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Can you tile the plane with closed balls?

Tommaso Russo
tommaso.russo.math@gmail.com

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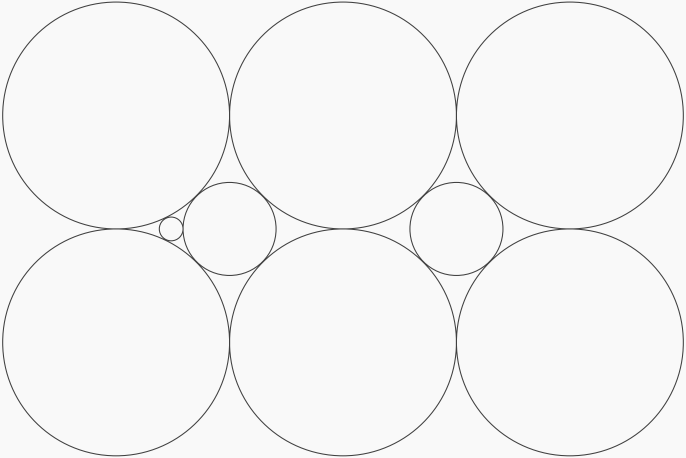
Tiling the plane

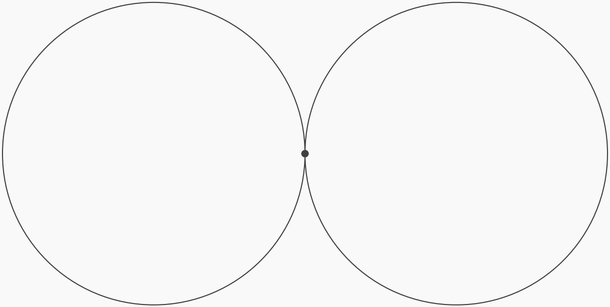
Can you tile the plane with balls?

- Are there closed balls $(B_j)_{j=1}^{\infty}$ with disjoint interiors s.t. $\mathbb{R}^2 = \bigcup B_j$?



Or maybe not?

- Assume $(B_i)_{i \in I}$ is a tiling.
- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.



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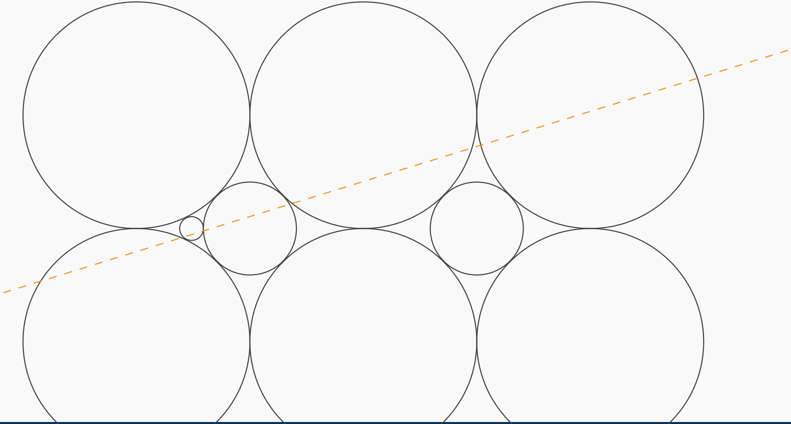

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- Then I is countable ($\mathrm{int}(B_i)$ are mutually disjoint open sets).
- $B_i \cap B_j = \{p_{ij}\}$ or empty.
- So there is a line L such that no p_{ij} belongs to L.
- $(B_k \cap L)_{k=1}^{\infty}$ are **disjoint** closed intervals that cover L.
- **Sierpinski (1918).** If a continuum is covered by countably many disjoint closed sets, then only one is not empty.
  - Continuum $\equiv$ compact, connected, Hausdorff.
- So, you can't tile the plane with (Euclidean) balls.
- I guess we see how to tile it with $\|\cdot\|_\infty$ balls.
- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.

Baby-S

- **Sierpinski-baby version.** You can't cover $\mathbb{R}$ by countably many disjoint compact intervals.
- Assume $I_k = [a_k, b_k]$ are disjoint intervals, $\mathbb{R} = \bigcup [a_k, b_k]$.
- $\mathcal{B} := \{a_k, b_k\}_{k=1}^{\infty}$.
- $\mathcal{B} \subseteq \mathcal{B}'$ (the set of accumulation points).

- $\mathcal{B}' \subseteq \mathcal{B}$ (if $x \notin \mathcal{B}$, there is k with $x \in (a_k, b_k)$).

- So $\mathcal{B} = \mathcal{B}'$ is perfect.
- Perfect sets aren't countable. ↯

Is this a planar result?

- The tiling is countable $\leftarrow \mathbb{R}^2$ is separable.
- Balls intersect in just one point $\leftarrow \mathbb{R}^2$ is strictly convex.

Thm. No separable strictly convex normed space has a tiling with balls.

- Did we actually use balls?

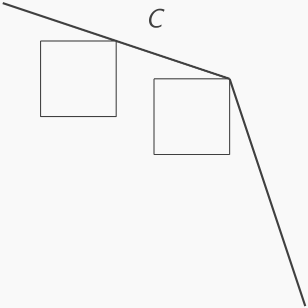
Thm. No separable normed space has a tiling with strictly convex bodies.

- c_0 has a tiling with balls.
- What happens without "separable"?
- No countable tiling can have disjoint tiles.
- Uncountable ones?
- Can we have "small" intersections?

A disjoint tiling from Badajoz

Klee's tiling

- **Klee (1981).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** balls of radius 1.
- The set of centers forms a **discrete** Chebyshev set (every point in the space has a unique point in C at minimal distance).
- **Problem.** Are Chebyshev sets in Hilbert spaces convex?
- $(\mathbb{R}^2, \|\cdot\|_\infty)$.



