Norming M-bases and $\mathcal{C}(\mathcal{K})$ spaces

Tommaso Russo
tommaso.russo.math@gmail.com

j./w. P. Hájek, J. Somaglia, and S. Todorčević

51^{th} Winter School in Abstract Analysis
Vlachovice, Czech Republic
January 13–20, 2024

Some basis stuff

- A sequence $(e_n)_{n=1}^{\infty}$ is a **Schauder basis** if for all $x \in \mathcal{X}$ there are unique scalars $(x_n)_{n=1}^{\infty}$ with

$$x = \sum_{n=1}^{\infty} x_n e_n \qquad \text{(the series converges in } \mathcal{X}\text{)}.$$

- Two drawbacks:
  - Schauder bases can only exist in separable spaces.
  - **Enflo ('73).** Not every separable Banach space has a Schauder basis.
- A system $\{e_n; \varphi_n\}_{n=1}^{\infty} \subseteq \mathcal{X} \times \mathcal{X}^*$ is a **Markushevich basis** (**M-basis**) if:
  (i) $\langle \varphi_k, e_n \rangle = \delta_{k,n}$,
  (ii) $\overline{\operatorname{span}}\{e_n\} = \mathcal{X}$,
  (iii) $\overline{\operatorname{span}}^{w^*}\{\varphi_n\} = \mathcal{X}^*$.
- Advantages:
  - **Markushevich ('43).** Every separable Banach space has an M-basis.
  - The definition extends to all Banach spaces (just change label!).
- Drawback: $\sum_{n=1}^{\infty} \langle \varphi_n, x \rangle e_n$ might not converge!

Let us welcome the main character

Example: The trigonometric system $\{t \mapsto e^{ikt}\}_{k\in\mathbb{Z}}$ is not a Schauder basis of $\mathcal{C}(\mathbb{T})$ (or $L^1(\mathbb{T})$), but it is an M-basis.

- **Johnson ('70).** ℓ_∞ has no M-basis.

We actually have more:

- If $\mathcal{X}^*$ is separable, $\mathcal{X}$ admits an M-basis with $\overline{\mathrm{span}}\{\varphi_\alpha\} = \mathcal{X}^*$.
- Every separable Banach space, for every $\varepsilon > 0$, admits an M-basis $\{e_n; \varphi_n\}_{n=1}^\infty$ with $\|e_n\| \cdot \|\varphi_n\| \leqslant 1 + \varepsilon$.
- Every separable Banach space admits a 1-norming M-basis.

Definition. An M-basis $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is **λ-norming** $(0 < \lambda \leqslant 1)$ if $\mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma}$ is a λ-norming subspace, namely if

$$\lambda\|x\| \leqslant \sup\{|\langle\varphi, x\rangle| \colon \varphi \in \mathrm{span}\{\varphi_\alpha\}_{\alpha\in\Gamma},\ \|\varphi\| \leqslant 1\}.$$

- $\approx$ there is a finitely supported Hahn–Banach theorem.

"Do WCG spaces!"
Viktor Klee $\rightarrow$ Vaclav Zizler, '69

- **John–Zizler ('74).** Every Banach space with norming M-basis has a PRI and a LUR norm.
- **Amir–Lindenstrauss ('68).** A Banach space $\mathcal{X}$ is **WCG** if it contains a weakly compact subset with dense linear span.
  - **Amir–Lindenstrauss ('68).** WCG spaces have a PRI.
  - **Troyanski ('71).** WCG spaces have a LUR norm.
- Perhaps **WCG** $\longleftrightarrow$ **norming M-basis**?
  - The canonical basis of $\ell_1(\Gamma)$ is 1-norming.
  - **John–Zizler ('74).** Does every WCG space have a norming M-basis?
- More recent results: WCG spaces, or spaces with norming M-basis, are **Plichko**. And Plichko spaces have a PRI and a LUR norm.

Theorem (Hájek, Advances '19)

There exists a WCG space with no norming M-basis.

- Actually, the example is a $\mathcal{C}(\mathcal{K})$ space.

