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Markov Processes and Related Fields

A Law of Large Numbers for the Range of Rotor Walks on Periodic Trees

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Abstract. The aim of the current work is to prove a law of large numbers for the range size of recurrent rotor walks with random initial configuration on a general class of trees, called *periodic trees* or *directed covers of graphs.* This generalizes [7, Theorem 1.1], but the proofs are of different nature and rely on generating functions.

KEYWORDS: rotor walk, range, rate of escape, periodic tree, Galton – Watson tree, generating function, law of large numbers, spectral radius, multitype branching process, recurrence, transience

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1. Introduction

In [7], we have considered rotor walks $(X_n)_{n\geq 0}$ with random initial configuration of rotors on regular trees and on Galton – Watson trees, and we have proven a law of large numbers for the size of the range and for the rate of escape. The proofs used there rely on on analyzing the behavior and the growth of certain branching processes that appear when looking at rotor walks on trees. We continue the investigation of the range of rotor walk in this work, and we prove similar results for positive recurrent rotor walks on periodic trees, but with completely different methods that involve generating functions.

Periodic trees are a straightforward generalization of regular trees. Using the methods developed here, one could also recover results from [7]. Nevertheless, the methods for periodic trees are technically more involved, and the results are stated in terms of spectral radii of adjacency matrices related to the periodic tree and to the random initial rotor configuration. Transience and recurrence of rotor walks on such trees was investigated in [6].

Before stating the main result, we introduce shortly the setting we are working on. Let G be a finite graph on N vertices, D its adjacency matrix, and T_i be a periodic tree (directed cover of G), with root of type $i \in \{1, \dots, \mathsf{N}\}$. A rotor configuration on T_i is a function which assigns to each vertex a rotor pointing to one of the neighbors, and the neighbors of each vertex are ordered counterclockwise. A rotor walk $(X_n^i)_{n\geq 0}$ on T_i is a process which starts at the root vertex, and at each time step it first rotates the rotor to the next neighbor in the counterclockwise order, and then it moves to the neighbor the rotor points at. A child of a vertex is called good if the rotor walk visits this child before returning to the parent vertex. The tree of good children of (X_n^i), which we denote by $\mathsf{T}_i^{\mathsf{good}}$, is a subtree of T_i, consisting of only good children. Suppose that (X_n^i) is a rotor walk on T_i, with $\mathcal{D}$-distributed random initial configuration of rotors; see the paragraph above Definition 2.1 for the precise definition of the distribution $\mathcal{D}$. Then $\mathsf{T}_i^{\mathsf{good}}$ is a *multitype branching process* (MBP), whose first moment matrix will be denoted by M. It has been shown in [6] that (X_n^i), for every $i \in \{1, \dots, \mathsf{N}\}$ is recurrent if and only if the spectral radius $\rho(M) \leq 1$. The range $R_n^i = \{X_1^i, \dots, X_n^i\}$ represents the number of distinct visited points by the walk (X_n^i) up to time n and $|R_n^i|$ its size. The walk (X_n^i) starts at the root of T_i, and the super index gives the dependence on the type i of root vertex. For the size $|R_n^i|$ of the rotor walk up to time n we prove the following law of large numbers.

Theorem 1.1. *Let $(X_n^i)_{n\geq 0}$ be a rotor walk with a $\mathcal{D}$-distributed random initial configuration of rotors on a periodic tree T_i, with root of type $i \in \{1, \dots, \mathsf{N}\}$. For all $i \in \{1, \dots, \mathsf{N}\}$, if $\rho(M) < 1$, then*

$$\frac{|R_n^i|}{n} \to \frac{1}{2}\left(1 - \frac{1}{\gamma}\right), \quad \textit{almost surely, as } n \to \infty,$$

where γ is the spectral radius of the matrix $I + (D - I)(I - M)^{-1}$.

In the null recurrent case, when $\rho(M) = 1$, we conjecture the following.

Conjecture 1.1. For all $i \in \{1, \dots, \mathsf{N}\}$, if $\rho(M) = 1$, then

$$\frac{|R_n^i|}{n} \to \frac{1}{2}, \quad \text{almost surely, as } n \to \infty.$$

Let us briefly comment on the proof of Theorem 1.1. For $\rho(M) < 1$, we use generating functions equalities and multitype Galton−Watson processes. For the remaining two cases (when the rotor walk is null recurrent ($\rho(M) = 1$) and transient ($\rho(M) > 1$)) one can also prove a law of large numbers for the size of the range, but the amount of technical calculations will be too high for the expected result. Range of rotor walks and its shape was considered also in [4] on comb lattices and on Eulerian graphs. It is conjectured in [8], that on $\mathbb{Z}^2$, the range of uniform rotor walks is asymptotically a disk, and its size if of order $n^{2/3}$. This is challenging, since on $\mathbb{Z}^2$ is not even known whether the uniform rotor walk is recurrent or transient. For recent results on rotor walks on transient graphs with initial rotor configuration sampled from the wired uniform spanning forest oriented toward infinity measure see [2, 3].

2. Preliminaries

Since we use the results from [6] on recurrence/transience of rotor walks on periodic trees, we keep the same notation as there.

2.1. Periodic trees

Periodic trees are also known in the literature as *directed covers of graphs* or *trees with finitely many cone types.* Such trees have interesting properties in what concerns both the behavior of random walks and of rotor walks on them. We add in the appendix an example of a rotor-recurrent periodic tree that contains rotor-transient subtrees.

Graphs and Trees. Let $\mathsf{G} = (V, E)$ be a locally finite and connected directed multigraph, with vertex set V and edge set E. For ease of presentation, we shall identify the graph G with its vertex set V, i.e. $i \in \mathsf{G}$ means $i \in V$. If (i, j) is an edge of G, we write $i \sim_{\mathsf{G}} j$, and write $\mathrm{d}(i, j)$ for the *graph distance*. Let $D = (d_{ij})_{i,j\in\mathsf{G}}$ be the *adjacency matrix* of G, where d_{ij} is the number of directed edges connecting i to j. We write d_i for the sum of the entries in the i-th row of D, that is $d_i = \sum_{j\in\mathsf{G}} d_{ij}$ is the *degree* of the vertex i. A *tree* $\mathcal{T}$ is a connected, cycle-free graph. A *rooted tree* is a tree with a distinguished vertex r, called *the root.* For a vertex $x \in \mathcal{T}$, denote by $|x|$ the *height* of x, that is the graph distance from the root to x. For $x \in \mathcal{T} \setminus \{r\}$, denote by $x^{(0)}$ its *ancestor*, which is the unique neighbor of x closer to the root r. We attach to $\mathcal{T}$ an additional vertex o to the root r, which will be considered in the following as a sink vertex. Additionally we fix a planar embedding of $\mathcal{T}$ and enumerate the neighbors of a vertex $x \in \mathcal{T}$ in counterclockwise order $\left(x^{(0)}, x^{(1)}, \ldots, x^{(d_x-1)}\right)$ beginning with the ancestor. We will call a vertex y a *descendant* of x, if x lies on the unique shortest path from y to the root r. A descendant of x, which is also a neighbor of x, will be called a *child.* The *principal branches* of $\mathcal{T}$ are the subtrees rooted at the children of the root r.

