Smoothness and completeness

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j/w Sheldon Dantas and Petr Hájek

Methods in Banach spaces
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Our main speakers will be

Igor Balla, Masaryk University, Brno, Czech Republik

Michael Dymond, University of Birmingham, United Kingdom

Vojtěch Kaluža, Institute of Science and Technology, Klosterneuburg, Austria

Noema Nicolussi, Technische Universität, Graz, Austria

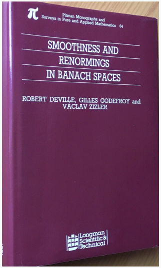
- https://www.uibk.ac.at/mathematik/functionalanalysis/analysis-seminar-innsbruck-2024/
- Christian Bargetz, Eva Kopecká, and Tommaso Russo

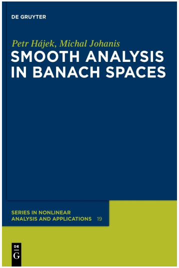
The setting

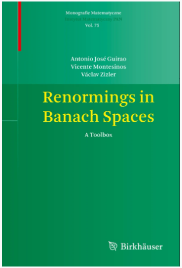
- Given a Banach space $\mathcal{X}$, is there a norm on $\mathcal{X}$ that is differentiable on a 'large' set?
- **Benyamini, Lindenstrauss**, *Geometric Nonlinear Functional Analysis*:
  - Does the existence of a smooth norm on some 'large' subset of a Banach space $\mathcal{X}$ imply that $\mathcal{X}$ is Asplund?
  - Is there a norm on ℓ_1 that is differentiable outside a countable union of closed hyperplanes?
- **Guirao, Montesinos, Zizler**, *Open problems...*, Problem 149: Does the space of finitely supported vectors in $\ell_1(\Gamma)$ have a C^1-smooth norm (when Γ is uncountable)?
- **Godefroy:** Assume that a norm on $\mathcal{X}$ has a point of differentiability in every subspace. Is $\mathcal{X}$ Asplund?
- Here, 'large' $\equiv$ containing a dense subspace.
- **Dantas, Hájek, R. (JMAA'20).** Given a Banach space $\mathcal{X}$, is there a dense subspace of $\mathcal{X}$ that admits a C^k-smooth norm?

Smooth references

- Deville, Godefroy, Zizler, *Smoothness and Renormings in Banach Spaces*, 1993.
- Hájek, Johanis, *Smooth analysis in Banach spaces*, 2014.
- Guirao, Montesinos, Zizler, *Renormings in Banach Spaces. A Toolbox*, 2022.







Smoothness 101

- Renorming theory = Find an equivalent norm on $\mathcal{X}$ with the strongest possible form of a certain property.
- Let $\mathcal{U} \subseteq \mathcal{X}$ be open. $f\colon \mathcal{U} \to \mathcal{Y}$ is differentiable at $x \in \mathcal{U}$ if there is $f'(x) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$f(x+h) = f(x) + f'(x)h + o(\|h\|) \quad \text{as} \quad h \to 0.$$

- C^k-smoothness, rules of calculus, Implicit Function theorem, ...

Lemma: 'If the unit ball looks smooth, the norm is smooth'.

Proof: Implicit function theorem + Minkowski functional.

- If $p \notin \mathbb{N}$, the ℓ_p norm is $C^{\lfloor p \rfloor}$-smooth, but not $C^{\lfloor p \rfloor + 1}$-smooth.
  If p is odd, it is C^{p-1}-smooth, but not C^p-smooth.
  If p is even, it is C^∞-smooth.
  - Namely, same differentiability of $t \mapsto |t|^p$.
- The c_0 norm is not C^1-smooth.
- Is there some renorming that is smooth?

Smoothness and structure

I keep recycling this slide from my first ever talk, back in '17

- If a separable Banach space $\mathcal{X}$ has a C^1-smooth norm, then $\mathcal{X}$ is Asplund (*i.e.*, $\mathcal{X}^*$ is separable).
  - No closed, inf-dim subspace of ℓ_1 has a C^1-smooth norm.
- For $p \notin 2\mathbb{N}$, ℓ_p has no $C^{\lceil p \rceil}$-smooth norm!
- **Meshkov (1978).** If $\mathcal{X}$ and $\mathcal{X}^*$ admit a C^2-smooth norm, then $\mathcal{X}$ is isomorphic to a Hilbert space.
- **Deville (1989).** If $\mathcal{X}$ has a C^∞-smooth norm, either it contains c_0, or it is super-reflexive and it contains ℓ_{2k}.
  - Can the first case actually happen? Does c_0 have a C^∞-smooth norm?
- **Pechanec, Whitfield, Zizler (1981) – Fabian, Zizler (1997).** If $\mathcal{X}$ has a LFC norm, then it is c_0-saturated and Asplund.