Asplund entered the chat

- **John–Zizler ('74).** If a Banach space $(\mathcal{X}, \|\cdot\|)$ has a 1-norming M-basis and $\|\cdot\|$ is Fréchet differentiable, then $\mathcal{X}$ is WCG.
- **Fabian ('87).** WCD + Asplund implies WCG.
  - $\mathcal{X}$ is Asplund if...
- **Godefroy ($\sim$'90).** Let $\mathcal{X}$ be an Asplund space with norming M-basis. Is $\mathcal{X}$ WCG?

Theorem (Hájek, R., Somaglia, Todorčević, Advances '21)

There exists an Asplund space $\mathcal{X}$ with a 1-norming M-basis such that $\mathcal{X}$ is not WCG.

- **Problem.** Is there a $\mathcal{C}(\mathcal{K})$ counterexample?
  - **(The same) Problem.** Let $\mathcal{K}$ be a scattered compact space such that $\mathcal{C}(\mathcal{K})$ has a norming M-basis. Is $\mathcal{K}$ Eberlein?

How could I give a talk with no ω_1?

- **Deville–Godefroy ('93).** A Valdivia compact space $\mathcal{K}$ is Corson iff it does not contain $[0, \omega_1]$.
- **Alster ('79).** A scattered Corson compact is Eberlein.
- Well, maybe you should consider the case $\mathcal{K} = [0, \omega_1]$...
- **Alexandrov–Plichko ('06).** $\mathcal{C}[0, \omega_1]$ admits no norming M-basis.

Theorem (R. and Somaglia, '23)

$C[0, \omega_1]$ embeds in no Banach space with a norming M-basis.

- So if $[0, \omega_1]$ is continuous image of $\mathcal{K}$, $\mathcal{C}(\mathcal{K})$ has no norming M-basis.
- If $\mathcal{K} = \mathcal{T}$ is a tree (with the coarse wedge topology), then: $\mathcal{T}$ scattered and $\mathcal{C}(\mathcal{T})$ with norming M-basis implies $\mathcal{T}$ Eberlein.

Hold on, did we do something?

- **Alexandrov–Plichko ('06).** $\mathcal{C}[0, \omega_1]$ admits no norming M-basis.
- **R.–Somaglia ('23).** $\mathcal{C}[0, \omega_1]$ does not embed in a Banach space with norming M-basis.
- Are they actually different results?

Problem

Let $\mathcal{X}$ be a Banach space with norming M-basis and $\mathcal{Y}$ be a subspace of $\mathcal{X}$. Must $\mathcal{Y}$ have a norming M-basis?

- **Vanderwerff–Whitfield–Zizler ('94).** Yes, if $\mathcal{Y}$ is WCG (WLD).
- ℓ_∞ does not embed in a space with norming M-basis (no LUR).
- **Kubiś ('07).** The analogue for Plichko spaces has negative answer.
- **Problem**, **Kalenda ('00).** Do all subspaces of $\ell_1(\Gamma)$ have a norming M-bases? Are they Plichko?

Forbidden area

Enter at your own risk

- **Problem.** Assume that a $\mathcal{C}(\mathcal{K})$ space has a norming M-basis. Must $\mathcal{K}$ be Valdivia?
- **Problem.** Let $\mathcal{K}$ be a scattered Valdivia compact such that $[0,\omega_1] \subseteq \mathcal{K}$. Is there a linear extension operator $E\colon \mathcal{C}([0,\omega_1]) \to \mathcal{C}(\mathcal{K})$?
    - Actually, what we really want is: does $\mathcal{C}([0,\omega_1])$ embed in $\mathcal{C}(\mathcal{K})$?
- If both answers are YES, there is no $\mathcal{C}(\mathcal{K})$ counterexample to Godefroy's problem.

(A few, recent) References

- P. Hájek
  Hilbert generated Banach spaces need not have a norming Markushevich basis
  Adv. Math. **351** (2019), 702–717.
- P. Hájek, T. Russo, J. Somaglia, and S. Todorčević
  An Asplund space with norming Markuševič basis that is not weakly compactly generated
  Adv. Math. **392** (2021), 108041.
- T. Russo and J. Somaglia
  Banach spaces of continuous functions without norming Markushevich bases
  Mathematika **69** (2023), 992–1010.

Thank you for your attention!