Directed Covers of Graphs. Suppose now that G is a finite, directed and strongly connected multigraph with adjacency matrix $D = (d_{ij})$. Let N be the cardinality of the vertices of G, and label the vertices of G by $\{1, 2, \ldots, \mathsf{N}\}$. The *directed cover* T of G is defined recursively as a rooted tree whose vertices are labeled by the vertex set $\{1, 2, \dots, \mathsf{N}\}$ of G. The root r of T is labeled with some $i \in \{1, 2, \dots, \mathsf{N}\}$. Recursively, if x is a vertex in T with label $i \in \mathsf{G}$, then x has d_{ij} descendants with label j. We define the *label function* $\tau : \mathsf{T} \to \mathsf{G}$ as the map that associates to each vertex in T its label in G. The label $\tau(x)$ of a vertex x will be also called the *type* of x. For a vertex $x \in \mathsf{T}$, we will not only need its type, but also the types of its children. In order to keep track of the type of a vertex and the types of its children we introduce the *generation function* $\chi = (\chi_i)_{i\in\mathsf{G}}$ with $\chi_i : \{1, \dots, d_i\} \to \mathsf{G}$. For a vertex x of type i, $\chi_i(k)$ represents the type of the k-th child $x^{(k)}$ of x, i.e.,

$$\text{if } \tau(x) = i \text{ then } \chi_i(k) = \tau(x^{(k)}), \text{ for } k = 1, \dots, d_i.$$

As the neighbors $\big(x^{(0)}, \dots, x^{(d_{\tau(x)})}\big)$ of any vertex x are drawn in counterclockwise order, the generation function χ also fixes the planar embedding of the tree and thus defines T uniquely as a planted plane tree. The tree T constructed in this way is called the *directed cover* of G. Such trees are also known as *periodic trees*, see [10]. The graph G is called the *base graph* or the *generating graph* for the tree T. We write T_i for a tree with root r of type i, that is $\tau(r) = i$, and we say that T_i is a periodic tree with N types of vertices and root of type $i \in \{1, 2, \dots, \mathsf{N}\}$.

2.2. Multitype branching processes

A multitype branching process (MBP) is a generalization of a Galton−Watson process, where one allows a finite number of distinguishable types of particles with different probabilistic behavior. The particle types will coincide with the different types of vertices in the periodic trees under consideration, and will be denoted by $\{1, \dots, \mathsf{N}\}$.

A *multitype branching process* is a Markov process $(\mathsf{Z}_n)_{n\in\mathbb{N}_0}$ such that the states $\mathsf{Z}_n = (\mathsf{Z}_{n,1}, \dots, \mathsf{Z}_{n,\mathsf{N}})$ are N-dimensional vectors with non-negative components. The initial state Z_0 is nonrandom. The i-th entry $\mathsf{Z}_{\mathsf{n},\mathsf{i}}$ of Z_n represents the number of particles of type i in the n-th generation. The transition law of the process is as follows. If $\mathsf{Z}_0 = \mathsf{e}_i$, where e_i is the N-dimensional vector whose $i-th$ component is 1 and all the others are 0, then Z_n has the generating function $\mathsf{f}(\mathsf{z}) = \big(f^1(\mathsf{z}), \dots, f^{\mathsf{N}}(\mathsf{z})\big)$ with

$$f^i(\mathsf{z}) = f^i(z_1, \dots, z_{\mathsf{N}}) = \sum_{s_1,\dots,s_{\mathsf{N}}\geq 0} p^i(s_1, \dots, s_{\mathsf{N}}) z_1^{s_1} \cdots z_{\mathsf{N}}^{s_{\mathsf{N}}}, \tag{2.1}$$

and $0 \leq z_1, \dots, z_{\mathsf{N}} \leq 1$, where $p^i(s_1, \dots, s_{\mathsf{N}})$ is the probability that a particle of type i has s_j children of type j, for $j = 1, \dots, \mathsf{N}$. For $\mathsf{i} = (i_1, \dots, i_{\mathsf{N}})$ and $\mathsf{j} = (j_1, \dots, j_{\mathsf{N}})$, the one-step transition probabilities are given by

$$\mathsf{p}(\mathsf{i}, \mathsf{j}) = \mathbb{P}[\mathsf{Z}_{n+1} = \mathsf{j} | \mathsf{Z}_n = \mathsf{i}] = \text{coefficient of } \mathsf{z}^{\mathsf{j}} \text{ in } \big(\mathsf{f}(\mathsf{z})\big)^{\mathsf{i}},$$

where $\big(\mathbf{f}(\mathbf{z})\big)^{\mathbf{i}} = \prod_{k=1}^{\mathsf{N}} f^k(\mathbf{z})^{i_k}$. For vectors z, s, we write $\mathsf{z}^{\mathsf{s}} = (z_1^{s_1}, \dots, z_{\mathsf{N}}^{s_{\mathsf{N}}})$. Let $M = (m_{ij})$ be the matrix of the first moments:

$$m_{ij} = \mathbb{E}\big[\mathsf{Z}_{1,j} | \mathsf{Z}_0 = \mathbf{e}_i\big] = \frac{\partial f^i(z_1, \dots, z_{\mathsf{N}})}{\partial z_j}\bigg|_{\mathsf{z}=\mathbf{1}}, \tag{2.2}$$

where $\mathbf{1} = (1, \dots, 1)$. Then m_{ij} represents the expected number of offsprings of type j of a particle of type i in one generation. If there exists an n such that $m_{ij}^{(n)} > 0$ for all i, j, then M is called *strictly positive* and the process Z_n is called *positive regular*. If each particle has exactly one child, then Z_n is called *singular*. The following is well known; see HARRIS [5].

Theorem 2.1. *Assume Z_n is positive regular and nonsingular, and let $\rho(M)$ be the spectral radius of M. If $\rho(M) \leq 1$, then the process Z_n dies with probability one. If $\mathsf{r}(M) > 1$, then Z_n survives with positive probability.*

We will also make use of the mixed second moments defined as following:

$$\sigma_{jk}^i = \mathbb{E}\big[\mathsf{Z}_{1,j}\mathsf{Z}_{1,k} | \mathsf{Z}_0 = \mathbf{e}_i\big] = \frac{\partial^2 f^i(z_1, \dots, z_{\mathsf{N}})}{\partial z_j \partial z_k}\bigg|_{\mathsf{z}=\mathbf{1}}. \tag{2.3}$$

2.3. Rotor walks

Let $\mathcal{T}$ be a rooted tree with root r and for each vertex x order its neighbors counterclockwise $\{x^0, \dots, x^{d_x-1}\}$. A *rotor configuration* is a function $\rho : \mathcal{T} \to \mathcal{T}$, with $\rho(x) \sim x$, for all $x \in \mathcal{T}$. By abuse of notation, we write $\rho(x) = i$ if the rotor at x points to the neighbor $x^{(i)}$. A *rotor walk* $(X_n)_{n \geq 0}$ is defined by the following rule. Let x be the current position of the walker, and $\rho(x) = i$ the state of the rotor at x. In one step the walker does the following: it increments the rotor at x to point to the next neighbor $x^{(i+1)}$ in the counterclockwise order of the neighbors of x, that is $\rho(x)$ is set to $i+1$ (with addition performed modulo d_x). Then it moves to position $x^{(i+1)}$. The rotor walk is obtained by repeatedly applying this rule. We suppose that it starts at the root, that is $X_0 = r$.

For a vertex $x \in \mathcal{T}$ define the set of *good children* as $\big\{x^{(k)} : \rho(x) < k \leq d_x\big\}$. This means that a particle performing rotor walk will first visit all its good children before visiting its ancestor. An infinite sequence of vertices $\big(x_n\big)_{n \in \mathbb{N}}$ with each vertex being a child of the previous one, is called a *live path* if for every $n \geq 0$ the vertex x_{n+1} is a good child of x_n. An *end* of $\mathcal{T}$ is an infinite sequence of vertices $x_1, x_2, \dots$ each being the ancestor of the next. An end is called *live* if the subsequence $(x_i)_{i \geq j}$ starting at one of its vertices is a live path. This definitions were introduced in [1].

