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Tiling Hilbert spaces

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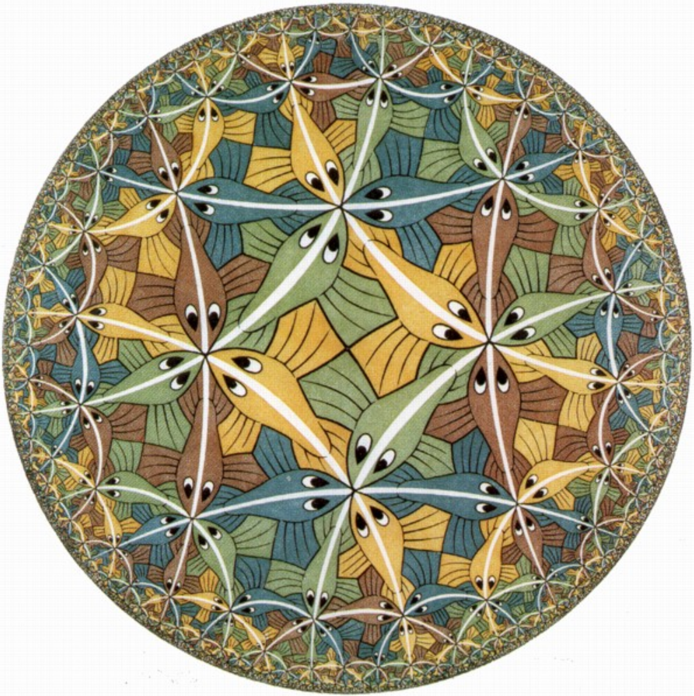
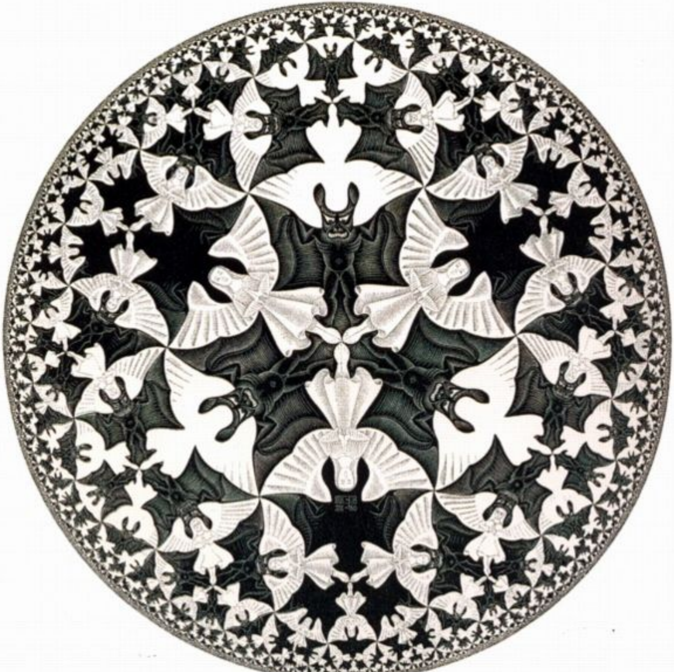
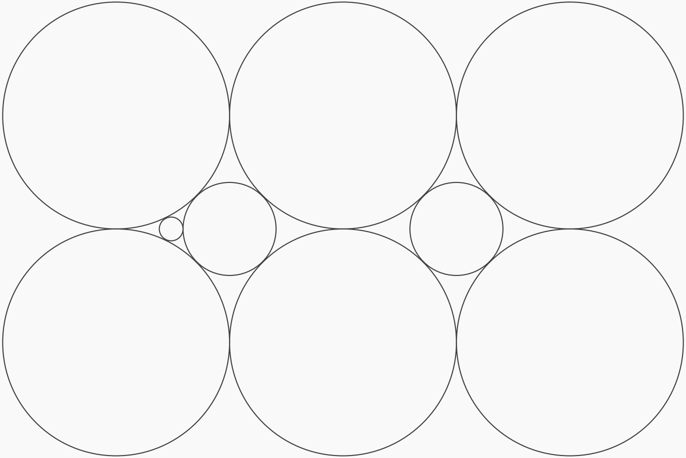




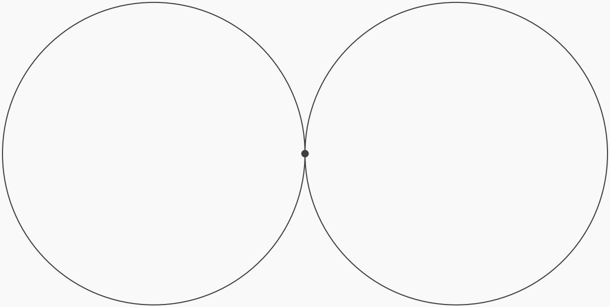
Can you tile the plane with balls?

