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Good packings of Banach spaces

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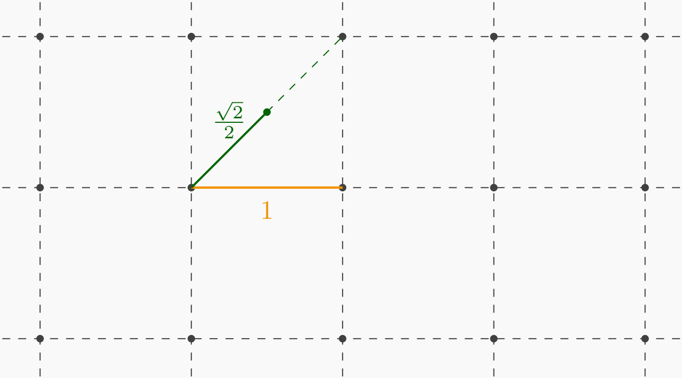
Maximal separated sets

- **Every maximal 1-separated set is 1-dense.**
  - $\mathcal{D}$ is *r*-**separated** if $\|d-h\| \geqslant r$ for $d \neq h \in \mathcal{D}$.
  - $\mathcal{D}$ is *R*-**dense** if for all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant R$.
- In $\mathbb{R}^2$, the integer grid $\mathbb{Z}^2$ is 1-separated and $\frac{\sqrt{2}}{2}$-dense.



Maximal separated sets

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  - $\mathcal{D}$ is **R-dense** if for all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant R$.
- In $\mathbb{R}^2$, the integer grid $\mathbb{Z}^2$ is 1-separated and $\frac{\sqrt{2}}{2}$-dense.
- In $(\mathbb{R}^2, \|\cdot\|_\infty)$, it $\mathbb{Z}^2$ is 1-separated and $\frac{1}{2}$-dense.
- **Can we always find a 1-separated set that is r-dense, $r<1$?**
  - Can we always find ad hoc constructions that improve Zorn's one?
  - Or are there spaces where the optimal net is a 'random' one?
- **Folklore.** In finite dimensions, yes.
  - WLOG, $\mathcal{X} = (\mathbb{R}^d, \|\cdot\|)$ and $\mathbb{Z}^d$ is 1-separated.
  - Take a maximal 1-separated set Λ in $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$.
  - By compactness, Λ is r-dense in $\mathbb{T}^d$, for some $r<1$.
  - Lift it to $\mathbb{R}^d$.
- **What about infinite-dimensional spaces?**
- A change of perspective: multiply by 2.

Packings

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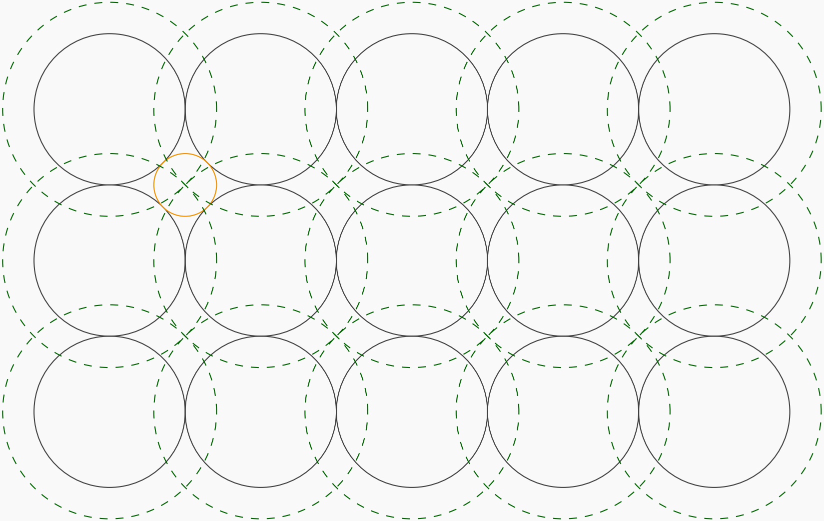
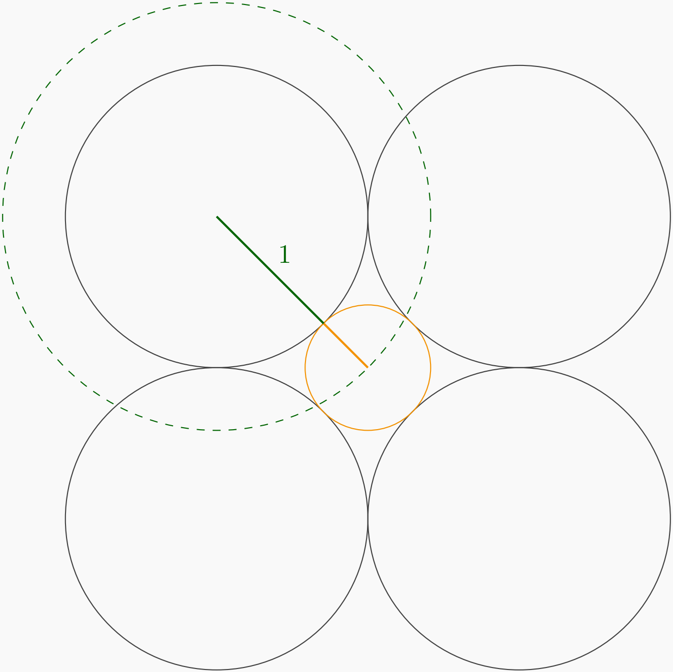
Packings

- A **packing** is a collection of non-overlapping balls of radius 1.
- **How to measure how optimal (or packed) a packing is?**
- In finite dimensions, compute its density.
- In infinite dimensions:
  - Compute the radius of the largest non-overlapping ball (the largest hole in the packing).
  - How much do we have to inflate the balls to cover the space?
  - First is second -1.
- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this.