Nondeterministic Rotor Configurations on Directed Covers. Let T_i be a periodic tree with N types of vertices, and root r of type i, to whom an additional sink vertex o is added. Moreover, let (X_n^i) be a rotor walk on T_i starting at r, with initial random configuration, distributed as following. Let $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_{\mathsf{N}})$ be a vector of probability distributions: for each $i \in \{1, \ldots, \mathsf{N}\}$, $\mathcal{D}_i$ is a probability distribution with values in $\{0, \ldots, d_i\}$. Consider a *random initial configuration* ρ of rotors on T_i, such that $(\rho(x))_{x\in\mathsf{T}_i}$ are independent random variables, and $\rho(x)$ has distribution $\mathcal{D}_j$ if the vertex x is of type j. Shortly

$$\rho(x) \overset{d}{\sim} \mathcal{D}_j \quad \Longleftrightarrow \quad \tau(x) = j. \tag{2.4}$$

If (2.4) is satisfied, we shall say that the rotor configuration ρ is $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_{\mathsf{N}})$-distributed, and we write $\rho \overset{d}{\sim} \mathcal{D}$.

Definition 2.1. For $i \in \{1, \ldots, \mathsf{N}\}$ and $k \in \{0, \ldots, d_i\}$ denote by $\mathfrak{C}_i^j(k)$ the number of good children with type j of a vertex x with type i, if the rotor $\rho(x)$ at x is in position k, i.e.,

$$\mathfrak{C}_i^j(k) = \#\big\{l \in \{k+1, \ldots, d_i\} : \chi_i(l) = j\big\}.$$

We have that $\sum_{j=1}^{\mathsf{N}} \mathfrak{C}_i^j(k) = d_i - k$. Using this definition we can now define a new MBP which models connected subtrees consisting of only good children. In this MBP, $p^i(s_1, \ldots, s_{\mathsf{N}})$ represents the probability that a vertex of type i has s_j good children of type j, with $j = 1, \ldots, \mathsf{N}$. Define the generating function of the MBP as in (2.1) and the probabilities p^i by

$$p^i(s_1, \ldots, s_m) = \begin{cases} \mathcal{D}_i(k) & \text{if for all } j = 1, \ldots, \mathsf{N} : s_j = \mathfrak{C}_i^j(k), \text{ and } k \in \{0, \ldots, d_i\}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

with $\mathcal{D}_i(k) = \mathbb{P}[\rho(x) = k]$, for $k \in \{0, \ldots, d_i\}$ and $i \in \{1, \ldots, \mathsf{N}\}$. In the following we always make the additional assumption that this MBP is *positive regular and nonsingular*, such that Theorem 2.1 can be applied. In particular, when the rotors point to every neighbor with positive probability these two conditions are always satisfied. Let $M(\mathcal{D})$ be the first moment matrix — as defined in (2.2) — of the MBP with offspring probabilities given in (2.5), and $\rho = \rho(M(\mathcal{D}))$ its spectral radius. It has been shown in [6] that the rotor walk (X_n^i) is recurrent if and only if $\rho \leq 1$, and transient if $\rho > 1$. For the MBP with offspring distribution as in (2.5), since every vertex in T_i has finite degree, also the entries σ_{jk}^i of the second moment matrix are finite, for all $i, j, k \in \{1, \ldots, \mathsf{N}\}$. If $\rho(M(\mathcal{D})) = 1$, we call (X_n) *null recurrent*, and we refer to the *critical case*. Otherwise, if $\rho(M(\mathcal{D})) < 1$, we say that (X_n) is positive recurrent.

3. Positive recurrent rotor walks

From now on, we let T_i be a periodic tree with N types, and root of type $i \in \{1, \ldots, \mathsf{N}\}$, and let D the $\mathsf{N} \times \mathsf{N}$-adjacency matrix of the finite graph which generates T_i. Let (X_n^i) be a rotor walk on T_i with $\mathcal{D}$-distributed initial configuration of rotors, as defined in (2.4). Denote by $\mathsf{T}_i^{\mathsf{good}}$ the tree of good children of (X_n^i), and the associated MBP (multitype branching process) with transition probabilities as in (2.5). We denote again by Z_n the size of the n-th generation of this MBP, that is, $\mathsf{Z}_n = (\mathsf{Z}_{n,1}, \dots, \mathsf{Z}_{n,\mathsf{N}})$, where $\mathsf{Z}_{n,i}$ represents the number of good children of type i in the n-th generation of the MBP. Finally, let $M = M(\mathcal{D})$ be the first moment matrix of Z_n.

In this section we handle the case $\rho(M) < 1$, when the rotor walk (X_n^i) is *positive recurrent*. We first look a look at the range $R_n^i = \{X_1^{(i)}, \dots, X_n^i\}$ at times (τ_k^i), when the rotor walk (X_n^i) returns to the sink o for the k-th time: set $\tau_0^i = 0$ and for $k \geq 1$ define

$$\tau_k^i = \inf\{n > \tau_{k-1}^i : X_n^i = o\}.$$

In the recurrent case, these stopping times are almost surely finite. Let us denote by $\mathsf{R}_k^i := R_{\tau_k^i}^i$. Then

$$\tau_k^i - \tau_{k-1}^i = 2|\mathsf{R}_k^i|. \tag{3.1}$$

The equation above has the following explanation: at time τ_k^i the walker is at the sink, all rotors in the explored part R_k^i of the tree T_i point towards the root, while all other rotors are still in their random initial configuration. Between two consecutive stopping times τ_{k-1}^i and τ_k^i, the walker performs a depth first search in the finite subtree induced by R_k^i, by visiting every child of a vertex from right to left order. In a depth first search of R_k^i is visited exactly two times. We first prove the following result.

Theorem 3.1. *Let γ be the spectral radius of the matrix $I+(D-I)(I-M)^{-1}$. For all $i = 1, \dots, \mathsf{N}$ we have the following strong law of large numbers at times $(\tau_k^i)_{k\geq 0}$:*

$$\lim_{k\to\infty} \frac{|\mathsf{R}_k^i|}{\tau_k^i} = \frac{1}{2}\left(1 - \frac{1}{\gamma}\right), \quad a.s.$$

In order to prove Theorem 3.1, we first look at the size of the tree $\mathsf{T}_i^{\mathsf{good}}$ of good children, tree which, in view of $\rho < 1$, dies out almost surely. Let $\mathbf{Y} = \sum_{\mathsf{n}=0}^{\infty} \mathsf{Z}_{\mathsf{n}}$ be the vector valued random variable counting the total number of vertices of $\mathsf{T}_i^{\mathsf{good}}$, separately for each type. Let $\mathbf{F}(\mathbf{z}) = \big(F^1(\mathbf{z}), \dots, F^{\mathsf{N}}(\mathbf{z})\big)$ with

$$F^i(\mathsf{z}) = \sum_{s_1,\dots,s_{\mathsf{N}}\geq 0} \mathbb{P}\big[\mathbf{Y} = (s_1, \dots, s_{\mathsf{N}}) | \mathsf{Z}_0 = \mathbf{e}_i\big] z_1^{s_1} \dots z_{\mathsf{N}}^{s_{\mathsf{N}}},$$

be the generating function of $\mathbf{Y}$. It follows (see e.g. [5, 13.2] for the case of Galton – Watson trees with just one type) that $F^i(\mathsf{z})$ satisfies the following functional equation

$$F^i(\mathbf{z}) = z_i \cdot f^i\big(\mathbf{F}(\mathbf{z})\big) = z_i \cdot f^i\big(F^1(\mathbf{z}), \dots, F^{\mathsf{N}}(\mathbf{z})\big), \tag{3.2}$$

for all $i \in \{1, \dots \mathsf{N}\}$, where $f(\mathsf{z}) = (f^1(\mathsf{z}), \dots, f^{\mathsf{N}}(\mathsf{z}))$ is the generating function of the MBP Z_n, as defined in (2.1). Let now $V = \big(v_{ij}\big)_{i,j=1,\dots,\mathsf{N}}$ be the first moment matrix of $\mathbf{Y}$, that is,

$$v_{ij} = \mathbb{E}\big[\mathbf{Y}_j | \mathsf{Z}_0 = \mathbf{e}_i\big] = \left.\frac{\partial F^i(\mathbf{z})}{\partial z_j}\right|_{\mathbf{z}=\mathbf{1}}.$$

