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An introduction to tilings of Banach spaces

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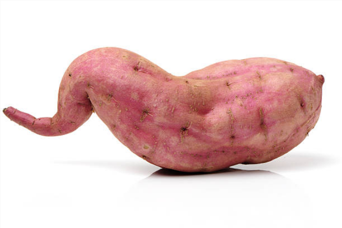
November 20, 2025
IMUS, Sevilla, Spain

Part I

Examples of tilings

Tilings and potatoes

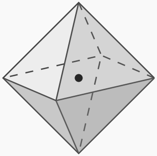
- A **tiling** of a normed space $\mathcal{X}$ is a collection $\mathcal{T}$ of subsets of $\mathcal{X}$ that have mutually disjoint interiors and that cover $\mathcal{X}$.
- We only consider tiles that are **bodies**: bounded, closed, **convex**, and with non-empty interior.



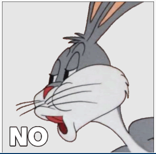
Examples

- In $\mathbb{R}^2$, we can tile by squares, or hexagons.

- In $\mathbb{R}^3$ by cubes. By octahedra?



- **Aristotle**: yes.

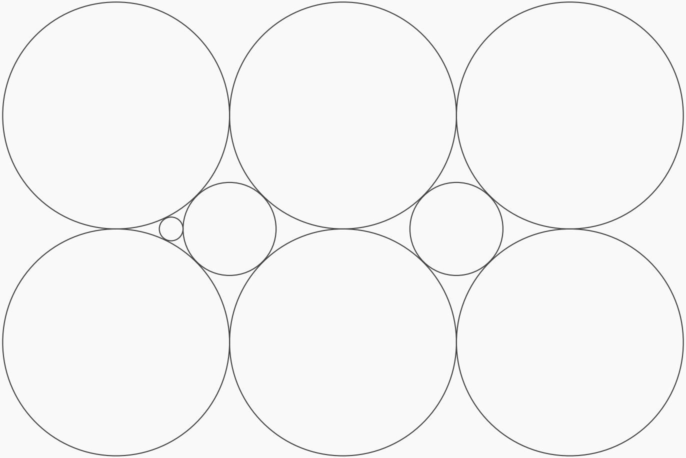


Part II

Tilings that don't exist

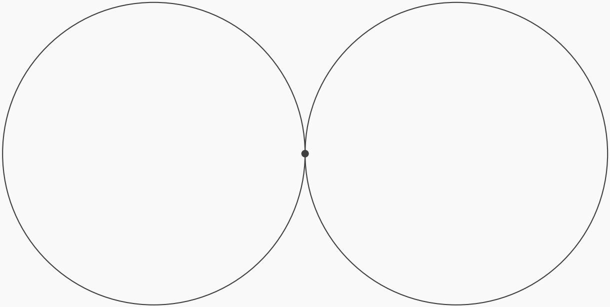
Can you tile the plane with balls?

- Are there closed balls $(B_j)_{j=1}^{\infty}$ with disjoint interiors s.t. $\mathbb{R}^2 = \bigcup B_j$?



Or maybe not?

- Assume $(B_i)_{i\in I}$ is a tiling.
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.



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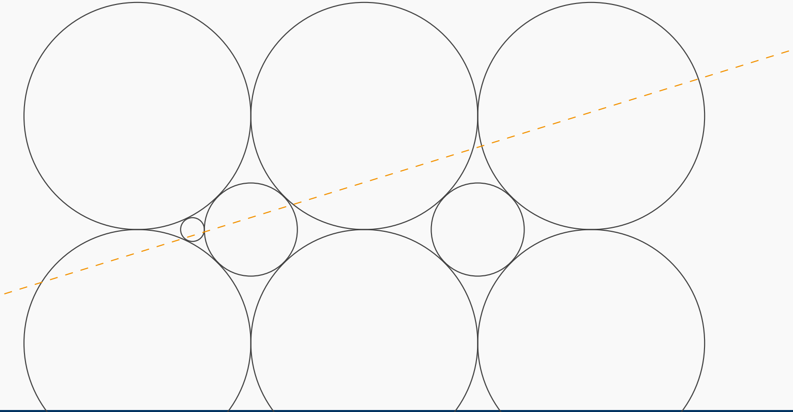
- So there is a line L such that no p_{ij} belongs to L.



