Packings and tilings: from Flatland to infinite dimensions

Tommaso Russo

Habilitation lecture

Innsbruck, November 5, 2025

Part I

Packing and tiling in $\mathbb{R}^n$: a quick glimpse

Packing problems

Kepler problem (1661)

What is the optimal way to stack cannonballs on a ship?

Gregory–Newton problem (1694)

Can a sphere in $\mathbb{R}^3$ touch 13 spheres of the same size?

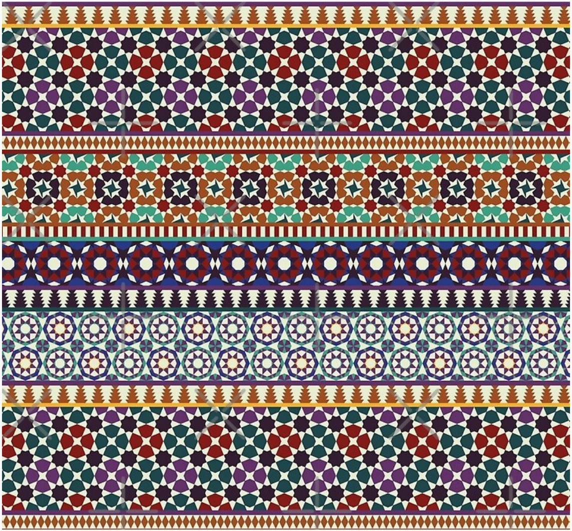
- **Gregory**: Yes.
- **Newton**: No, at most 12.
- δ_n: the density of the optimal packing of spheres in $\mathbb{R}^n$.
- δ_n^*: with the extra requirement that the centers are a lattice (namely, a subgroup).

A few values

n	δ_n^*	Author	δ_n	Author
2	$\frac{\pi}{\sqrt{12}}$	Lagrange, 1773	$\frac{\pi}{\sqrt{12}}$	Thue, 1892
3	$\frac{\pi}{\sqrt{18}}$	Gauss, 1831	$\frac{\pi}{\sqrt{18}}$	Hales, 2005 *Annals*
4	$\frac{\pi^2}{16}$	Korkin, Zolotarev, 1872	???	
5	$\frac{\pi^2}{15\sqrt{2}}$	Korkin, Zolotarev, 1877	???	
6	$\frac{\pi^3}{48\sqrt{3}}$	Blichfeldt, 1925	???	
7	$\frac{\pi^3}{105}$	Blichfeldt, 1926	???	
8	$\frac{\pi^4}{384}$	Blichfeldt, 1934	$\frac{\pi^4}{384}$	Viazovska, 2017 *Annals*
24	$\frac{\pi^{12}}{12!}$	Cohn, Kumar, 2009 *Annals*	$\frac{\pi^{12}}{12!}$	Cohn et.al. 2017 *Annals*

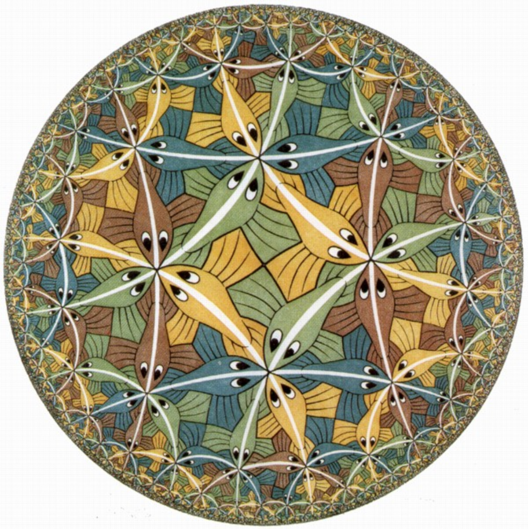
- **Algorithm**: for infinitely many *Annals* papers.