Lemma 3.1. *We have $V = (I - M)^{-1}$ where I is the identity matrix, and the spectral radius of V is given by*

$$\rho(V) = \frac{1}{1 - \rho(M)} > 1. \tag{3.3}$$

Proof. Differentiating the functional equation (3.2) and using that $f^i(\mathbf{1}) = 1$ and $\mathbf{F}(\mathbf{1}) = \mathbf{1}$ gives

$$\begin{aligned} v_{ij} &= \left.\left\{ z_i \sum_{a=1}^{\mathsf{N}} \frac{\partial f^i}{\partial F^a}\big(\mathbf{F}(\mathbf{z})\big) \frac{\partial F^a}{\partial z_j}(\mathbf{z}) + \delta_{ij} \cdot f^i\big(\mathbf{F}(\mathbf{z})\big) \right\}\right|_{\mathbf{z}=\mathbf{1}} \\ &= \sum_{a=1}^{\mathsf{N}} \left.\frac{\partial f^i}{\partial z_a}(\mathbf{z})\right|_{\mathbf{z}=\mathbf{1}} \left.\frac{\partial F^a}{\partial z_j}(\mathbf{z})\right|_{\mathbf{z}=\mathbf{1}} + \delta_{ij} \\ &= \sum_{a=1}^{\mathsf{N}} m_{ia} v_{aj} + \delta_{ij}. \end{aligned}$$

Thus we get the matrix equality $V = M \cdot V + I$, where I is the identity matrix, and this implies that

$$V = (I - M)^{-1}.$$

The inverse of $I - M$ exists, since by assumption $\rho(M) < 1$, and the equation for the spectral radius follows immediately. □

We next look at the mixed second moments of $\mathbf{Y}$, for which we prove the following.

Lemma 3.2. $\mathbb{E}\big[\mathbf{Y}_j \mathbf{Y}_k | \mathsf{Z}_0 = \mathbf{e}_i\big] < \infty$ *for all* $i, j, k \in \{1, \dots, \mathsf{N}\}$.

Proof. Let

$$\xi^i_{jk} = \mathbb{E}\big[\mathbf{Y}_j \mathbf{Y}_k | \mathsf{Z}_0 = \mathbf{e}_i\big] = \left.\frac{\partial^2 F^i(z_1, \dots, z_{\mathsf{N}})}{\partial z_j \partial z_k}\right|_{\mathbf{z}=\mathbf{1}}.$$

From the functional equation (3.2) we get

$$\xi^i_{jk} = \frac{\partial}{\partial z_k} \left.\left\{ z_i \sum_{a=1}^{\mathsf{N}} \frac{\partial f^i}{\partial F^a}\big(\mathbf{F}(\mathbf{z})\big) \frac{\partial F^a}{\partial z_j}(\mathbf{z}) + \delta_{ij} \cdot f^i\big(\mathbf{F}(\mathbf{z})\big) \right\}\right|_{\mathbf{z}=\mathbf{1}}$$

$$= \left\{ \sum_{a=1}^{\mathsf{N}} \frac{\partial f^i}{\partial F^a}\big(\mathbf{F}(\mathbf{z})\big) \left(\delta_{ij} \frac{\partial F^a}{\partial z_j}(\mathbf{z}) + z_i \frac{\partial^2 F^a}{\partial z_j \partial z_k}(\mathbf{z}) + \delta_{ik} \frac{\partial F^a}{\partial z_k}(\mathbf{z}) \right) \right.$$
$$\left. + z_i \sum_{a,b=1}^{\mathsf{N}} \frac{\partial^2 f^i}{\partial F^a \partial F^b}\big(\mathbf{F}(\mathbf{z})\big) \frac{\partial F^a}{\partial z_j}(\mathbf{z}) \frac{\partial F^b}{\partial z_k}(\mathbf{z}) \right\} \Bigg|_{\mathbf{z}=\mathbf{1}} .$$

Using the fact that $\mathbf{F}(\mathbf{1}) = \mathbf{1}$, the above equation simplifies to

$$\xi^i_{jk} = \sum_{a=1}^{\mathsf{N}} m_{ia}\big(\delta_{ik} v_{aj} + \xi^a_{jk} + \delta_{ij} v_{ak}\big) + \sum_{a,b=1}^{\mathsf{N}} \sigma^i_{ab} v_{aj} v_{bk}. \tag{3.4}$$

For each k, we introduce the matrices $S_k = \big(\xi^i_{jk}\big)_{i,j=1,\dots,\mathsf{N}}$ and $\Gamma_k = \big(\gamma^i_{jk}\big)_{i,j=1,\dots,\mathsf{N}}$, where

$$\gamma^i_{jk} = \sum_{a=1}^{\mathsf{N}} m_{ia}\big(\delta_{ik} v_{aj} + \delta_{ij} v_{ak}\big) + \sum_{a,b=1}^{\mathsf{N}} \sigma^i_{ab} v_{aj} v_{bk} < \infty.$$

From (3.4) it follows that $S_k = MS_k + \Gamma_k$ and thus $S_k = (I - M)^{-1}\Gamma_k$. In particular $\xi^i_{jk} < \infty$ for all i, j, k, which concludes the proof. $\square$

Since by Lemma 3.1 also the first moment of the total population size exists, we get the following.

Corollary 3.1. $\mathrm{Var}(\mathbf{Y}|\mathsf{Z}_0 = \mathbf{e}_i) < \infty$.

Definition 3.1. For a finite subset $G \subset \mathsf{T}_i$ of the vertex set of T_i, the (multi-type) cardinality of G is defined as $\#G = (g_1, \dots, g_{\mathsf{N}})$, where $g_k = \#\big\{v \in G : v \text{ has type } k\big\}$.

Definition 3.2. Let G be a connected subset of T_i containing the root. Denote by $\partial_o G$ the set of leaves of G, that is, $\partial_o G = \big\{w \in \mathsf{T}_i \setminus G : \exists v \in G \text{ s.t. } v \sim w\big\}$.

We will use the following simple fact.