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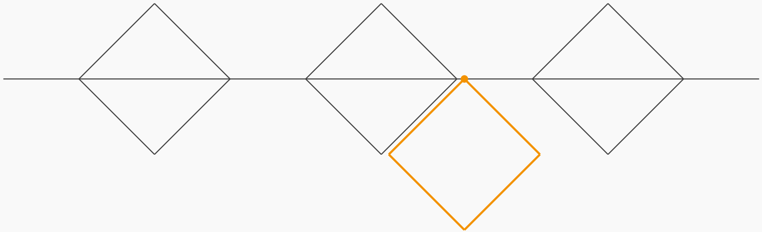
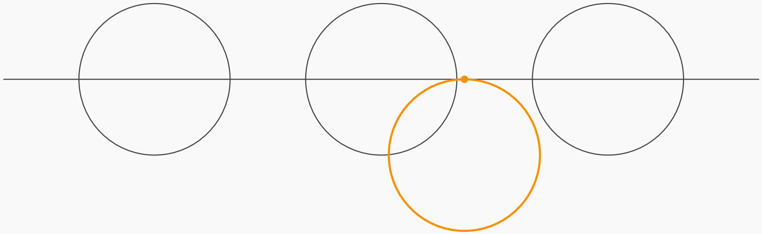
- $(\mathbb{R}^2, \|\cdot\|_\infty)$.
- **De Bernardi, Veselý (2017).** A tiling of $\ell_1(\mathbb{R})$ with **disjoint** LUR (in particular, strictly convex) bodies.
  - So, in nonseparable spaces there are (even disjoint!!) tilings by strictly convex bodies.
- **De Bernardi, Veselý (2017).** LUR Banach spaces don't have tilings by balls.
  - So, the above LUR bodies can't be balls.

Klee's proof in one picture

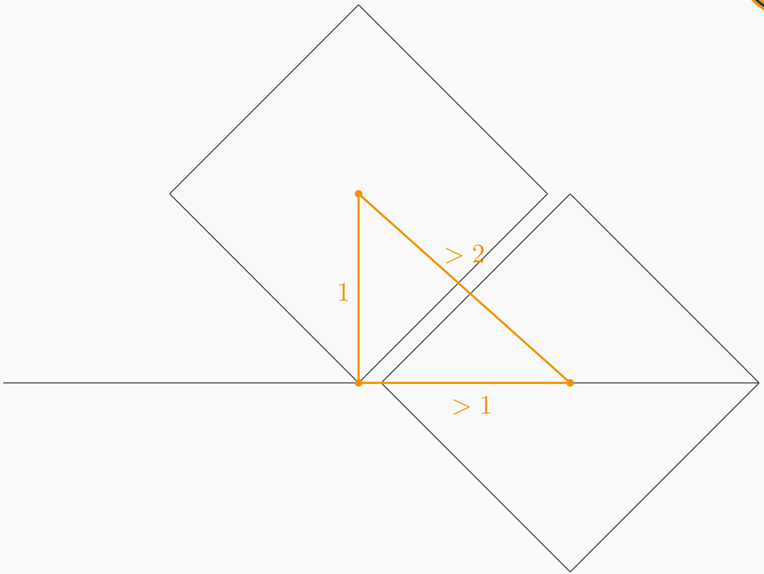


Klee's proof in one picture



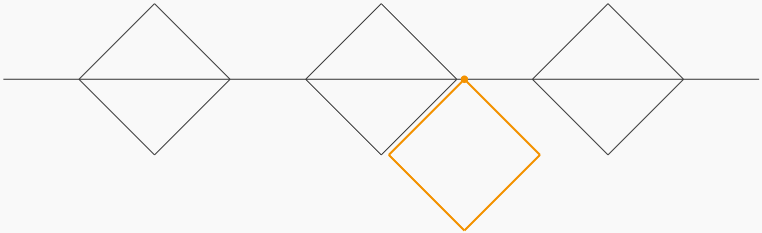
Klee's proof in one picture

The same, just bigger



How do you actually use that?

- $\ell_1(\mathbb{R}) \equiv \ell_1([0,1]) \subseteq \mathcal{C}([0,1])^* \subseteq \ell_\infty$.
- So, $|\ell_1(\mathbb{R})| = \mathfrak{c}$. Write $\ell_1(\mathbb{R}) = \{u_\alpha\}_{\alpha<\mathfrak{c}}$.
- By (long) induction. If $(B_\alpha)_{\alpha<\gamma}$ already cover u_γ, ✓.
- If not, let c_α be the center of B_α.
  - Find a subspace that contains all c_α and u_γ.
  - There is $\tilde{\gamma}$ with $u_\gamma(\tilde{\gamma}) = 0$ and $c_\alpha(\tilde{\gamma}) = 0$.
- Take $B_\gamma := B(u_\gamma + e_{\tilde{\gamma}})$.
  - This ball contains u_γ
  - and touches that subspace only in one point.



I have no idea if I will have time

- **Marchese, Zanco (2012).** Every Banach space has a tiling by convex bodies such that every tile intersects at most two other tiles.
- Locally finite tiling $\equiv$ every point has a neighbourhood that intersects finitely many tiles.
- **Fonf (1990).** A separable Banach space has a locally finite tiling if and only if it is isomorphically polyhedral.
- **Preiss (2010).** ℓ_2 has a tiling with equi-bounded outer and inner radii.
- **Deville, Mar Jimenez (2021).** The same in every separable Banach space, but with starshaped tiles.
- A shared Overleaf file with De Bernardi and Somaglia.
- Maybe next year?

Thank you for your attention!