Klee's tiling of $\ell_1(\Gamma)$. Variations on a theme

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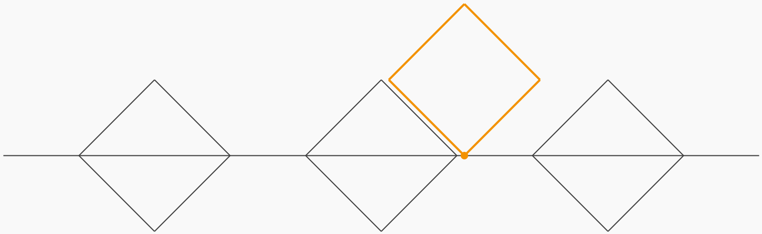
Klee's tiling

- **Klee (1981).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** balls of radius 1.

Thm. **Klee, Maluta, Zanco (1986).** **Separable** rotund Banach spaces don't have tilings with balls.

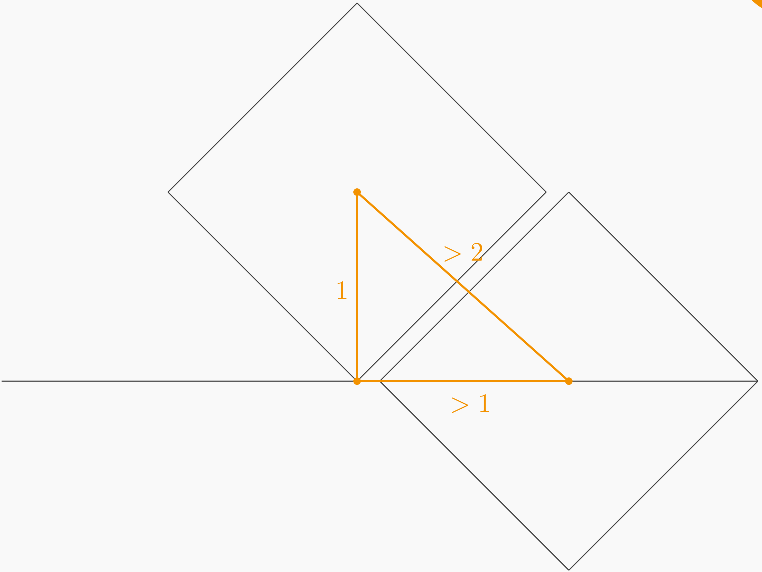
- **Klee, Tricot (1987).** Nor do **separable** smooth ones.
- **De Bernardi, Veselý (2017).** LUR Banach spaces don't have tilings by balls.
  - Nor do Fréchet smooth ones.
- **Problem.** Can a rotund/smooth Banach space have a tiling with balls?
- **De Bernardi, Veselý (2017).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** LUR (in particular, rotund) bodies.
- The set $\mathcal{D}$ of centers is $(2+)$-separated and 1-dense.
  - If $d \neq h \in \mathcal{D}$, then $\|d - h\| > 2$.
  - For all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x - d\| \leqslant 1$.
- In $\ell_p(\mathbb{R})$ a $(2^{1/p}+)$-separated and 1-dense set.

Klee's proof in one picture



Klee's proof in one picture

The same, just bigger



And two variations

Keep assuming that $\Gamma^\omega = \Gamma$

Klee: In $\ell_2(\Gamma)$ a $(\sqrt{2}+)$-separated and 1-dense set.

Theorem (De Bernardi, R., Somaglia)

$\ell_2(\Gamma)$ contains a $(\sqrt{2}+)$-separated and 1-dense **subgroup**.

- So, there is a symmetric body whose translates tile $\ell_2(\Gamma)$.
- **There exists a reflexive Banach space (isomorphic to $\ell_2(\Gamma)$) that is tiled by balls of radius 1.**
- **Fonf, Lindenstrauss (1998).** Can a reflexive space be tiled by translates of a convex body?
  - Repeated in **Guirao, Montesinos, Zizler (2016)** *Open problems...*
- In every infinite-dimensional Banach space $\mathcal{X}$ there is a 1-separated and $(1+\varepsilon)$-dense subgroup.
  - **Dilworth, Odell, Schlumprecht, Zsák (2008).** $\mathcal{X}$ separable.

A disjoint tiling from Badajoz