Lemma 3.3. *Let G be a finite connected subset of T_i containing the root. Then*

$$\#\partial_o G = (D - I) \cdot \#G + \mathsf{e}_i.$$

Proof. Let $D = \big(d_{kl}\big)_{k,l=1,\dots,\mathsf{N}}$ be the adjacency matrix of the finite graph which generates the periodic tree T_i. Let H be the set of children of all vertices in G. By the definition of periodic trees, any vertex of type k has d_{kl} children of type l, hence $\#H = D \cdot \#G$. We can then recover the set of leaves of G by

$$\partial_o G = H \setminus G \cup \{\text{root of } \mathsf{T}_i\},$$

and the claim follows. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Write $\mathsf{R}^i_k = R^i_{\tau^i_k}$ for the range of (X^i_n) up to time τ^i_k. From [6], we know that R^i_1 is a subcritical multitype Galton – Watson tree with offspring probabilities as given in (2.5) and first moment matrix M. At time τ^i_k the rotor walk is at the sink, all rotors in the visited set R^i_k point towards the root, while all other rotors are still in their initial $\mathcal{D}$-distributed random configuration. Thus between times τ^i_k and τ^i_{k+1} the rotor walk performs a depth first search of $\mathsf{R}^i_k \cup \partial_o \mathsf{R}^i_k$ visiting every child of each vertex from right to left. Whenever it reaches a leaf vertex $v \in \partial_o \mathsf{R}^i_k$ of type l, it performs an independent finite excursion starting at v, excursion which has the same distribution as R^l_1. Let $L^i_k = \partial_o \mathsf{R}^i_k$. Then, by Lemma 3.3 we have

$$\# L^i_k = (D - I) \cdot \# \mathsf{R}^i_k + \mathbf{e}_i. \tag{3.5}$$

Let $\mathbf{L}_0 = \mathbf{e}_i$ if the root of the tree is of type i, and let $\mathbf{L}_k = \# L^i_k$ be the (multitype)-cardinality of the leaves of the range after the k-th return to the sink, that is, the j-th entry in $\mathbf{L}_k$ represents the number of leaves of type j in $\partial_o \mathsf{R}^i_k$. By Lemma 3.1, we get

$$\mathbb{E}\big[\mathbf{L}_{1,j} | \mathbf{L}_0 = \mathbf{e}_i\big] = \big(I + (D - I)(I - M)^{-1}\big)_{ji}.$$

Then $\big(\mathbf{L}_k\big)_{k \geq 0}$ is a multitype Galton – Watson process with first moment matrix

$$\Gamma = \big(I + (D - I)(I - M)^{-1}\big)^T. \tag{3.6}$$

We also need to check the finiteness of the second cross moments. Let $C_i = \big(c^i_{jl}\big)_{j,l=1,\ldots,\mathsf{N}}$ be the matrix of second cross moments

$$c^i_{jl} = \mathbb{E}\big[\mathbf{L}_{1,j} \mathbf{L}_{1,l} | \mathbf{L}_0 = \mathbf{e}_i\big].$$

Since $\mathrm{Var}(\mathbf{L_1} | \mathbf{L}_0 = \mathbf{e}_i) = C_i - \mathbb{E}\big[\mathbf{L}_1 | \mathbf{L}_0 = \mathbf{e}_1\big] \mathbb{E}\big[\mathbf{L}_1^T | \mathbf{L}_0 = \mathbf{e}_1\big]$, and the first moments of $\mathbf{L}_1$ exist by Lemma 3.1 and Lemma 3.3 it suffices to check the existence of the variance. We have

$$\begin{aligned} \mathrm{Var}(\mathbf{L_1} | \mathbf{L}_0 = \mathbf{e}_i\big) &= \mathrm{Var}\big((D - I) \# \mathsf{R}^i_1 + \mathbf{e}_i\big) \\ &= (D - I) \mathrm{Var}\big(\# \mathsf{R}^i_1\big) (D - I)^T, \end{aligned}$$

which together with Corollary 3.1 implies that all matrix elements of C_i are finite. Let $\gamma > 1$ be the spectral radius of Γ and $\mathbf{u} > 0$ be the corresponding Perron-Frobenius eigenvector. Since C_i is finite, Kesten-Stigum Theorem [9] for the branching process $\mathbf{L}_k$ implies the existence of an almost surely positive random variable W such that

$$\gamma^{-k} \mathbf{L}_k \to W \mathbf{u}, \tag{3.7}$$

almost surely as $k \to \infty$. Let $e = (u, v)$ where u, v are vertices of the tree with $v \sim u$, be an edge of the tree. The type $\iota(e)$ of the edge e is defined as the type of the endvertex of the edge which is further away from the sink. For each time $n \geq 0$ we let $\boldsymbol{\psi}(n) \in \mathbb{N}_{\geq 0}^{\mathsf{N}}$ be the vector of the number of edges of each type that are traversed by the rotor walk up to time n. That is, $\boldsymbol{\psi}(n) = \big(\psi_1, \dots, \psi_{\mathsf{N}}\big)$, with

$$\psi_l = \#\big\{l = 1, \dots, n : \ \iota(\{X_{l-1}^i, X_l^i\}) = l\},$$

which satisfies $\|\boldsymbol{\psi}(n)\|_1 = n$. Moreover, if we define $\boldsymbol{\tau}_k = \boldsymbol{\psi}(\tau_k)$, then for all $k \geq 1$

$$\boldsymbol{\tau}_k^{\boldsymbol{i}} - \boldsymbol{\tau}_{k-1}^{\boldsymbol{i}} = 2\#\mathsf{R}_k^i, \tag{3.8}$$

which together with (3.5) yields

$$(D - I)\big(\boldsymbol{\tau}_k^{\boldsymbol{i}} - \boldsymbol{\tau}_{k-1}^{\boldsymbol{i}}\big) + 2\mathbf{e}_i = (D - I)2\#\mathsf{R}_k^i + \mathbf{e}_i = 2\mathbf{L}_k,$$

assuming the initial state $\mathbf{L}_0 = \mathbf{e}_i$ for the branching process. Multiplying the previous equation by γ^{-k} and using (3.7) we get

$$(D - I)\left(\frac{\boldsymbol{\tau}_k^{\boldsymbol{i}}}{\gamma^k} - \frac{1}{\gamma} \cdot \frac{\boldsymbol{\tau}_{k-1}^{\boldsymbol{i}}}{\gamma^{k-1}}\right) + 2\gamma^{-k}\mathbf{e}_i = 2\gamma^{-k}\mathbf{L}_k \to 2W\mathbf{u},$$

almost surely as $k \to \infty$. Writing $\boldsymbol{\sigma}_k = \boldsymbol{\tau}_k^i / \gamma^k$, the previous equation reduces to

$$\boldsymbol{\sigma}_k - \frac{1}{\gamma}\boldsymbol{\sigma}_{k-1} \to 2(D - I)^{-1} W\mathbf{u},$$

and the limit vector $2(D - I)^{-1}W\mathbf{u}$ is almost surely positive. Denoting by $\boldsymbol{\sigma}^\star$ the common limit of $\boldsymbol{\sigma}_k$ and $\boldsymbol{\sigma}_{k-1}$, we have

$$\boldsymbol{\sigma}^\star = \lim_{k \to \infty} \frac{\boldsymbol{\tau}_k^i}{\gamma^k} = 2\left(1 - \frac{1}{\gamma}\right)^{-1}(D - I)^{-1}W\mathbf{u} > 0.$$

Therefore, also τ_k^i grows exponentially with rate γ, thus, the almost sure limit

$$\lim_{k \to \infty} \frac{\tau_k^i}{\gamma^k} = \lim_{k \to \infty} \frac{\big\|\boldsymbol{\tau}_k^i\big\|_1}{\gamma^k}, \tag{3.9}$$

exists and is almost surely positive. Now since $|\mathsf{R}_k^i| = \big|R_{\tau_k^i}^i\big| = \|\#\mathsf{R}_k^i\|_1$, the total size of the range up to time of the k-th return τ_k^i is $\tau_k^i - \tau_{k-1}^i = 2|\mathsf{R}_k^i|$, which dividing by τ_k^i gives

$$\frac{|\mathsf{R}_k^i|}{\tau_k^i} = \frac{1}{2}\left(1 - \frac{\tau_{k-1}^i}{\tau_k^i}\right) = \frac{1}{2}\left(1 - \frac{1}{\gamma} \cdot \frac{\tau_{k-1}^i}{\gamma^{k-1}} \cdot \frac{\gamma^k}{\tau_k^i}\right).$$

Using now (3.9) we get the almost sure limit

$$\lim_{k\to\infty}\frac{|\mathsf{R}_k^i|}{\tau_k^i}=\frac{1}{2}\left(1-\frac{1}{\gamma}\right),$$

which completes the proof. □

The generalization of the previos result to all times follows the lines of the similar result for regular trees in [7]. For sake of completeness, we adapt the result to periodic trees.

