universität innsbruck

Lattice tilings of Hilbert spaces

Tommaso Russo
tommaso.russo.math@gmail.com

j./w. C.A. De Bernardi and J. Somaglia
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Discrete subgroups of Banach spaces

- **Rogers (1984).** Every infinite-dimensional Banach space contains a 1-separated and $(3/2+\varepsilon)$-dense subgroup.
  - $\mathcal{D}$ is ***r*-separated** if $\|d-h\| \geqslant r$ for $d \neq h \in \mathcal{D}$.
  - $\mathcal{D}$ is ***R*-dense** if for all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant R$.
- **Swanepoel (2009).** Can you get $(1+\varepsilon)$-dense?
- **Dilworth, Odell, Schlumprecht, Zsák (2008).** Yes, if $\mathcal{X}$ separable.
  - *The following result is of interest in nonlinear functional analysis.*
- *I wonder whether separability is necessary* (Doucha, by email).

Theorem (De Bernardi, R., Somaglia)

Every infinite-dimensional Banach space $\mathcal{X}$ contains a 1-separated and $(1+\varepsilon)$-dense subgroup. Further, it is generated by the elements of norm at most $2+\varepsilon$.

- **Steprāns (1985).** Discrete subgroups of normed spaces are **free**.
  - Uses Shelah's Singular Compactness theorem and Fodor's pressing-down lemma.
  - We have *a proof without any logic* (cit. Fabian).

And tilings of Hilbert spaces

- $3/2 \rightsquigarrow 1$. **Was it really worth the effort?**
- A simple constructive proof by induction, only using Riesz' lemma.

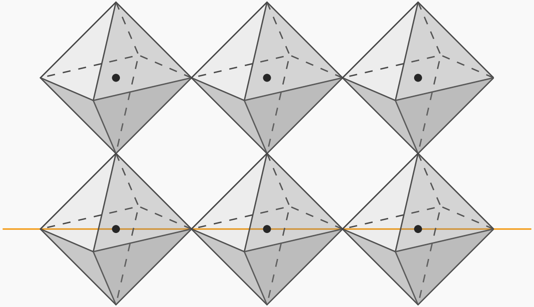
Theorem (De Bernardi, R., Somaglia)

If $\kappa^\omega = \kappa$, $\ell_p(\kappa)$ contains a $2^{1/p}$-separated and 1-dense subgroup $\mathcal{D}$.

  - Which has applications to the theory of **tilings**.
  - The constant $2^{1/p}$ is optimal.
- **Case $p = 2$.**
- The **Voronoi cells** generated by $\mathcal{D}$ are convex and $\mathcal{D}$-invariant.
- So, there is a symmetric, bounded convex body whose translates tile $\ell_2(\kappa)$.
- **There exists a reflexive Banach space (isomorphic to $\ell_2(\kappa)$) that admits a tiling by balls of radius 1.**
  - And the centers form a group (*i.e.*, the tiling is **lattice**).
- **Fonf, Lindenstrauss (1998).** Can a reflexive Banach space be tiled by translates of a bounded convex body?
  - Repeated in **Guirao, Montesinos, Zizler (2016)**, *Open problems...*

Tilings with diamonds

- **Case $p = 1$.**
- **Klee (1981).** A tiling of $\ell_1(\kappa)$ with **disjoint** balls of radius 1.
- **$\ell_1(\kappa)$ admits a lattice tiling by balls of radius 1.**
  - Lattice tilings with balls cannot be disjoint.
  - Each point belongs to at most two tiles and two tiles intersect at most in some vertex.



Thank you for your attention!