- Are there closed balls $(B_j)_{j=1}^{\infty}$ with disjoint interiors s.t. $\mathbb{R}^2 = \bigcup B_j$?



Or maybe not?

- Assume $(B_i)_{i \in I}$ is a tiling.
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.



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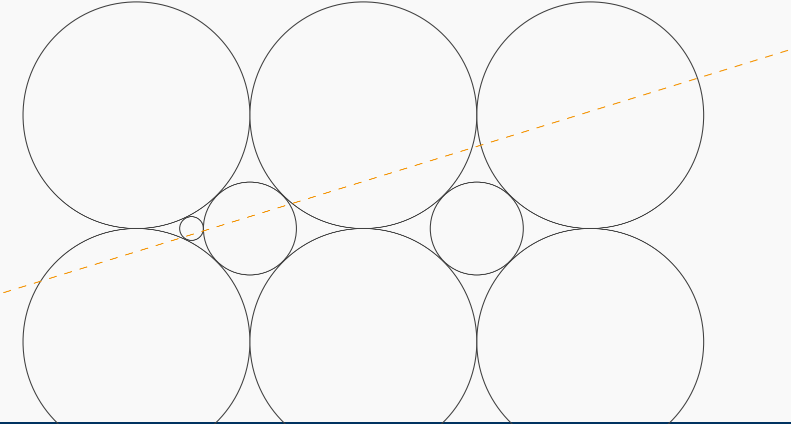
- So there is a line L such that no p_{ij} belongs to L.



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- Assume $(B_i)_{i\in I}$ is a tiling.
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.
- So there is a line L such that no p_{ij} belongs to L.
- $(B_k \cap L)_{k=1}^{\infty}$ are **disjoint** closed intervals that cover L.
- **Sierpinski (1918).** If a continuum is covered by countably many disjoint closed sets, then only one is not empty.
  - Continuum $\equiv$ compact, connected, Hausdorff.
- So, you can't tile the plane with (Euclidean) balls.
- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.
  - **Exercise.** (It's a school, after all.)

Is this a planar result?

- The tiling is countable $\leftarrow \mathbb{R}^2$ is separable.
- Balls intersect in just one point $\leftarrow \mathbb{R}^2$ is strictly convex.

Thm. **Klee, Maluta, Zanco (1986).** No separable normed space has a tiling with strictly convex bodies.

  - ℓ_2 doesn't have a tiling with balls.
  - c_0 (and ℓ_∞) have a tiling with balls.

- **Klee, Tricot (1987).** Separable smooth Banach spaces don't have tilings with balls.
- **De Bernardi, Veselý (2017).** LUR Banach spaces don't have tilings by balls.
  - Nor do Fréchet smooth Banach spaces.
- **Problem.** Can a strictly convex/smooth Banach space have a tiling with balls?
- **Preiss (2010).** ℓ_2 has a normal tiling (*i.e.*, inner and outer radii are equi-bounded).

A disjoint tiling from Badajoz

Klee's tiling

- **Klee (1981).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** balls of radius 1.
- The set of centers is $(2+)$-separated and 1-dense.
  - In $\ell_p(\mathbb{R})$ a $(2^{1/p}+)$-separated and 1-dense set.
- But centers do not form a subgroup.
- **De Bernardi, Veselý (2017).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** LUR (in particular, strictly convex) bodies.

Theorem (De Bernardi, R., Somaglia)

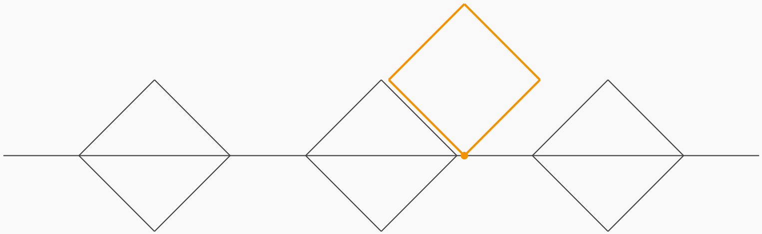
$\ell_p(\mathbb{R})$ contains a $2^{1/p}$-separated and 1-dense subgroup.

