Packings and tilings in Banach spaces

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Part I

Discrete subgroups of Banach spaces

Discrete subgroups

- **Rogers (1984).** Every infinite-dimensional Banach space contains a 1-separated and $(3/2+\varepsilon)$-dense subgroup.
  - $\mathcal{D}$ is **r-separated** if $\|d-h\| \geqslant r$ for $d \neq h \in \mathcal{D}$.
  - $\mathcal{D}$ is **R-dense** if for all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant R$.
- **Swanepoel (2009).** Can you get $(1+\varepsilon)$-dense?
- Every maximal 1-separated set is 1-dense.
- **Dilworth, Odell, Schlumprecht, Zsák (2008).** Yes, if $\mathcal{X}$ separable.
  - *The following result is of interest in nonlinear functional analysis.*
- *I wonder whether separability is necessary* (Doucha, by email).

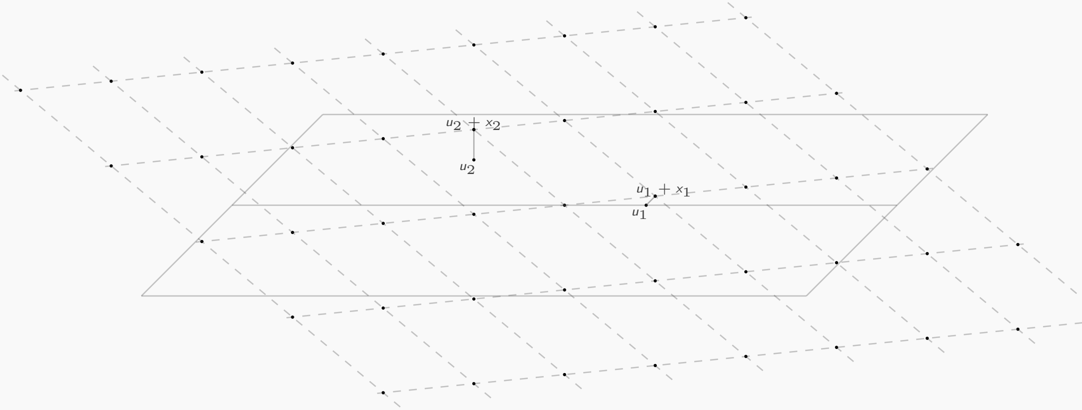
Theorem (De Bernardi, R., Somaglia)

Every infinite-dimensional Banach space $\mathcal{X}$ contains a 1-separated and $(1+\varepsilon)$-dense subgroup.

- **Steprāns (1985).** Discrete subgroups of normed spaces are **free**.
  - Uses Shelah's Singular Compactness theorem and Fodor's pressing-down lemma.
  - We have *a proof without any logic* (cit. Fabian).

How to construct them?

- Suppose $\mathcal{X}$ is separable, $(u_k)_{k=1}^{\infty}$ is dense in $\mathcal{X}$.



How to construct them?

- Suppose $\mathcal{X}$ is separable, $(u_k)_{k=1}^{\infty}$ is dense in $\mathcal{X}$.
- Construct a chain $(\mathcal{D}_k)_{k=1}^{\infty}$ of finitely generated subgroups such that
  1. $\mathcal{D}_k$ is 1-separated
  2. $\operatorname{dist}(u_k, D_k) \leqslant 1$.
- Then $\bigcup_{k=1}^{\infty} \mathcal{D}_k$ does the job.
- Having $\mathcal{D}_k$, $E := \operatorname{span}\{\mathcal{D}_k, u_{k+1}\}$ is finite dim.
- By Riesz' lemma, there is $x_{k+1} \in \mathcal{X}$ with $\|x_{k+1}\| = 1$ and $\operatorname{dist}(x_{k+1}, E) \geqslant 1$.
- Define