Proof of Theorem 1.1.. Write again $\mathsf{R}_k^i = R^i_{\tau_k^i}$ for the range of (X_n^i) up to time τ_k^i. For every n, let

$$k=\max\{j:\tau_j^i<n\},$$

so that $\tau_k^i < n \le \tau_{k+1}^i$ and $X_n^i \in \mathsf{R}_{k+1}^i := R^i_{\tau_{k+1}}$. For each $k = 1, 2, \dots,$ we partition the time intervals $(\tau_k^i, \tau_{k+1}^1]$ into finer intervals, on which the behavior of the range can be controlled. Recall, from the proof of Theorem 3.1, that $L_k^i = \partial_o \mathsf{R}_k^i$ represents the set of leaves of R_k^i, and $\mathbf{L}_k = \# L_k^i$ represents the multitype cardinality of L_k^i, which is a multitype Galton−Watson process with first moment matrix Γ as in (3.6) and spectral radius $\gamma > 1$. We order the vertices in $L_k^i = \partial_o \mathsf{R}_k^i = \{x_1, \dots, x_{|L_k^i|}\}$ from right to left, and introduce the the following two (finite) sequences of stopping times $(\eta_k(j))$ and $(\theta_k(j))$ of random length $|L_k^i| + 1$, as following: let $\theta_k(0) = \tau_k^i$ and $\eta_k(|L_k^i| + 1) = \tau_{k+1}^i$ and for $j = 1, 2, \dots, |L_k^i|$

$$\begin{aligned}\eta_k(j) &= \min\{l > \theta_k(j-1):\ X_l^i = x_j\},\\ \theta_k(j) &= \min\{l > \eta_k(j):\ X_l^i = x_j \text{ and } \rho(X_l^i) = x_j^{d_{x_j}}\}.\end{aligned}\tag{3.10}$$

That is, for each leaf x_j, the time $\eta_k(j)$ represents the first time the rotor walk reaches x_j, and $\theta_k(j)$ represents the last time the rotor walk returns to x_j after making a full excursion in the subtree rooted at x_j. Then

$$(\tau_k^i,\tau_{k+1}^i]=\left\{\cup_{j=1}^{|L_k^i|+1}\big(\theta_k(j-1),\eta_k(j)\big]\right\}\cup\left\{\cup_{j=1}^{|L_k^i|}\big(\eta_k(j),\theta_k(j)\big]\right\}.$$

For leaves x_j of type l, with $l = 1, \dots, \mathsf{N}$, the increments $(\theta_k(j) - \eta_k(j))$ are i.i.d random variables, and distributed according to τ_1^l, which represents the time the rotor walk, started at the root of type l of a periodic tree T_l, needs to return to the sink for the first time. Once the rotor walk reaches the leaf x_j for the first time at time $\eta_k(j)$, the subtree rooted at x_j was never visited before by a rotor walk. Even more, the tree of good children with root x_j is a subcritical multitype Galton−Watson tree, which dies out almost surely. Thus, the rotor walk on this subtree is recurrent, and it returns to x_j at time $\theta_k(j)$ for the last time. Then $\theta_k(j) - \eta_k(j)$ represents the length of this excursion which has expectation $\mathbb{E}[\tau_1^l] = 2\mathbb{E}[|\mathsf{R}_1^l(x_j)|]$, and $\mathsf{R}_1^l(x_j)$ has the same distribution as R_1^l which has the first moment matrix M, with $\rho(M) < 1$, associated with transition probabilities (2.5). In the time intervals $(\theta_k(j-1), \eta_k(j)]$, the rotor walk leaves the leaf x_{j-1} and returns to the confluent between x_{j-1} and x_j, from where it continues its journey until it reaches x_j. Then $\eta_k(j) - \theta_k(j-1)$ is the time the rotor walk needs to reach the new leaf x_j after leaving x_{j-1}. In this time intervals, the range does not change, since (X_n^i) makes steps only in R_k^i. Depending on the position of X_n^i, we shall distinguish two cases:

Case 1: There exists a $j \in \{1, 2, \ldots, |L_k^i|\}$ such that $n \in (\eta_k(j), \theta_k(j)]$.

Case 2: There exists a $j \in \{1, 2, \ldots, |L_k^i|+1\}$ such that $n \in (\theta_k(j-1), \eta_k(j)]$.

Case 1: if $n \in (\eta_k(j), \theta_k(j)]$ for some j, then $\eta_k(j) < n \leq \theta_k(j)$

$$\frac{|R^i_{\eta_k(j)}|}{\theta_k(j)} \leq \frac{|R^i_n|}{n} \leq \frac{|R^i_{\theta_k(j)}|}{\eta_k(j)}, \tag{3.11}$$

and we show that, as $k \to \infty$, the difference $\dfrac{|R^i_{\theta(j)}|}{\eta(j)} - \dfrac{|R^i_{\eta(j)}|}{\theta(j)} \to 0$, almost surely. We use the following relations: for all $i = 1, \ldots, \mathsf{N}$ and all $j = 1, 2, \ldots, |L_k^i|$, if the type of x_j is $l \in \{1, \ldots, \mathsf{N}\}$

$$\begin{aligned} |R^i_{\theta_k(j)}| &= |R^i_{\eta_k(j)}| + |\mathsf{R}_1^l(x_j)| \\ \theta_k(j) &= \eta_k(j) + \tau_1^l(x_j), \end{aligned}$$

where $\tau_1^l(x_j)$ is a random variable which is distributed as τ_1^l, which is finite almost surely. Moreover, $\mathsf{R}_1^l(x_j)$, represents the range of the rotor walk started at x_j which has type l, until the first return to x_j; $\mathsf{R}_1^l(x_j)$ has the same distribution as R_1^l, the range of the rotor walk started at the root r of type l of a periodic tree T_l, until the first return τ_1^l. Then,

$$\begin{aligned} 0 &\leq \frac{|R^i_{\theta_k(j)}|}{\eta_k(j)} - \frac{|R^i_{\eta_k(j)}|}{\theta_k(j)} \\ &= \frac{\tau_1^l(x_j)|\mathsf{R}_1^l|}{\eta_k(j)(\eta_k(j)+\tau_1^l(x_j))} + \frac{|\mathsf{R}_1^l|}{\eta_k(j)+\tau_1^l(x_j)} + \frac{\tau_1^l(x_j)|R^i_{\eta_k(j)}|}{\eta_k(j)(\eta_k(j)+\tau_1^l(x_j))} \\ &\leq \frac{2|\mathsf{R}_1^l|}{\eta_k(j)} + \frac{\tau_1^l(x_j)|R^i_{\eta_k(j)}|}{(\eta_k(j))^2}. \end{aligned}$$

