Tiling with infinite-dimensional oranges

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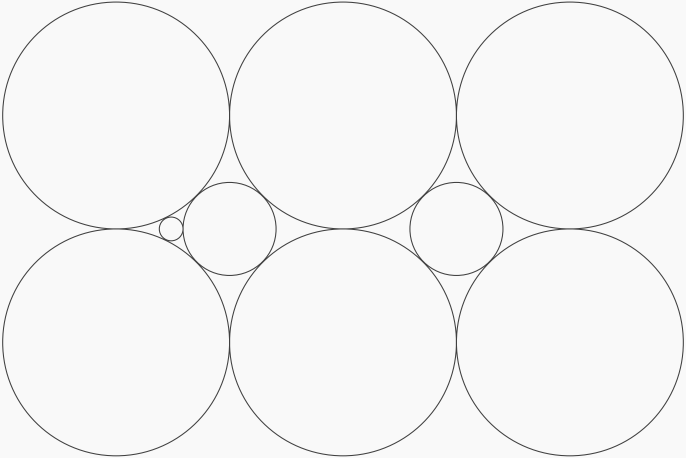
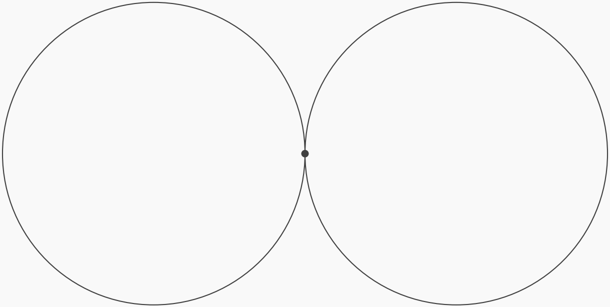
Can you tile the plane with balls?

- Are there closed balls $(B_j)_{j=1}^{\infty}$ with disjoint interiors s.t. $\mathbb{R}^2 = \bigcup B_j$?



Or maybe not?

- Assume $(B_i)_{i\in I}$ is a tiling.
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.



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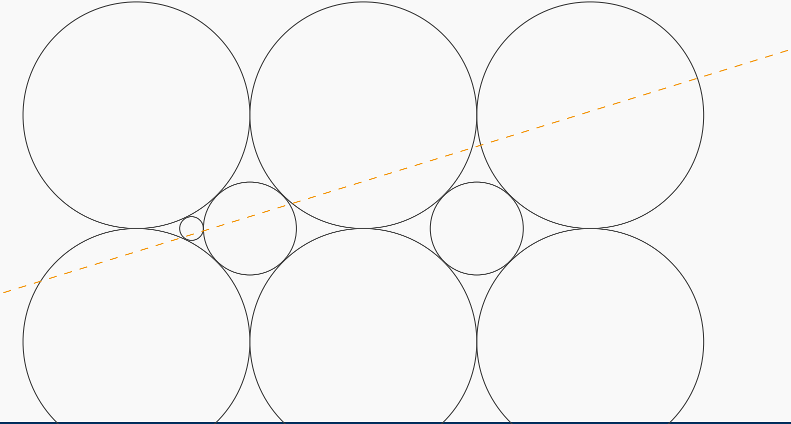
- So there is a line L such that no p_{ij} belongs to L.



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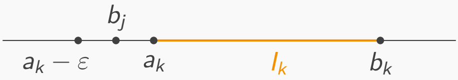
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.
- So there is a line L such that no p_{ij} belongs to L.
- $(B_k \cap L)_{k=1}^{\infty}$ are **disjoint** closed intervals that cover L.
- **Sierpinski (1918).** If a continuum is covered by countably many disjoint closed sets, then only one is not empty.
  - Continuum $\equiv$ compact, connected, Hausdorff.
- So, you can't tile the plane with (Euclidean) balls.
- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.

Baby-S

- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.
- Assume $I_k = [a_k, b_k]$ are disjoint intervals, $\mathbb{R} = \bigcup [a_k, b_k]$.
- $\mathcal{B} := \{a_k, b_k\}_{k=1}^{\infty}$.
- $\mathcal{B} \subseteq \mathcal{B}'$ (the set of accumulation points).