Tilings and patterns

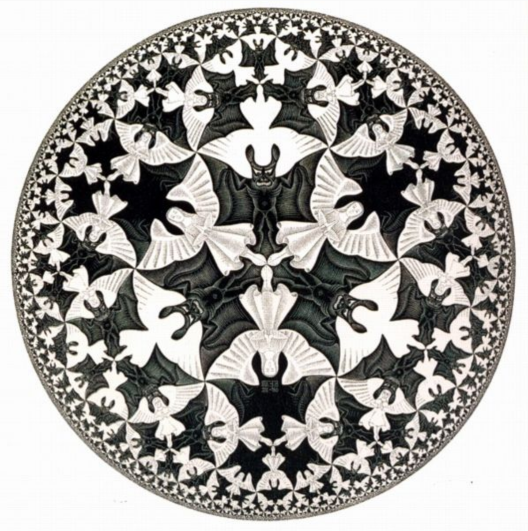


Tilings and patterns

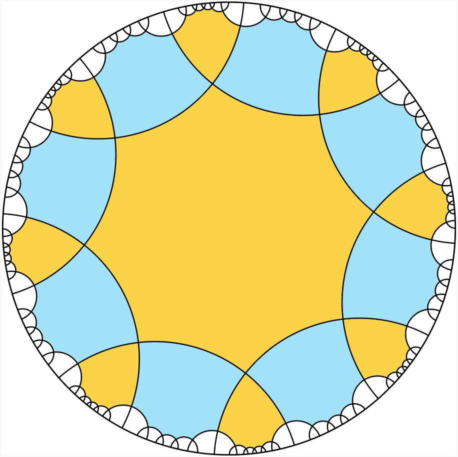
Tilings and patterns



Tilings and patterns



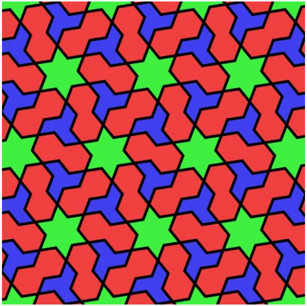
Tilings and patterns



C. Bargetz, F. Luggin, and T. Russo, *Tilings of the hyperbolic space and Lipschitz functions*, Canad. J. Math. (online first).

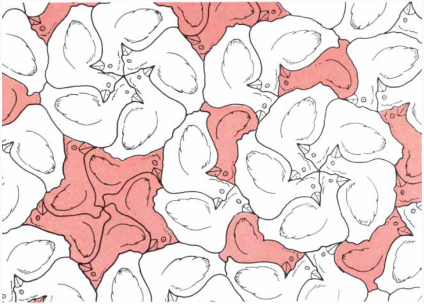
Tilings and patterns

- **Venkov–McMullen.** If a convex set C tiles $\mathbb{R}^n$ by translations, then it also generates a lattice tiling.
- **Stein–Szabó.** Counterexamples without convexity.
- *Periodic* tiling.



Tilings and patterns

- **Venkov–McMullen.** If a convex set C tiles $\mathbb{R}^n$ by translations, then it also generates a lattice tiling.
- **Stein–Szabó.** Counterexamples without convexity.
- *Periodic* tiling.
- The *Aperiodic tiling conjecture*: is there a finite set of tiles that tiles, but only aperiodically?
- Penrose' 'kites' and 'darts'.



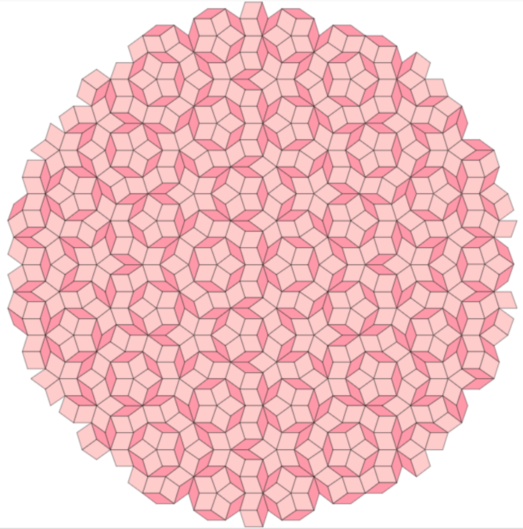
Tilings and patterns

- **Venkov–McMullen.** If a convex set C tiles $\mathbb{R}^n$ by translations, then it also generates a lattice tiling.
- **Stein–Szabó.** Counterexamples without convexity.
- *Periodic* tiling.
- The *Aperiodic tiling conjecture*: is there a finite set of tiles that tiles, but only aperiodically?
- Penrose' 'kites' and 'darts'.
- The *Einstein problem*: Is there a single tile...?
- **Smith–Myers–Kaplan–Goodman-Strauss (2024).** The 'hat' tiles the plane by rotations and translations, but only aperiodically.