  $$\mathcal{D}_{k+1} := \mathcal{D}_k \oplus (x_{k+1} + u_{k+1})\mathbb{Z}.$$

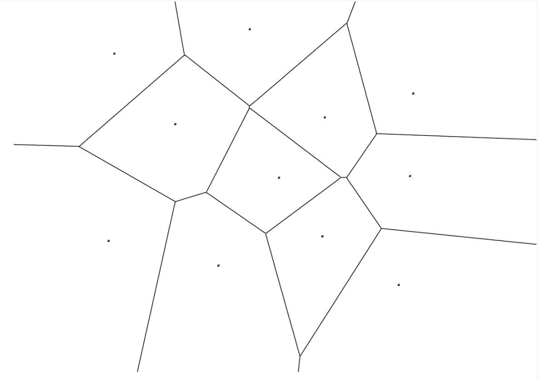
  And check that it is 1-separated.
- An elementary constructive proof that only uses Riesz' lemma.

Part II

Tilings

Tilings by Voronoi cells

- A **tiling** of $\mathcal{X}$ is a family of bodies that cover $\mathcal{X}$ and have mutually disjoint interiors.
  - **Body** $\equiv$ closed, convex, bounded, and with non-empty interior.
- **If $\kappa^\omega = \kappa$, $\ell_2(\kappa)$ contains a $\sqrt{2}$-separated and 1-dense subgroup $\mathcal{D}$.**
  - The constant $\sqrt{2}$ is optimal, as we will see later.



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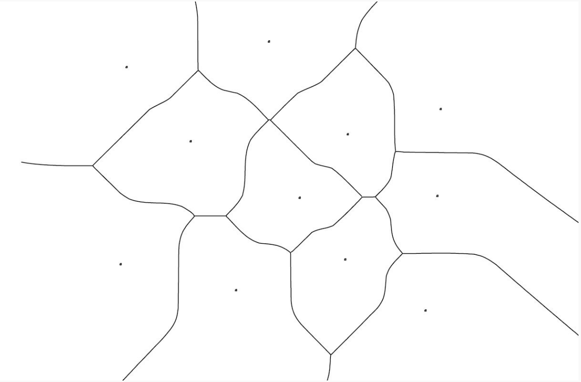

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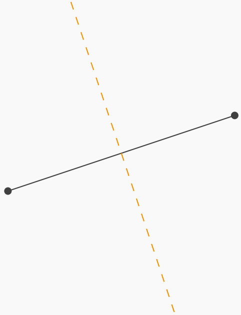

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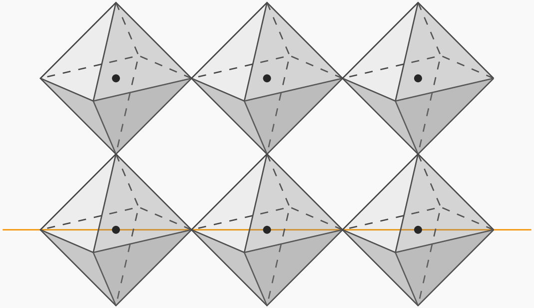
- **If $\kappa^\omega = \kappa$, $\ell_2(\kappa)$ contains a $\sqrt{2}$-separated and 1-dense subgroup $\mathcal{D}$.**
  - The constant $\sqrt{2}$ is optimal, as we will see later.
- The **Voronoi cells** generated by $\mathcal{D}$ are convex and $\mathcal{D}$-invariant.
- Moreover, they form a tiling (because of the value $\sqrt{2}$).
- So, $\ell_2(\kappa)$ can be tiled by translates of one Voronoi cell.
- **There exists a reflexive Banach space (isomorphic to $\ell_2(\kappa)$) that admits a tiling by balls of radius 1.**
  - And the centers form a group (*i.e.*, the tiling is **lattice**).
- **Fonf, Lindenstrauss (1998).** Can a reflexive Banach space be tiled by translates of a bounded convex body?

Tilings with diamonds

- **If $\kappa^\omega = \kappa$, $\ell_1(\kappa)$ contains a 2-separated and 1-dense subgroup $\mathcal{D}$.**
- **$\ell_1(\kappa)$ admits a lattice tiling by balls of radius 1.**
- **Klee (1981).** A tiling of $\ell_1(\kappa)$ with **disjoint** balls of radius 1.
  - Lattice tilings with balls cannot be disjoint.
- Each point belongs to at most two tiles and two tiles intersect at most in some vertex.