Sheldon: *What the hell is this?*

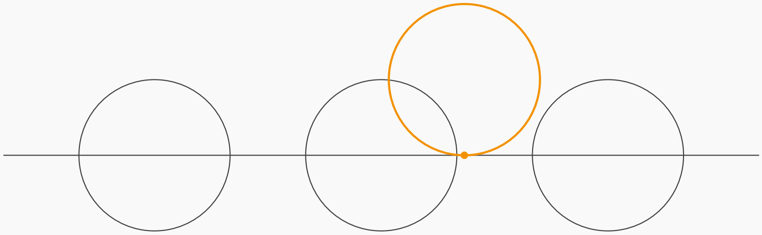
Pepe: *WHY?*

- We'll get there, suspense.

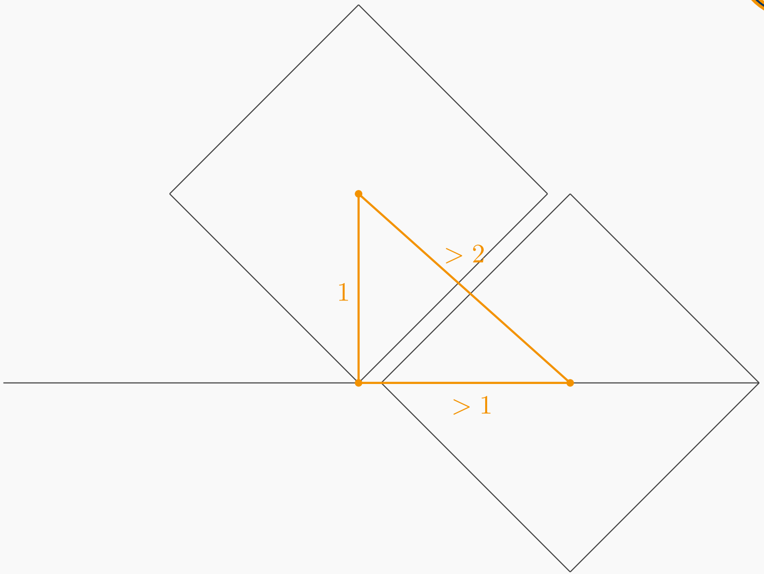
Klee's proof in one picture



Klee's proof in one picture

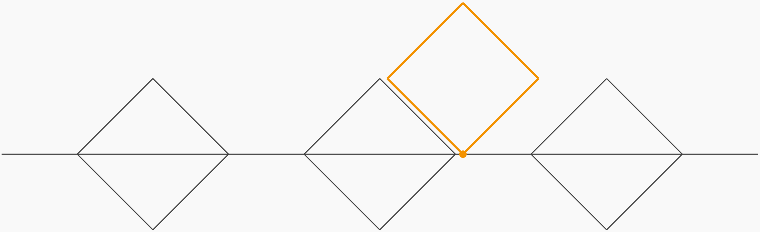


Klee's proof in one picture
The same, just bigger



Is this the solution to an exercise?

- $\ell_1(\mathbb{R}) \equiv \ell_1([0,1]) \subseteq \mathcal{C}([0,1])^* \subseteq \ell_\infty$.
- So, $|\ell_1(\mathbb{R})| = \mathfrak{c}$. Write $\ell_1(\mathbb{R}) = \{u_\alpha\}_{\alpha<\mathfrak{c}}$.
- By (long) induction. If $(B_\alpha)_{\alpha<\gamma}$ already cover u_γ, ✓.
- If not, let c_α be the center of B_α.
  - Find a subspace that contains all c_α and u_γ.
  - There is $\tilde{\gamma}$ with $u_\gamma(\tilde{\gamma}) = 0$ and $c_\alpha(\tilde{\gamma}) = 0$.
- Take $B_\gamma := B(u_\gamma + e_{\tilde{\gamma}})$.
  - This ball contains u_γ
  - and touches that subspace only in one point.



Back to that subgroup, please

- $\ell_2(\mathbb{R})$ contains a $\sqrt{2}$-separated and 1-dense subgroup $\mathcal{D}$.
- The Voronoi cells generated by $\mathcal{D}$ are convex
- and invariant under $\mathcal{D}$.
- So, there is a symmetric, bounded convex body whose translates tile $\ell_2(\mathbb{R})$.
- **There exists a reflexive Banach space (isomorphic to $\ell_2(\mathbb{R})$) that is tiled by balls of radius 1.**
  - (and the centers form a group).
- **Fonf, Lindenstrauss (1998).** Can a reflexive space be tiled by translates of a convex body?
  - Repeated in **Guirao, Montesinos, Zizler (2016)** *Open problems...*
- In every infinite-dimensional Banach space $\mathcal{X}$ there is a 1-separated and $(1+\varepsilon)$-dense subgroup.
  - **Dilworth, Odell, Schlumprecht, Zsák (2008).** $\mathcal{X}$ separable.

Thank you for your attention!