Tilings and patterns

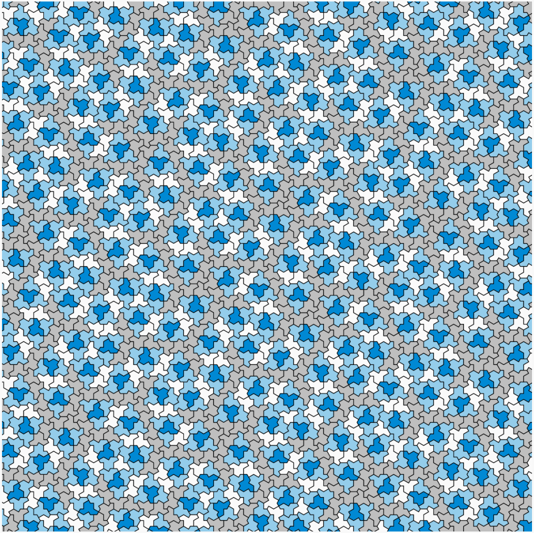
- **Venkov–McMullen.** If a convex set C tiles $\mathbb{R}^n$ by translations, then it also generates a lattice tiling.
- **Stein–Szabó.** Counterexamples without convexity.
- *Periodic* tiling.
- The *Aperiodic tiling conjecture*: is there a finite set of tiles that tiles, but only aperiodically?
- Penrose' 'kites' and 'darts'.
- The *Einstein problem*: Is there a single tile...?
- **Smith–Myers–Kaplan–Goodman-Strauss (2024).** The 'hat' tiles the plane by rotations and translations, but only aperiodically.
- **Greenfeld–Tao (2024).** Only by translations.
- **Tilings in finite dimensions tend to have some patterns.**

Tilings and patterns

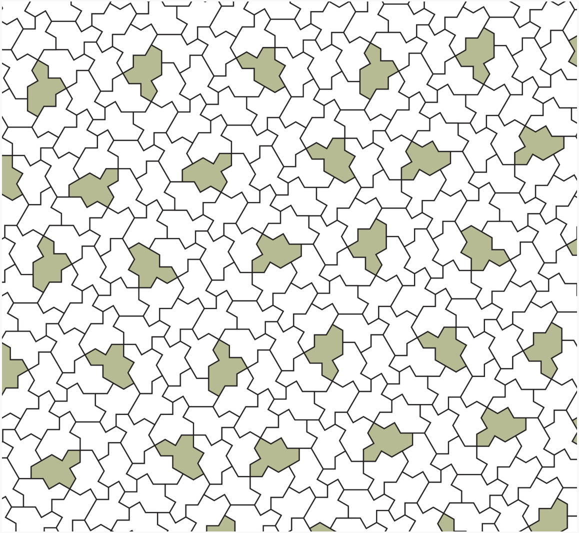


Tilings and patterns

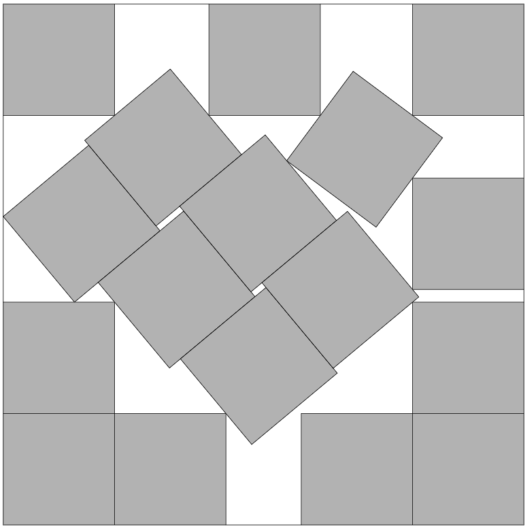
Tilings and patterns



Tilings and patterns

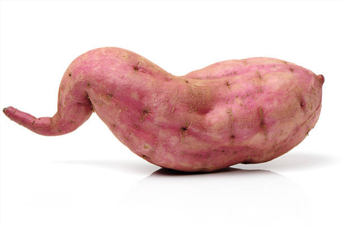


Tilings and patterns



Convex potatoes

From now on, we only consider tiles that are *bodies*/potatoes, namely bounded, closed, convex, and with non-empty interior.