As $k \to \infty$, $\eta_k(j) \to \infty$, but $\tau_1^l(x_j)$ and $|\mathsf{R}_1^l|$ are finite, almost surely, therefore the first term in the last inequality above converges to 0 almost surely, and we still have to prove convergence to 0 of the second term. But we know that $\tau_k^i < \eta_k(j) < \tau_{k+1}^i$, therefore

$$0 \leq \frac{\tau_1^l(x_j)|R^i_{\eta_k(j)}|}{(\eta_k(j))^2} \leq \frac{\tau_1^l(x_j)}{\tau_k^i} \cdot \left(\frac{|\mathsf{R}_k^i|}{\tau_k^i} + \frac{\sum_{l=1}^{j}(\theta_k(l)-\eta_k(l))}{2\tau_k^i} \right) \to 0, \tag{3.12}$$

almost surely. Since $\tau_1^l(x_j)$ is finite almost surely, and $\tau_k^i \to \infty$ as $k \to \infty$, it is clear that the first fraction goes to 0, while $|\mathsf{R}_k^i|/\tau_k^i$ converges to $(1-1/\gamma)/2$ by Theorem 3.1. The last term in the equation above is bounded from above by $(\tau_{k+1}^i - \tau_k^i)/(2\tau_k^i)$ which in view of (3.1) and Theorem 3.1 converges almost surely to $\gamma < \infty$ as $k \to \infty$. This shows that

$$\frac{|R^i_{\theta_k(j)}|}{\eta_k(j)} - \frac{|R^i_{\eta_k(j)}|}{\theta_k(j)} \to 0, \quad \text{almost surely, as } k \to \infty,$$

therefore there exists $\alpha_0 = \lim |R^i_{\theta_k(j)}|/\eta_k(j)$, and both the right and the left hand side in (3.11) converge to $\alpha_0 < \infty$ almost surely, which implies that $|R_n^i|/n$ converges to α_0 almost surely, as well. But, along the subsequence (τ_k^i), we have from Theorem 3.1, that

$$\frac{|\mathsf{R}_k^i|}{\tau_k^i} \to \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)$$

almost surely, as $k \to \infty$, therefore $\alpha_0 = (1-1/\gamma)/2$.

Case 2: if $n \in (\theta_k(j-1), \eta_k(j)]$ for some $j \in \{1, 2, \ldots, |L_k^i| + 1\}$, and since in this time interval the rotor walk X_n^i visits no new vertices and moves only in R_k^i, we have

$$\frac{|R^i_{\theta_k(j-1)}|}{n} = \frac{|R_n^i|}{n} = \frac{|R^i_{\eta_k(j)}|}{n}. \tag{3.13}$$

In view of $\lim |R^i_{\theta_k(j)}|/\eta_k(j) \to (1-1/\gamma)/2$ almost surely and

$$\frac{|R^i_{\theta_k(j)}|}{\eta_k(j)} = \frac{|R^i_{\theta_k(j)}|}{\theta_k(j) - \tau_1^l(x_j)} = \frac{|R^i_{\theta_k(j)}|}{\theta_k(j)} \frac{1}{1 - \frac{\tau_1^l(x_j)}{\theta_k(j)}},$$

and $\tau_1^l(x_j)/\theta_k(j) \to 0$, it follows that $|R^i_{\theta_k(j)}|/\theta_k(j)$ converges almost surely to $(1-1/\gamma)/2$. The same argument can be applied for the almost sure convergence of $|R^i_{\eta_k(j)}|/\eta_k(j)$ to $(1-1/\gamma)/2$. Finally, in view of (3.13)

$$\frac{|R^i_{\eta_k(j)}|}{\eta_k(j)} \le \frac{|R^i_{\eta_k(j)}|}{n} = \frac{|R_n^i|}{n} = \frac{|R^i_{\theta_k(j-1)}|}{n} \le \frac{|R^i_{\theta_k(j-1)}|}{\theta_k(j-1)},$$

both the lower bound and the upper bound in the previous equation converge almost surely to $(1-1/\gamma)/2$. Thus, also in this case $|R_n^i|/n$ converges almost surely to $(1-1/\gamma)/2$, and the claim follows. □

4. Palindromic trees

If we look at periodic trees with a strong mirror symmetry we can give a more geometric interpretation of the limit in Theorem 3.1.

Definition 4.1. A periodic tree T with production rule $\chi_i(k)$ is called palindromic if the word $\big(\chi_i(1), \chi_i(2), \dots, \chi_i(d_i)\big)$ is a palindrome for all types $i \in \{1, \dots, \mathsf{N}\}$, that is,

$$\chi_i(k) = \chi_i(d_i + 1 - k),$$

for all $k \in \{1, \dots, d_i\}$.

Let (X_n) be a rotor walk on T, and denote by T^{good} the tree of good children for (X_n).

Lemma 4.1. *Let T be a palindromic periodic tree, and (X_n) a rotor walk with uniform initial rotor configuration. Let D be the adjacency matrix of the graph which generates T and M the first moment matrix of T^{good}. Then*

$$D = 2M.$$

Proof. Recall that $D = \big(d_{ij}\big)$ with $d_{ij} = \sum_{k=1}^{d_i} \mathbb{1}\{\chi_i(k) = j\}$. For uniformly distributed rotors the first moment matrix $M = \big(m_{ij}\big)$ is given by

$$m_{ij} = \frac{1}{d_{i+1}} \sum_{k=1}^{d_i} k\mathbb{1}\{\chi_i(k) = j\}.$$

Fix $i \in \{1, \dots, \mathsf{N}\}$ assuming d_i is even. We can split m_{ij} into two summands as follows

$$\begin{aligned} m_{ij} &= \frac{1}{d_i + 1}\left(\sum_{k=1}^{d_i/2} k\mathbb{1}\{\chi_i(k) = j\} + \sum_{k=d_i/2+1}^{d_i} k\mathbb{1}\{\chi_i(k) = j\}\right) \\ &= \frac{1}{d_i + 1}\left(\sum_{k=1}^{d_i/2} k\mathbb{1}\{\chi_i(k) = j\} + \sum_{k=d_i/2+1}^{d_i} k\mathbb{1}\{\chi_i(d_i + 1 - k) = j\}\right), \end{aligned}$$

where in the second line we use that T is palindromic. Changing the order of summation of the second sum and using the substitution $l = d_i + 1 - k$ gives

$$\begin{aligned} m_{ij} &= \frac{1}{d_i + 1}\left(\sum_{k=1}^{d_i/2} k\mathbb{1}\{\chi_i(k) = j\} + \sum_{l=1}^{d_i/2} (d_i + 1 - l)\mathbb{1}\{\chi_i(l) = j\}\right) \\ &= \frac{1}{d_i + 1} \sum_{k=1}^{d_i/2} (k + d_i + 1 - k)\mathbb{1}\{\chi_i(k) = j\} \\ &= \sum_{k=1}^{d_i/2} \mathbb{1}\{\chi_i(k) = j\} = \frac{1}{2} d_{ij}, \end{aligned}$$

where the last identity is again due to the palindromic property of T. If d_i is odd, we rewrite m_{ij} into three summands

$$m_{ij} = \frac{1}{d_i+1}\Bigg(\sum_{k=1}^{(d_i-1)/2} k\mathbb{1}\{\chi_i(k)=j\} + \frac{d_i+1}{2}\mathbb{1}\Big\{\chi_i\Big(\frac{d_i+1}{2}\Big)=j\Big\} + \sum_{k=(d_i+3)/2}^{d_i} k\mathbb{1}\{\chi_i(k)=j\}\Bigg).$$

By the palindromic property of T, and setting $l = d_i+1-k$ we can transform the third summand in the last equation

$$\sum_{k=(d_i+3)/2}^{d_i} k\mathbb{1}\{\chi_i(k)=j\} = \sum_{k=(d_i+3)/2}^{d_i} k\mathbb{1}\{\chi_i(d_i+1-k)=j\} = \sum_{l=1}^{(d_i-1)/2} (d_i+1-l)\mathbb{1}\{\chi_i(l)=j\}.$$

Thus

$$m_{ij} = \frac{1}{2}\mathbb{1}\Big\{\chi_i\Big(\frac{d_i+1}{2}\Big)=j\Big\} + \frac{1}{d_i+1}\sum_{k=1}^{(d_i-1)/2}(k+d_i+1-k)\mathbb{1}\{\chi_i(k)=j\}$$
$$= \frac{1}{2}\mathbb{1}\Big\{\chi_i\Big(\frac{d_i+1}{2}\Big)=j\Big\} + \sum_{k=1}^{(d_i-1)/2}\mathbb{1}\{\chi_i(k)=j\} = \frac{1}{2}d_{ij},$$

and this implies $2M = D$. □

The following is obvious.