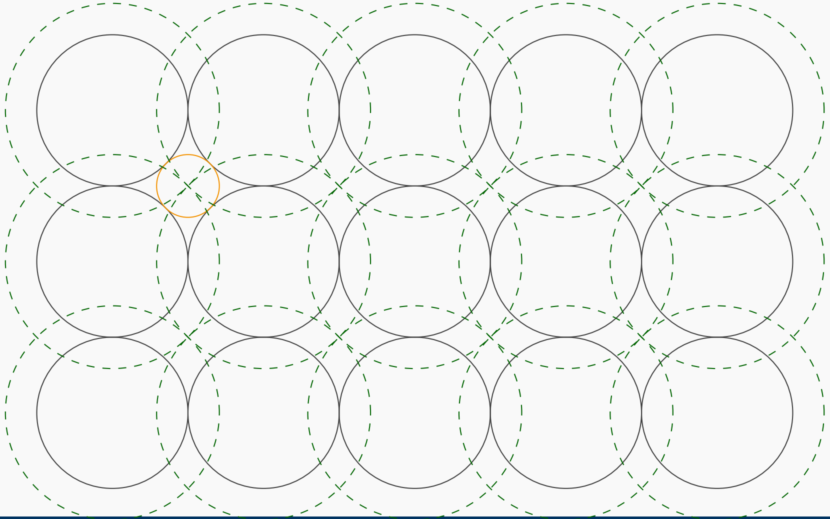
Part III

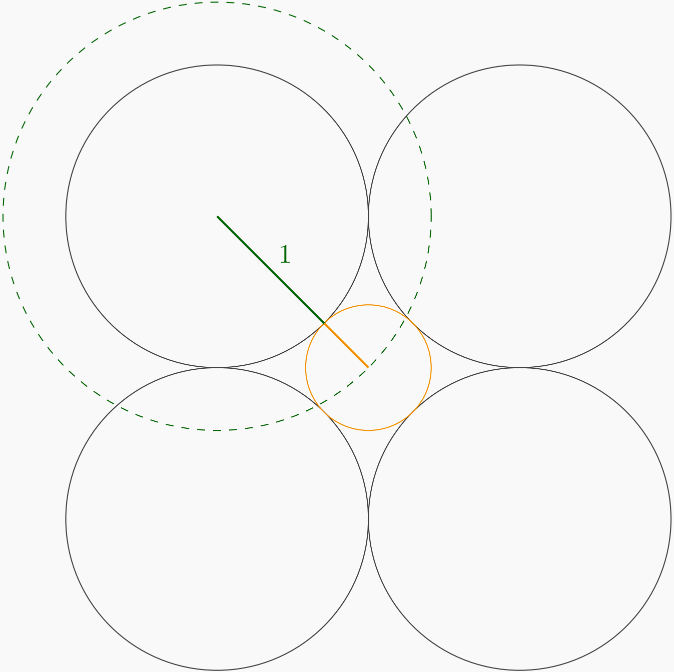
Packings

When tilings don't exist

- **Not all Banach spaces admit a tiling by balls.**
- **Klee, Tricot (1987).** **Separable** smooth Banach spaces don't have tilings with balls.
- **Klee, Maluta, Zanco (1986).** Nor do **separable** rotund normed spaces.
- **De Bernardi, Veselý (2017).** LUR Banach spaces don't have tilings by balls.
  - Nor do Fréchet smooth Banach spaces.
- A **packing** (of balls) is a collection of non-overlapping balls of radius 1.
- **How to measure how optimal (or packed) a packing is?**
  - Compute the radius of the largest non-overlapping ball (the largest hole in the packing).
  - How much do we have to inflate the balls to cover the space?
  - First is second -1.
- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ measures this.