Part II

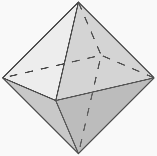
Aristotle, Hilbert, and Voronoi go to a restaurant

Polyhedra and tilings

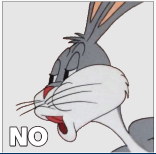
In this part we consider tilings by translates of just one body.

- In $\mathbb{R}^2$, we can tile by squares, or hexagons.


- In $\mathbb{R}^3$ by cubes. By octahedra?

- **Aristotle**: yes.



Cubes in infinite dimensions

- Can we tile by cubes in infinite dimensions?
- $C := \{(x_j) \in \ell_2 \colon |x_j| \leqslant 1\}$.
- C is not bounded: $\|(\underbrace{1, \dots, 1}_{n}, 0, 0, 0, \dots)\| = \sqrt{n}$.
- But C is bounded in c_0

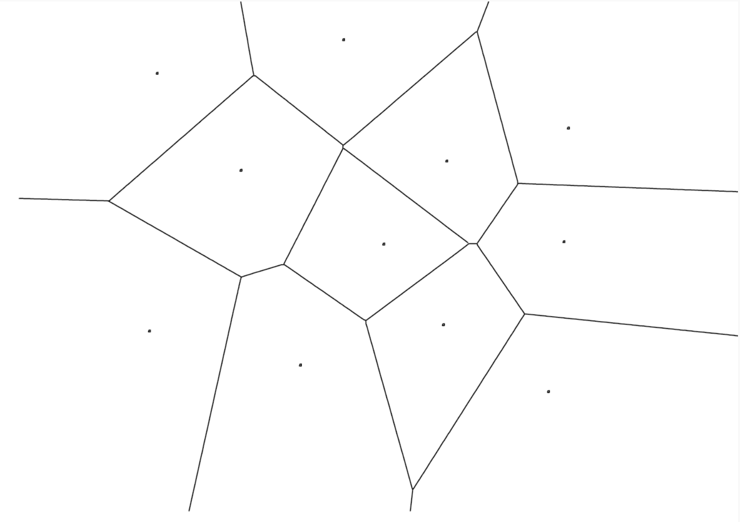
$$c_0 := \{(x_j) \in \mathbb{R}^{\mathbb{N}} \colon x_j \to 0\} \qquad \|(x_j)\| = \max |x_j|.$$

- c_0 can be tiled by translates of its unit ball.
- **Fonf, Lindenstrauss (1998).** Can you tile a reflexive space by translates of a body?

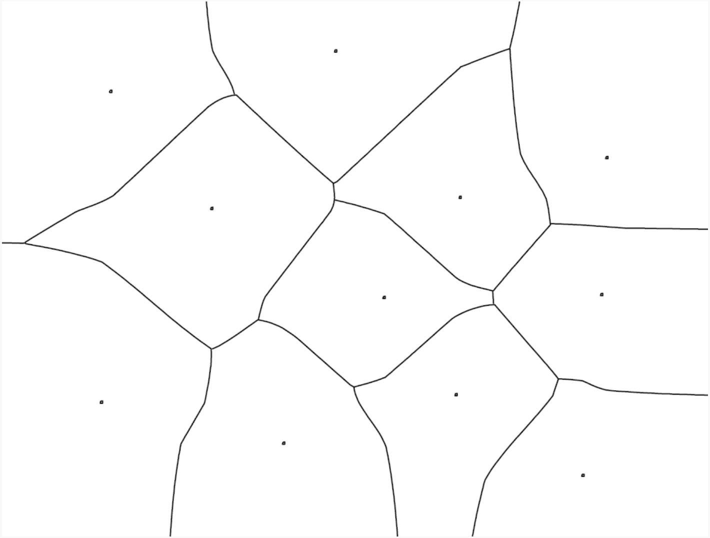
Theorem (j./w. De Bernardi and Somaglia, `arXiv.2505.04267`)

Yes. Actually, some Hilbert spaces.