Lemma 4.2. *Let D be a $(\mathsf{N}\times\mathsf{N})$-matrix with spectral radius ψ. Let γ be the spectral radius of $I + (D-I)(I-\alpha^{-1}\cdot D)^{-1}$, for some real number α. If the spectral radius ψ of D is not equal to α then*

$$\gamma = 1 + \frac{\psi - 1}{1-\psi/\alpha} = \frac{(\alpha-1)\psi}{\alpha-\psi}.$$

Theorem 4.1. *Let T be a palindromic periodic tree, and consider a rotor walk (X_n) on T with uniform initial rotor configuration. Let $\mathsf{br}(T)$ be the branching number of T. Then (X_n) is recurrent if and only if $\mathsf{br}(T)\le 2$ and in the positive recurrent case (when $\mathsf{br}(T)<2$) we have:*

$$\lim_{n\to\infty}\frac{|R_n|}{n} = \frac{\mathsf{br}(T)-1}{\mathsf{br}(T)}, \quad \textit{almost surely.}$$

Proof. Recall that the branching number of T is equal to the spectral radius of the matrix D. By Lemma 4.1, $D = 2M$. In the positive recurrent case, we can apply Lemma 4.2 with $\alpha = 2$, which gives $\gamma = \frac{\mathbf{br(T)}}{2-\mathbf{br(T)}}$, which together with Theorem 3.1 completes the proof. □

Appendix A. Rotor-recurrent trees

Definition A.1. We call a tree T *rotor-recurrent* if the rotor walk (X_n) on T with uniform initial rotor configuration is recurrent. If (X_n) is transient on T, we call T *rotor-transient.*

Not much is known about the recurrence and transience of rotor walks on graphs other than trees. But even on trees we can give examples of unusual properties of rotor-recurrence that confirm the fact that a general theory of rotor-recurrence will necessarily be much more involved than the theory of recurrence of random walks. In [6] the authors give an example of a tree that is rotor-recurrent or rotor-transient depending on its planar embedding into the plane. This suggests that the underlying graph for the rotor walk does not provide enough information for stating a rotor-recurrence criteria. The whole ribbon structure of the graph, which determines the way the rotors turn, will be needed for that. The rotor-recurrence has even more peculiar properties as the following example shows.

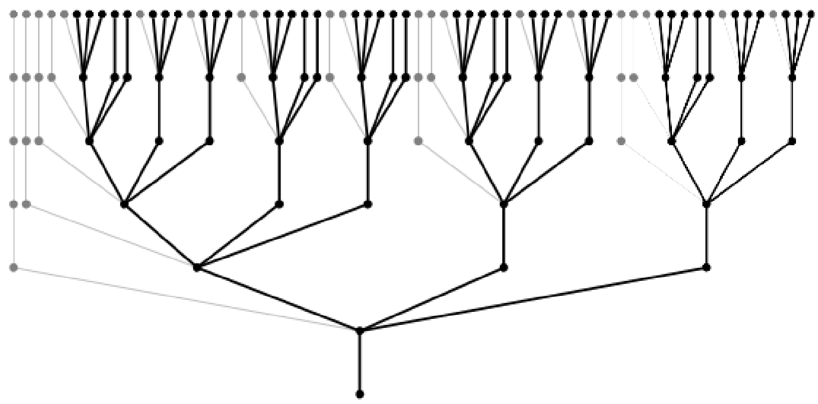

Figure 1. A rotor-recurrent tree that contains a transient subtree, which is drawn in black.

Example A.1. Let T be the direct cover given by the following generator χ. We use the notation of [6]; note the planar embedding of the tree, and thus the rotor-mechanism, already specified by the table χ.

$$\begin{array}{cc|cccc} & & & k \rightarrow & & \\ & \chi_i(k) & 1 & 2 & 3 & 4 \\ \hline & 1 & 2 & 2 & 1 & 3 \\ & 2 & 1 & & & \\ i & 3 & 4 & & & \\ \downarrow & 4 & 5 & & & \\ & 5 & 2 & & & \end{array}$$

The rotor walk on T has the first moment matrix M given by

$$M = \begin{pmatrix} 3/5 & 3/5 & 4/5 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 0 \end{pmatrix},$$

with spectral radius $\rho(M) = 0.967$. By [6, Theorem 3.5], since $\rho(M) < 1$, T is rotor-recurrent. We now construct a subtree $\bar{\mathsf{T}}$ from T by deleting all vertices of type 3 and all their descendants. Thus $\bar{\mathsf{T}}$ is the direct cover defined by the generator $\bar{\chi}$ as follows:

$$\begin{array}{cc|ccc} & & & k \rightarrow & \\ & \bar{\chi}_i(k) & 1 & 2 & 3 \\ \hline & 1 & 2 & 2 & 1 \\ \overset{i}{\downarrow} & 2 & 1 & & \end{array}$$

The rotor walk on $\bar{\mathsf{T}}$ has first moment matrix $\bar{M}$ given by

$$\bar{M} = \begin{pmatrix} 3/4 & 3/4 \\ 1/2 & 0 \end{pmatrix},$$

which has spectral radius $\rho(\bar{M}) = 1.093$. Hence, by [6, Theorem 3.5], $\bar{\mathsf{T}}$ is rotor-transient.

Remark. There exist rotor-recurrent graphs that contain rotor-transient subgraphs.

References

[1] O. Angel and A.E. Holroyd (2011) Rotor walks on general trees. *SIAM J. Discrete Math.* **25** (1), 423–446. `arXiv:1009.4802`.

[2] Swee Hong Chan (2019) Infinite-step stationarity of rotor walk and the wired spanning forest. `arXiv:1909.13195`.

[3] Swee Hong Chan (2019) Rotor walks on transient graphs and the wired spanning forest. *SIAM J. Discrete Math.* **33** (4), 2369–2393. `arXiv:1809.09790`.

[4] L. Florescu, L. Levine and Y. Peres (2016) The range of a rotor walk. *Amer. Math. Monthly* **123** (7), 627–642.

[5] T.E. Harris (2002) *The Theory of Branching Processes.* Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY. Corrected reprint of the 1963 original [Springer, Berlin; MR0163361 (29 #664)].

[6] W. Huss and E. Sava (2012) Transience and recurrence of rotor-router walks on directed covers of graphs. *Electron. Commun. Probab.* **17**, no. 41, 1–13. arXiv:1203.1477.

[7] W. Huss and E. Sava-Huss (Apr 2019) Range and speed of rotor walks on trees. *Journal of Theoretical Probability.* arXiv:1805.05746.

[8] R. Kapri and D. Dhar (Nov 2009) Asymptotic shape of the region visited by an eulerian walker. *Phys. Rev. E* **80**, 051118. arXiv:0906.5506.

[9] H. Kesten and B.P. Stigum (10 1966) A limit theorem for multidimensional Galton – Watson processes. *Ann. Math. Statist.* **37** (5), 1211–1223.

[10] R. Lyons and Y. Peres (2016) *Probability on Trees and Networks.* Cambridge Series in Statistical and Probabilistic Mathematics **42**. Cambridge University Press, New York.