$\Longrightarrow$ Smooth norms on a Banach space don't come for free.

- All these proofs require $\mathcal{X}$ to be complete (variational principles).

An incomplete slide
To be completed?

$\Longrightarrow$ Smooth norms on a Banach space don't come for free.

- All these proofs require $\mathcal{X}$ to be complete (variational principles).
- Do the results require completeness?

Ex: If a normed space $\mathcal{X}$ has a smooth norm, then the completion...

- Let's take a normed space $\mathcal{X}$ that is really not complete.
- Let $\mathcal{X}$ be a normed space with a countable algebraic basis.
  - **Vanderwerff (1992).** $\mathcal{X}$ has a C^1-smooth norm.
  - **Hájek (1995).** $\mathcal{X}$ has a C^∞-smooth norm.
  - **Dantas, Hájek, R. (JMAA'20).** $\mathcal{X}$ has an analytic norm.
- As a consequence:
  Every separable Banach space $\mathcal{X}$ admits a dense subspace with an analytic norm.
  - Analytic $\equiv$ locally a power series $\Longrightarrow$ C^∞-smooth.
  - Take the linear span of a dense sequence in $\mathcal{X}$.

Strong maxima

A.k.a. The secret of smoothness unveiled

- $c_0(\Gamma)$ has a C^∞-smooth norm.
- Normed spaces of countable dim have a C^∞-smooth norm.

Theorem (Dantas, Hájek, R., JMAA'20)

$\ell_\infty^F := span\{\mathbb{1}_A \colon A \subseteq \mathbb{N}\}$ has a C^∞-smooth norm.

- Take a sequence $\varepsilon_j \searrow 0$ and define $T\colon \ell_\infty^F \to \ell_\infty$ by

$$(x_j)_{j=1}^\infty \mapsto ((1+\varepsilon_j)\cdot x_j)_{j=1}^\infty.$$

- Look at the picture. ■
- Take $\mathcal{X} \subseteq \ell_\infty$ of countable dimension, $\mathcal{X} = \text{span}\{e_j\}_{j=1}^\infty$.
- Take $\{v_j\}_{j=1}^\infty \subseteq \ell_\infty^F$ with v_j 'very close' to e_j.
- $\mathcal{X}$ is isomorphic to $\text{span}\{v_j\}_{j=1}^\infty$ (small perturbation lemma).
  - Well, I'm cheating a bit, $\{e_j\}_{j=1}^\infty$ has to be an M-basis.

Fundamental biorthogonal systems

Theorem (Dantas, Hájek, R., IMRN'23)

Let $\mathcal{X}$ be a Banach space with a fundamental biorthogonal system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$. Consider $\mathcal{Y} := \mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$. Then:

(i) *$\mathcal{Y}$ admits a polyhedral and LFC norm.*

(ii) *$\mathcal{Y}$ admits a C^∞-smooth and LFC norm.*

(iii) *$\mathcal{Y}$ admits a C^1-smooth LUR norm.*

Moreover, such norms are dense.
Further, $\mathcal{Y}$ admits locally finite C^∞-smooth partitions of unity (hence smooth approximation of functions).

The norm $\|\cdot\|$ is *LFC* on $\mathcal{X}$ if for each $x \in \mathcal{S}_\mathcal{X}$ there exist an open nhood $\mathcal{U}$ of x, functionals $\varphi_1, \ldots, \varphi_k \in \mathcal{X}^*$, and $G\colon \mathbb{R}^k \to \mathbb{R}$ such that

$$\|y\| = G\big(\langle\varphi_1, y\rangle, \ldots, \langle\varphi_k, y\rangle\big) \qquad \text{for every } y \in \mathcal{U}.$$

Ex: The c_0 norm. You didn't erase the pic, did you?