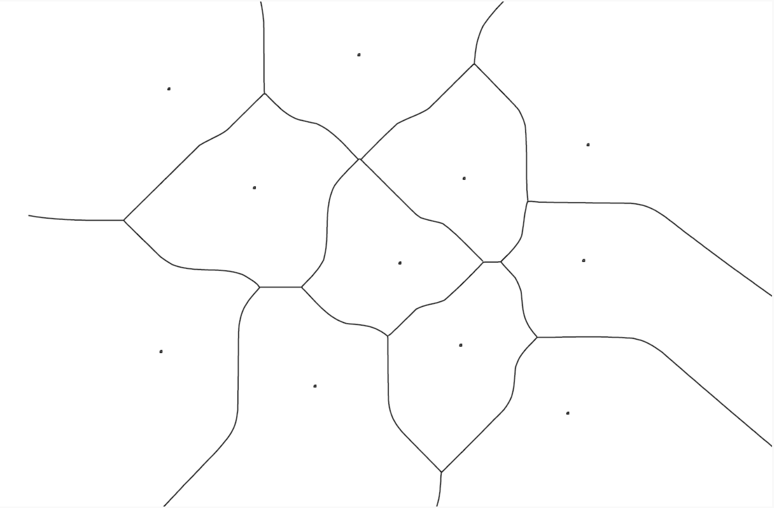
Voronoi



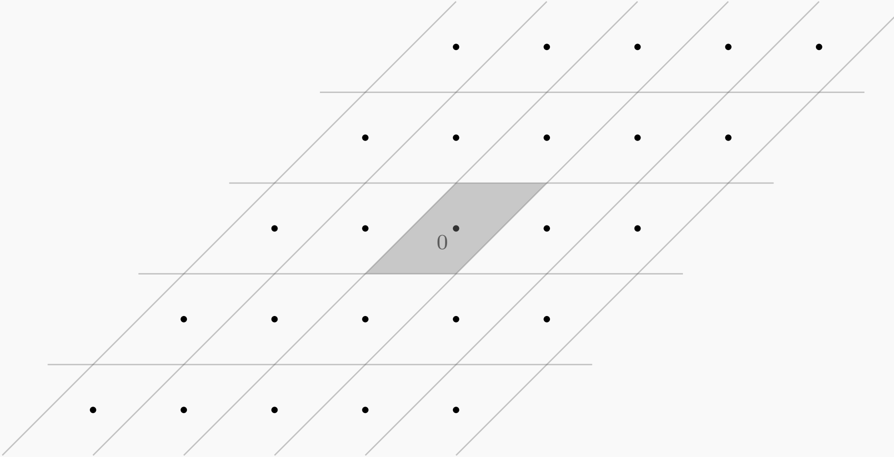
Voronoi



Voronoi

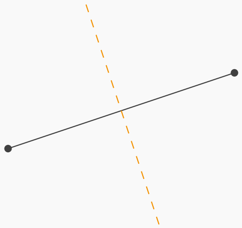


Voronoi



What could go wrong?

- Voronoi cells in Hilbert spaces are convex.



- Why does every point belong to a cell?
  - Why is there always a closest restaurant?
  - In finite dimensions $\rightsquigarrow$ compactness.
  - In infinite dimensions, overcoming this is the main issue.

