7]. In this work we will contribute to the development of the field by taking an optimisation approach, which will lead to an online algorithm for learning analysis operators

II. The target function

Suppose we are given a batch of signals $y_1, \dots, y_N \in \mathbb{R}^d$ which are ℓ-cosparse with respect to some (unknown) operator $\Omega \in \mathbb{R}^{K \times d}$. Our goal is to learn the operator Ω from the data.

Define $\mathcal{A} = \{\Gamma \in \mathbb{R}^{K \times d} \colon \|\gamma_k\|_2 = 1\}$, where γ_k denotes the k-th row of Γ and $\mathcal{X}_\ell = \{(x_1, x_2, \dots, x_N) \in \mathbb{R}^{K \times N} \colon |\text{supp}(x_n)| = K - \ell\}$. If we collect the given data $y_1 \dots, y_N$ in a matrix $Y = (y_1, \dots, y_N) \in \mathbb{R}^{d \times N}$, then the unknown operator Ω should satisfy $\|\Omega Y - X\|_F = 0$ for some $X \in \mathcal{X}_\ell$ since every column x_n can be used to zero out all non-zero entries in Ωy_n. This suggests to find Ω via the following opimisation program,

$$\hat{\Omega} = \operatorname*{argmin}_{\Gamma \in \mathcal{A}} \min_{X \in \mathcal{X}_\ell} \|\Gamma Y - X\|_F^2. \tag{1}$$

This can be rewritten as

$$\hat{\Omega} = \operatorname*{argmin}_{\Gamma \in \mathcal{A}} \sum_{n=1}^{N} \min_{J \subset [K] \colon |J| = \ell} \|\Gamma_J y_n\|_2^2, \tag{2}$$

where Γ_J denotes the submatrix of Γ consisting only of the rows of Γ indexed by J. So in order to find an estimate for Ω, we need to minimize the function f_N over $\mathcal{A}$, where

$$f_N(\Gamma) = \sum_{n=1}^{N} \min_{J \subset [K] \colon |J| = \ell} \|\Gamma_J y_n\|_2^2. \tag{3}$$

III. The ISAOL algorithm

Our next step consists in finding an efficient way to minimize the target function given in equation (3). In the spirit of online learning, we will employ a stochastic gradient descent algorithm in order to achieve this goal.

The derivative of f_N with respect to some row γ_k of Γ is given by

$$\frac{\partial f_N}{\partial \gamma_k}(\Gamma) = \gamma_k \underbrace{\sum_{n \colon k \in J_n} 2 y_n y_n^\star}_{=: A_k(\Gamma)}, \tag{4}$$

where $J_n = \operatorname{argmin}_{|J| = \ell} \|\Gamma_J y_n\|_2^2$. Using this expression we now perform a semi-implicit gradient step followed by a projection onto the unit sphere for each of the rows of Γ, i.e. $\bar{\gamma}_k = \gamma_k - \alpha \bar{\gamma}_k A_k(\Gamma)$. Setting $\alpha = 1$, yields the Implicit Sequential Analysis Operator Learning (ISAOL) algorithm:

ISAOL(Γ,ℓ,Y) - (one iteration)

- For all $n \in [N]$
  - Find $J_n = \operatorname{argmin}_{|J| = \ell} \|\Gamma_J y_n\|_2^2$
  - For all $k \in J_n$ update $A_k = A_k + y_n y_n^\star$
- Set $\bar{\gamma}_k = \gamma_k (I + A_k)^{-1}$
- Output $\bar{\Gamma} = (\frac{\bar{\gamma}_1}{\|\bar{\gamma}_1\|_2}, \dots, \frac{\bar{\gamma}_K}{\|\bar{\gamma}_K\|_2})^\star$

Note that this algorithm is sequential with respect to the data y_n, since every data point is used to perform the update of the matrices A_k, but not needed anymore later in the algorithm.

Figure 2 shows the result of performing analysis operator learning with the ISAOL algorithm for image patches taken from Figure 1.

IV. Conclusion

We presented a new sequential algorithm for learning analysis operators in an online setting along with some numerical experiments showing the performance of the algorithm in various situations. The problem of multiple recovery of the same rows can for example be tackled by introducing a replacement strategy, [8].

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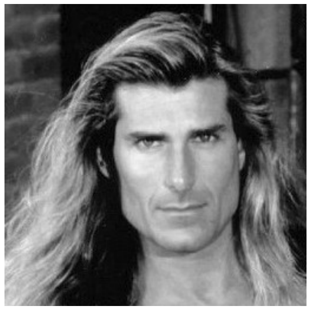

Fig. 1. The image, from which our analysis operator was learned.

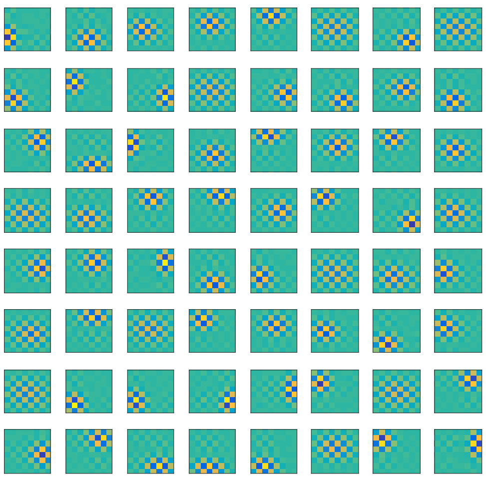

Fig. 2. The recovered analysis operator for Fabio. We took the cosparsity parameter to be $\ell = 57$.

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