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- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.
- So there is a line L such that no p_{ij} belongs to L.
- $(B_k \cap L)_{k=1}^{\infty}$ are **disjoint** closed intervals that cover L.
- **Sierpinski (1918).** If a continuum is covered by countably many disjoint closed sets, then only one is not empty.
  - **Continuum** $\equiv$ compact, connected, Hausdorff.
- So, you can't tile the plane with (Euclidean) balls.
- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.

Baby-S

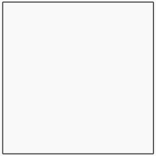
- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.
- Assume $I_k = [a_k, b_k]$ are disjoint intervals, $\mathbb{R} = \bigcup [a_k, b_k]$.
- $\mathcal{B} := \{a_k, b_k\}_{k=1}^{\infty}$.
- $\mathcal{B} \subseteq \mathcal{B}'$ (the set of accumulation points).

- $\mathcal{B}$ is closed (if $x \notin \mathcal{B}$, there is k with $x \in (a_k, b_k)$).

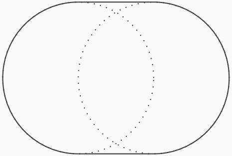
- So $\mathcal{B} = \mathcal{B}'$ is **perfect**.
- Perfect subsets of $\mathbb{R}$ aren't countable. ↯

Smooth and rotund bodies

- **Smooth** = No corners;
- **Rotund** = No segments.



Not smooth, not rotund



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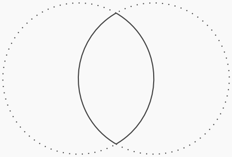

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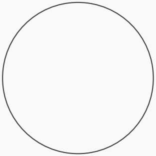


Smooth, rotund

Smooth/rotund tilings don't exist

- **Klee, Maluta, Zanco (1986).** **Separable** normed spaces do not admit tilings by rotund bodies.
- **Klee, Tricot (1987).** **Separable** smooth **Banach** spaces don't have tilings with smooth bodies.
- **De Bernardi, Veselý (2017).**
  - No Banach space admits a tiling by Fréchet smooth bodies.
  - LUR Banach spaces do not have tilings by balls.
  - $\ell_1(\kappa)$, for $\kappa^\omega = \kappa$, admits a tiling by LUR bodies.

Part III

Tilings that exist

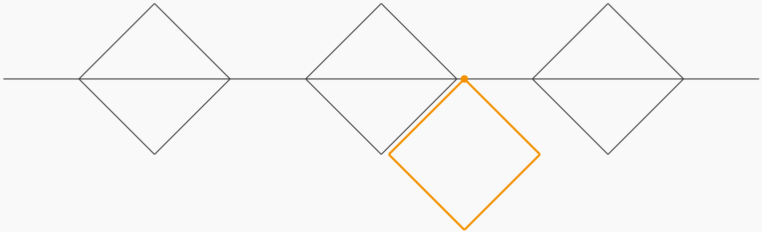
Local finiteness

- A point x_0 is a **singular point** for a covering $\mathcal{T}$ if every neighbourhood of x_0 intersects infinitely many elements of $\mathcal{T}$.
- A covering is **locally finite** if it has no singular point.
- **Corson (1961).** Infinite-dimensional reflexive Banach spaces do not admit locally finite coverings.
- **Fonf, Zanco (2006).** If a Banach space $\mathcal{X}$ admits a locally finite covering, then it is c_0-saturated.
- **Fonf (1990).** A separable Banach space admits a locally finite tiling if and only if it is isomorphically polyhedral.