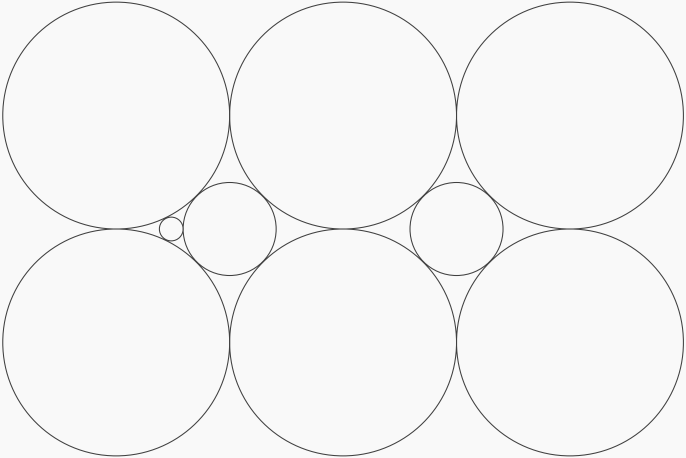
    $\rightsquigarrow$ Measures of non-compactness.

Part III

Smooth tiles

Back to Flatland

- **Which shapes can the tiles have?**
- Is it possible to tile the plane with balls?

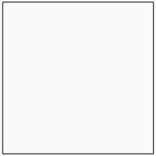


Back to Flatland

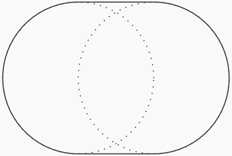
- **Which shapes can the tiles have?**
- Is it possible to tile the plane with balls?
- Does this work, any guesses?
- **It doesn't.**
- **Exercise.**
  - *Hint:* The most useful tool from Analysis 4, a.k.a. Baire theorem.

Smooth and rotund bodies

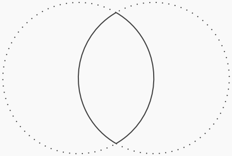
- **Smooth** = No corners;
- **Rotund** = No segments.



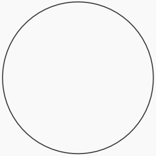
Not smooth, not rotund



Smooth, not rotund



Not smooth, rotund



Smooth, rotund

Can you get smooth tilings?

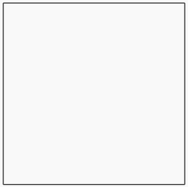
- **Which shapes can the tiles have?**
- Is it possible to tile the plane with balls?
- Does this work, any guesses?
- **It doesn't.**
- **Exercise.**
  - *Hint:* The most useful tool from Analysis 4, a.k.a. Baire theorem.

Thm. **Klee, Maluta, Zanco (1986).** No separable normed space has a tiling with rotund bodies.

Theorem (j./w. De Bernardi and Somaglia, in progress)

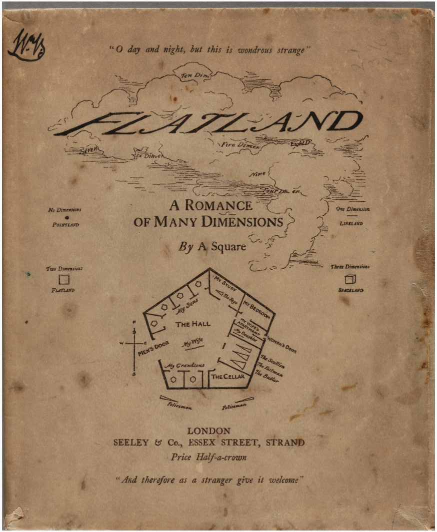
Some (separable) inner product spaces have tilings by smooth bodies.

Does a square have corners?

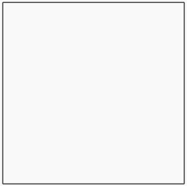


Lemma (Abbott, 1884)

Does a square have corners?



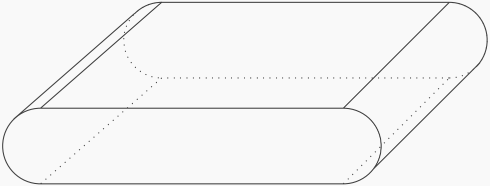
Does a square have corners?



Lemma (Abbott, 1884)

Yes, this is a smooth body (in 3 dimensions).

Does a square have corners?