Can we hope for more?

- **Dantas, Hájek, R. (JMAA'20).** No dense subspace of $c_0(\omega_1)$ admits an analytic norm.
  - There is a dense subspace of $\ell_1(\mathfrak{c})$ with an analytic norm.
  - There is a dense subspace of $\ell_1(\Gamma)$ with a C^∞-smooth norm.
  - Can it be made analytic when $|\Gamma| \geqslant \mathfrak{c}^+$?
- **Fabian, Whitfield, Zizler (1983).** Let $\mathcal{Y}$ be a normed space with a $C^{1,+}_{\rm loc}$-smooth (*e.g.*, C^2-smooth) LUR norm $\|\cdot\|$. Then the completion of $\mathcal{Y}$ is super-reflexive.
- What about dense subspaces that are not the span of a fundamental biorthogonal system?
  - **Hájek, R. (JFA'20).** Different dense subspaces of a Banach space can be extremely different.
  - **Dantas, Hájek, R. (REMC'24+).**
- **Main problem.** Is there a Banach space $\mathcal{X}$ such that no dense subspace of $\mathcal{X}$ has a C^k-smooth norm?

How general is the result?
Enter at your own risk

$\{e_\alpha;\varphi_\alpha\}_{\alpha\in\Gamma}\subseteq\mathcal{X}\times\mathcal{X}^*$ is a *fundamental biorthogonal system* for $\mathcal{X}$ if

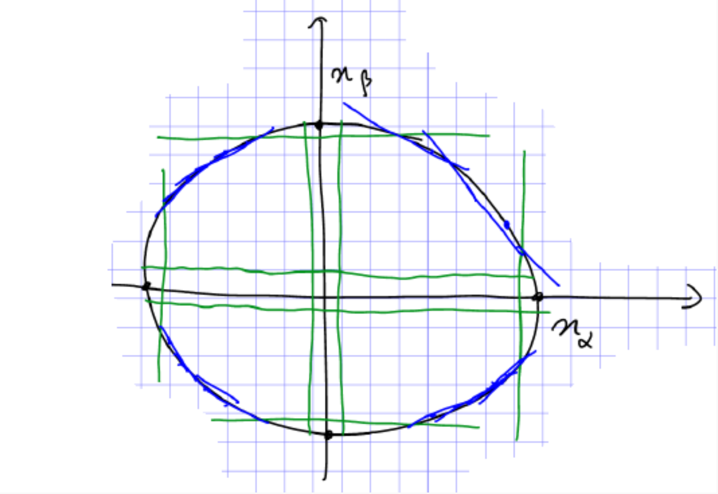
- $\langle\varphi_\beta,e_\alpha\rangle=\delta_{\alpha,\beta}$,
- $\mathrm{span}\{e_\alpha\}_{\alpha\in\Gamma}$ is dense in $\mathcal{X}$.

Which Banach spaces admit a fundamental biorthogonal system?

- **Short answer:** many (most?).
- Plichko spaces (*e.g.*, WCG, reflexive, $c_0(\Gamma)$, $L_1(\mu)$ for any measure, $\mathcal{C}(\mathcal{K})$ for $\mathcal{K}$ Valdivia, or compact Abelian group),
- **Kalenda (2020).** Every space with projectional skeleton (duals of Asplund spaces, preduals of Von Neumann algebras, preduals of JBW*-triples),
- $\ell_\infty(\Gamma)$, $\ell_\infty^c(\Lambda)$ when $|\Lambda|\leqslant\mathfrak{c}$,
- $\mathcal{C}(\mathcal{T})$, when $\mathcal{T}$ is a tree,
- **Davis, Johnson (1973).** $\mathcal{X}$ with $\mathrm{dens}\,\mathcal{X}=\kappa$ that has a WCG quotient of density κ,
- **Todorčević (2006).** All Banach spaces of density ω_1, under MM.

One more pic

- **Troyanski (1970).** $\mathcal{Y}$ has a LUR norm (and LUR norms are dense).