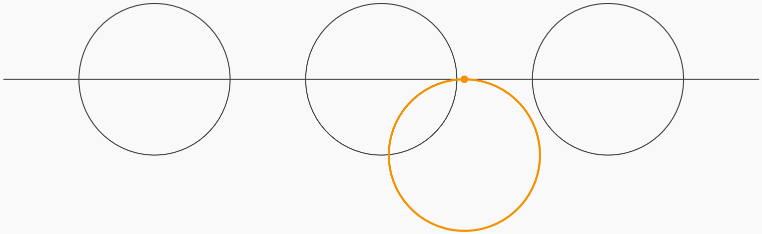
Tilings that exist

- **Klee (1981).** $\ell_1(\kappa)$, for $\kappa^\omega = \kappa$, has a **disjoint** tiling by unit balls.
- **Fonf, Pezzotta, Zanco (1997).**
  - ℓ_∞ admits a countable tiling.
  - Every Banach space admits a tiling that is **bounded below**: there is $r > 0$ such that all tiles contain a ball of radius r.
- **Preiss (2010).** ℓ_2 admits a **normal** tiling: there are $r, R > 0$ such that all tiles contain a ball of radius r and have diameter at most R.
- **Marchese, Zanco (2012).** Every Banach space admits a tiling where each point belongs to at most two bodies.

Klee's proof in one picture



Klee's proof in one picture



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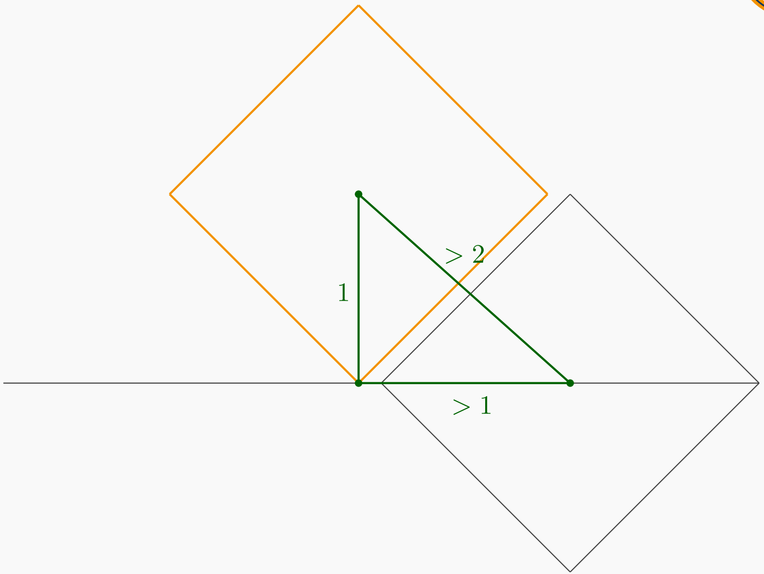
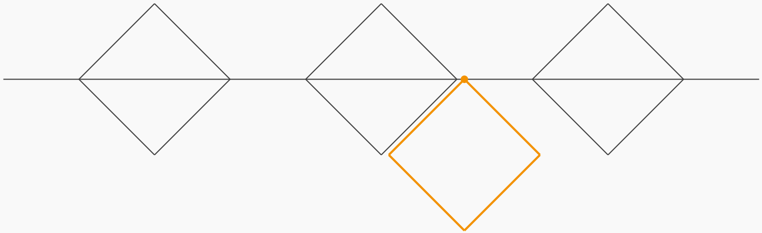


How do you actually use that?

- $\ell_1(\mathbb{R}) \equiv \ell_1([0,1]) \subseteq \mathcal{C}([0,1])^* \subseteq \ell_\infty$.
- So, $|\ell_1(\mathbb{R})| = \mathfrak{c}$. Write $\ell_1(\mathbb{R}) = \{u_\alpha\}_{\alpha<\mathfrak{c}}$.
- By (long) induction. If $(B_\alpha)_{\alpha<\gamma}$ already cover u_γ, ✓.
- If not, let c_α be the center of B_α.
  - Find a subspace that contains all c_α and u_γ.
  - There is $\tilde{\gamma}$ with $u_\gamma(\tilde{\gamma}) = 0$ and $c_\alpha(\tilde{\gamma}) = 0$.
- Take $B_\gamma := B(u_\gamma + e_{\tilde{\gamma}})$.
  - This ball contains u_γ
  - and touches that subspace only in one point.



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That's all folks!