When tilings don't exist

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Two formulas

- The **simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma(\mathcal{X}) := \inf\{r > 0 \colon \text{there exists } \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense}\}.$$

$$\frac{2}{\gamma(\mathcal{X})} = \sup\{r > 0 \colon \text{there exists } \mathcal{D} \subseteq \mathcal{X} \text{ 1-dense and } r\text{-separated}\}.$$

- The **lattice simultaneous covering and packing constant** $\gamma(\mathcal{X})$ of $\mathcal{X}$ is

$$\gamma^*(\mathcal{X}) := \inf\{r > 0 \colon \text{there exists } \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

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Two formulas

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$$\gamma^*(\mathcal{X}) := \inf\{r > 0\colon \text{there exists } \mathcal{D} \subseteq \mathcal{X} \text{ 2-separated and } r\text{-dense } \textbf{subgroup}\}.$$

- If $\mathcal{X}$ has a tiling by balls of radius 1, $\gamma(\mathcal{X}) = 1$.
- Taking any maximal packing $\rightsquigarrow \gamma(\mathcal{X}) \leqslant 2$.
- By the result from the beginning, $\gamma^*(\mathcal{X}) \leqslant 2$.
- **Casini, Papini, Zanco (1986).** $\gamma(\mathcal{X}) \geqslant \frac{2}{K(\mathcal{X})}$.
- To sum it up

$$1 \leqslant \frac{2}{K(\mathcal{X})} \leqslant \gamma(\mathcal{X}) \leqslant \gamma^*(\mathcal{X}) \leqslant 2.$$

A problem of Swanepoel

- **Swanepoel (2009).** $\gamma(\ell_p) = \frac{2}{2^{1/p}}$ which equals $\frac{2}{K(\ell_p)}$.
- **Swanepoel (2009).** Is it true that for all Banach spaces

$$\gamma(\mathcal{X}) = \frac{2}{K(\mathcal{X})}?$$

Theorem (De Bernardi, R., Sezgek, Somaglia)

If the unit ball of $\mathcal{X}$ admits a LUR point, then $\gamma(\mathcal{X}) > 1$.

- Consider $\ell_1 \oplus_2 \mathbb{R}$. Its unit ball has a LUR point, but $K(\ell_1 \oplus_2 \mathbb{R}) = 2$.
- Every Banach space $\mathcal{X}$ is isomorphic to a Banach space $\mathcal{Y}$ with $K(\mathcal{Y}) = 2$ and $\gamma(\mathcal{Y}) > 1$.
  - So, there are reflexive (even isomorphic to ℓ_2) counterexamples to Swanepoel's question.

Some known values

- For $1 \leqslant p < \infty$ and every infinite κ

$$\gamma(\ell_p(\kappa)) = \gamma^*(\ell_p(\kappa)) = \frac{2}{2^{1/p}}.$$

- For $1 \leqslant p < \infty$ and q the conjugate index of p

$$\max\left\{\frac{2}{2^{1/p}}, \frac{2}{2^{1/q}}\right\} \leqslant \gamma(L_p([0,1])) \leqslant \gamma^*(L_p([0,1])) \leqslant \frac{2}{2^{1/p}}.$$

- If $\mathcal{X}$ is separable and octahedral, or $\mathcal{X} = \mathcal{C}(\mathcal{K})$ with $\mathcal{K}$ zero-dimensional

$$\gamma(\mathcal{X}) = \gamma^*(\mathcal{X}) = 1.$$

- This applies, *e.g.*, to: $\mathcal{C}([0,1])$, $\mathcal{C}(2^\omega)$, $\mathcal{C}(\mathcal{K})$ for $\mathcal{K}$ countable (or scattered).
- Some Lipschitz-free spaces, spaces of Lipschitz functions, tensor products, ...

And unknown ones

- Is there a Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) = 2$?
- Is there a Banach space $\mathcal{X}$ with $\gamma(\mathcal{X}) \neq \gamma^*(\mathcal{X})$?
- What are the exact values of $\gamma(\ell_1 \oplus_2 \mathbb{R})$ and $\gamma^*(\ell_1 \oplus_2 \mathbb{R})$?
  - They are > 1 (LUR point).
- What are the exact values of $\gamma(\ell_1 \oplus_2 \ell_1)$ and $\gamma^*(\ell_1 \oplus_2 \ell_1)$?
- And many more

Thank you for your attention!