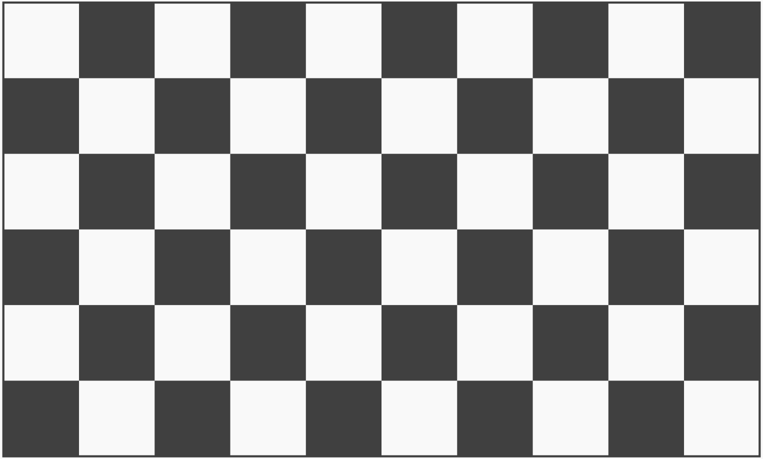


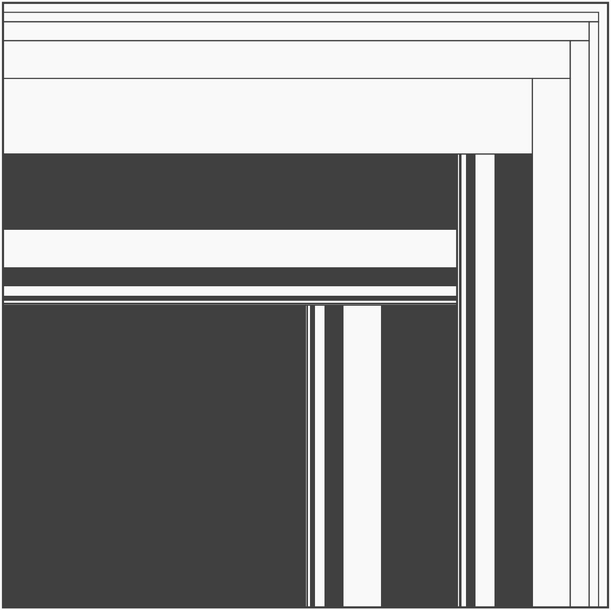
Lemma (Abbott, 1884)

There is a 3-dimensional smooth body that has a square as a face.

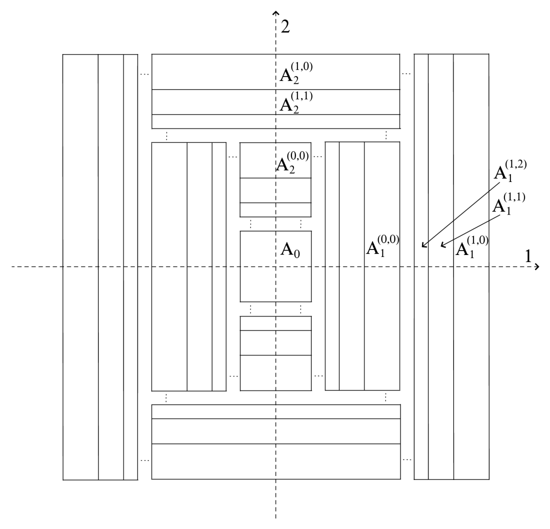
Tiling the plane

Tiling the plane

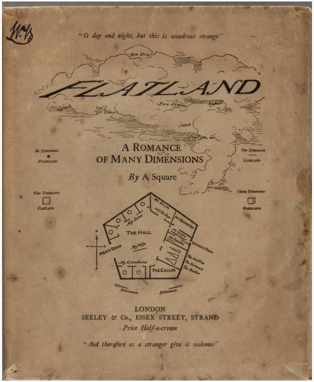




Tiling the plane



Tiling the plane



120957072

Something to take home

- Algorithm for infinitely many *Annals* papers.
- Keeping a list of your favourite restaurants is useful.
- **Some Hilbert spaces admit lattice tilings.**
  - Namely, tilings by translates of one body (and the set of translations is a group).
- Political pamphlets/book for kids can give good ideas too.
- **Some (separable) inner product spaces admit tilings by smooth bodies.**

Thank you for your attention!