- $\mathcal{B}' \subseteq \mathcal{B}$ (if $x \notin \mathcal{B}$, there is k with $x \in (a_k, b_k)$).

- So $\mathcal{B} = \mathcal{B}'$ is perfect.
- Perfect sets aren't countable. ↯

Is this a planar result?

- The tiling is countable $\leftarrow \mathbb{R}^2$ is separable.
- Balls intersect in just one point $\leftarrow \mathbb{R}^2$ is rotund.

Thm. **Klee, Maluta, Zanco (1986).** No **separable** normed space has a tiling with rotund bodies.

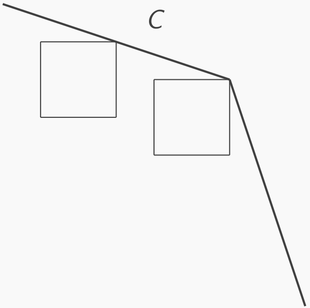
- ℓ_2 doesn't have a tiling with balls.
- c_0 (and ℓ_∞) have a tiling with balls.

- **Klee, Tricot (1987).** **Separable** smooth Banach spaces don't have tilings with balls.
- **De Bernardi, Veselý (2017).** LUR Banach spaces don't have tilings by balls.
  - Nor do Fréchet smooth Banach spaces.
- **Problem.** Can a rotund/smooth Banach space have a tiling with balls?
- **Preiss (2010).** ℓ_2 has a normal tiling (*i.e.*, inner and outer radii are equi-bounded).

A disjoint tiling from Badajoz

Klee's tiling

- **Klee (1981).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** balls of radius 1.
- The set of centers forms a **discrete** Chebyshev set (every point in the space has a unique point in C at minimal distance).
- **Problem.** Are Chebyshev sets in Hilbert spaces convex?
- $(\mathbb{R}^2, \|\cdot\|_\infty)$.



Klee's tiling

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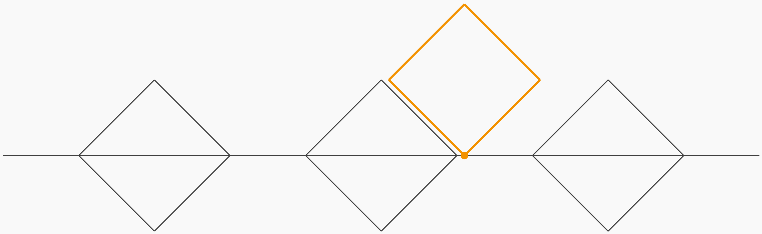
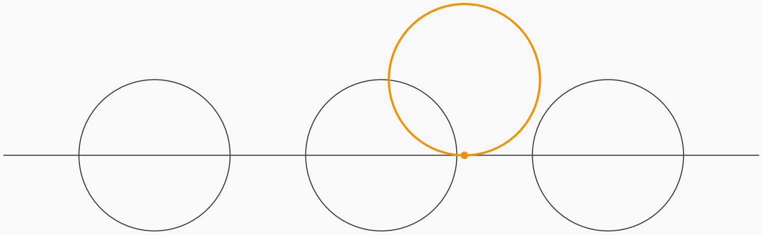
- **Problem.** Are Chebyshev sets in Hilbert spaces convex?
- $(\mathbb{R}^2, \|\cdot\|_\infty)$.
- The set $\mathcal{D}$ of centers is $(2+)$-separated and 1-dense.
  - If $d \neq h \in \mathcal{D}$, then $\|d-h\| > 2$.
  - For all $x \in \mathcal{X}$ there is $d \in \mathcal{D}$ with $\|x-d\| \leqslant 1$.
- In $\ell_p(\mathbb{R})$ a $(2^{1/p}+)$-separated and 1-dense set.
- **De Bernardi, Veselý (2017).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** LUR (in particular, rotund) bodies.