Packings

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Two formulas

- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma(\mathcal{X}) \coloneqq \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

- The **lattice simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma^*(\mathcal{X}) \coloneqq \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

Two formulas

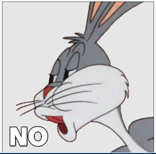
$$\gamma(\mathcal{X}) := \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\gamma^*(\mathcal{X}) := \inf\{r > 0 \colon \exists\, \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

- If $\mathcal{X}$ has a packing that also covers, $\gamma(\mathcal{X}) = 1$.
- Taking any maximal packing $\rightsquigarrow \gamma(\mathcal{X}) \leqslant 2$.
- **De Bernardi, R., Somaglia (2026+).** $\gamma^*(\mathcal{X}) \leqslant 2$.
- **Casini, Papini, Zanco (1986).** $\gamma(\mathcal{X}) \geqslant \frac{2}{K(\mathcal{X})}$.
- To sum it up

$$(\textbf{good}) \quad 1 \leqslant \frac{2}{K(\mathcal{X})} \leqslant \gamma(\mathcal{X}) \leqslant \gamma^*(\mathcal{X}) \leqslant 2 \quad (\textbf{bad})$$

- **Is it true that $\gamma(\mathcal{X}) < 2$ for all spaces?**
  - **Are we smarter than Zorn?**



A problem of Swanepoel

- **Our original motivation:**
- **Swanepoel (2009).** $\gamma(\ell_p) = \frac{2}{2^{1/p}}$ which equals $\frac{2}{K(\ell_p)}$.
- **Swanepoel (2009).** Is it true that for all Banach spaces

$$\gamma(\mathcal{X}) = \frac{2}{K(\mathcal{X})}?$$

- **If the unit ball of $\mathcal{X}$ admits a LUR point, then $\gamma(\mathcal{X}) > 1$.**
- Consider $\ell_1 \oplus_2 \mathbb{R}$. Its unit ball has a LUR point, but $K(\ell_1 \oplus_2 \mathbb{R}) = 2$.
- Every Banach space $\mathcal{X}$ is isomorphic to a Banach space $\mathcal{Y}$ with $K(\mathcal{Y}) = 2$ and $\gamma(\mathcal{Y}) > 1$.
  - **So, there are reflexive (even isomorphic to ℓ_2) counterexamples to Swanepoel's question.**

Some known values

- For $1 \leqslant p < \infty$ and every infinite κ

$$\gamma(\ell_p(\kappa)) = \gamma^*(\ell_p(\kappa)) = \frac{2}{2^{1/p}}.$$

- If $\mathcal{X}$ is **separable** and octahedral, or $\mathcal{X} = \mathcal{C}(\mathcal{K})$ with $\mathcal{K}$ zero-dimensional

$$\gamma(\mathcal{X}) = \gamma^*(\mathcal{X}) = 1.$$

- This applies, *e.g.*, to: $L_1(\mu)$, $\mathcal{C}([0,1])$, $\mathcal{C}(2^\omega)$, $\mathcal{C}(\mathcal{K})$ for $\mathcal{K}$ countable (or scattered).
- Some Lipschitz-free spaces, spaces of Lipschitz functions, tensor products, ...

ϕ-octahedral spaces

- A **modulus** is a function $\phi\colon (0,\infty) \to (0,\infty)$ such that $\phi(0^+) = 0$ and $t \mapsto \frac{\phi(t)}{t}$ is increasing.
- A normed space $\mathcal{X}$ is **ϕ-octahedral** if for every $\varepsilon > 0$ and every finite-dimensional subspace $\mathcal{Z}$ of $\mathcal{X}$ there is $x \in S_{\mathcal{X}}$ with

$$\|z + \lambda x\| \geqslant (1-\varepsilon)\big(1 + \phi(|\lambda|)\big) \qquad z \in S_{\mathcal{Z}},\ \lambda \neq 0.$$

- Octahedral spaces are ϕ-octahedral; **so are uniformly convex ones**.
- **Take $p \in [1,\infty)$, $p_k \to \infty$. Then**

$$\mathcal{X}_p = \left(\bigoplus_{k=1}^{\infty} \ell_{p_k}(\omega_k)\right)_{\ell_p}$$

satisfies $\gamma(\mathcal{X}_p) = 2$.
- **There are reflexive (resp. octahedral) spaces with $\gamma(\mathcal{X}) = 2$.**

Unknown values

- Is there a **separable** Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
- Is there a Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?
- What are the exact values of $\gamma(\ell_1 \oplus_2 \mathbb{R})$ and $\gamma^*(\ell_1 \oplus_2 \mathbb{R})$?
  - They are > 1 (LUR point).
- What are the exact values of $\gamma(\ell_1 \oplus_2 \ell_1)$ and $\gamma^*(\ell_1 \oplus_2 \ell_1)$?
- **And many more...**

Thank you for your attention!