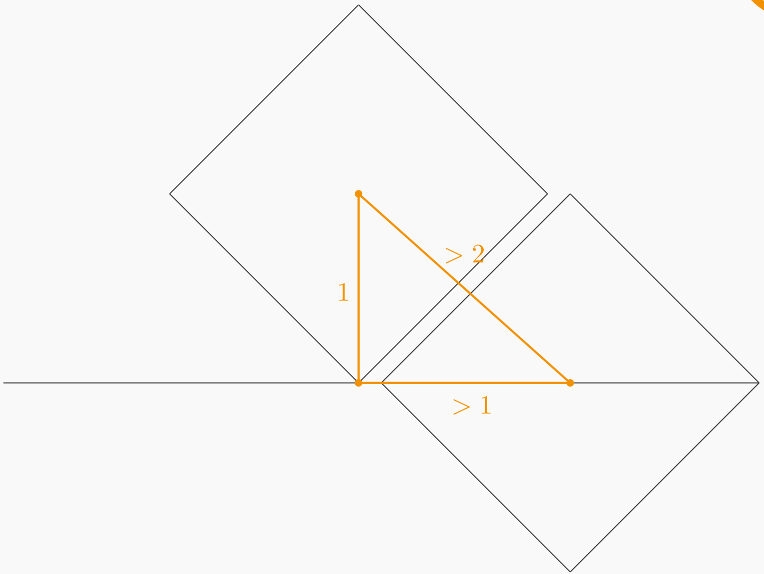
Klee's proof in one picture



Klee's proof in one picture

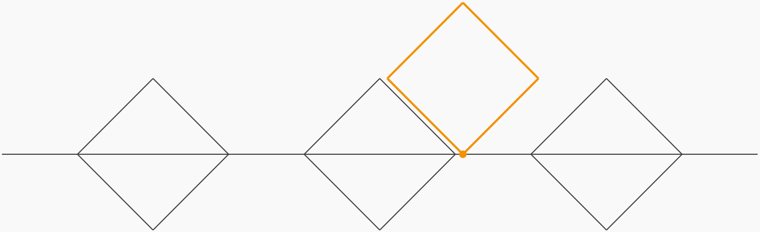


Klee's proof in one picture
The same, just bigger



Nice pics, but how do you **prove** it?

- For a Banach space $\mathcal{X}$, $|\mathcal{X}| = \mathrm{dens}(\mathcal{X})^\omega$.
- So, $|\ell_1(\mathbb{R})| = \mathfrak{c}$. Write $\ell_1(\mathbb{R}) = \{u_\alpha\}_{\alpha<\mathfrak{c}}$.
- By (long) induction. If $(B_\alpha)_{\alpha<\gamma}$ already cover u_γ, ✓.
- If not, let c_α be the center of B_α.
    - Find a subspace that contains all c_α and u_γ.
    - There is $\tilde{\gamma}$ with $u_\gamma(\tilde{\gamma}) = 0$ and $c_\alpha(\tilde{\gamma}) = 0$.
- Take $B_\gamma := B(u_\gamma + e_{\tilde{\gamma}})$.
    - This ball contains u_γ
    - and touches that subspace only in one point.



A little algebraic drop

- **Klee (1981).** A $(2+)$-separated and 1-dense set $\mathcal{D}$ in $\ell_1(\mathbb{R})$.
- This set is not an (additive) subgroup of $\ell_1(\mathbb{R})$.
- A tiling $\mathcal{T}$ is **group-generated** (or **lattice**) if there is a subgroup $\mathcal{D}$ of $\mathcal{X}$ such that

$$\mathcal{T} = \{d + B_{\mathcal{X}}\}_{d \in \mathcal{D}}.$$

  - So, a tiling by balls of radius 1 with centers being a group.

Theorem (De Bernardi, R., Somaglia)

If $\mathcal{X}$ admits a group-generated tiling, $\mathrm{dens}(\mathcal{X})$-many tiles intersect $B_{\mathcal{X}}$.

- Group-generated tilings are not disjoint.
- And in infinite dimensions they are not star-finite.
- Who cares? Please, wait for the next talk...

Thank you for your attention!