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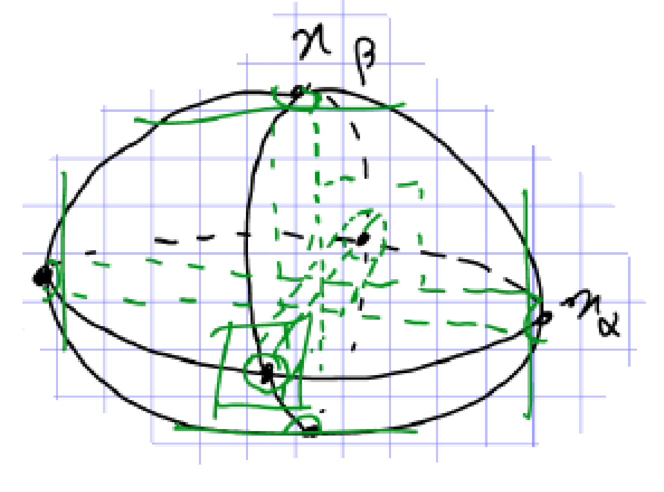

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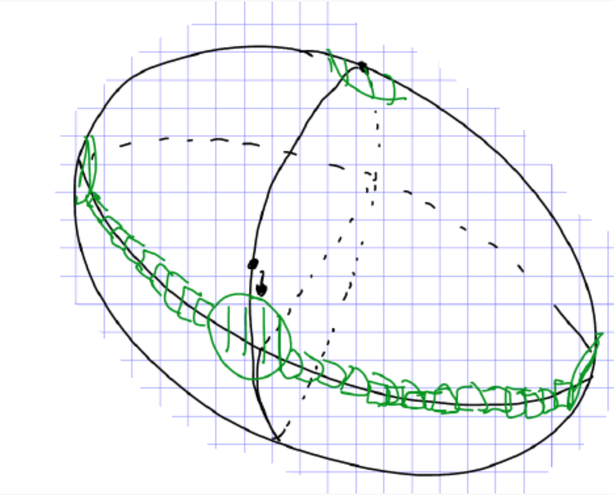

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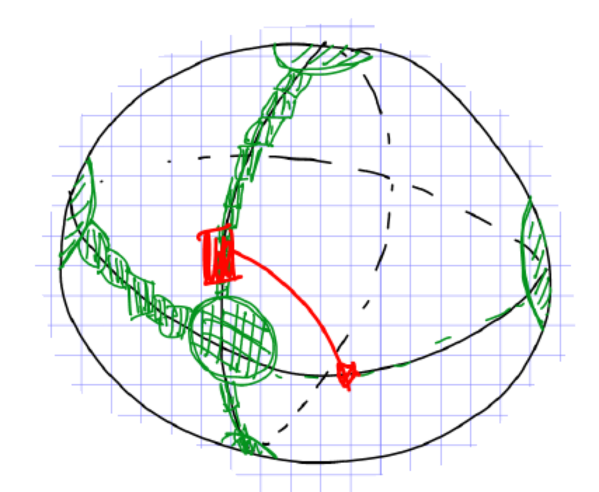

Claim 3.1. Let $\varepsilon > 0$ be fixed. Then there are nets $(\varepsilon_F)_{F\in\mathcal{F}^{<\omega}}$ and $(\theta_F)_{F\in\mathcal{F}^{<\omega}}$ of positive reals and sets $(\Omega_F)_{F\in\mathcal{F}^{<\omega}}$, where Ω_F is a finite set of slices of $\mathcal{B}_{\mathcal{Y}}$, such that

(i) $\theta_F \leqslant \varepsilon_F \leqslant \varepsilon$ for every $F \in \mathcal{F}^{<\omega}$;

(ii) $\varepsilon_F \leqslant \frac{1}{4nM}$ for every $F \in \mathcal{F}^n$;

(iii) $\varepsilon_F \leqslant \frac{1}{4nM}\theta_G$ for every $F \in \mathcal{F}^n$ and every $G \in \mathcal{F}^{<\omega}$ with $G \subsetneq F$;

(iv) if $S \in \Omega_F$, then $-S \in \Omega_F$ as well;

(v) $\operatorname{diam}(S) < \varepsilon_F$ for every $S \in \Omega_F$;

(vi) if $S \in \Omega_F$, then S is of the form $S = S(x, \psi, \delta)$, for some $\delta > 0$, some functional $\psi \in \mathcal{S}_{\mathcal{Y}^*}$ that is norming for x, where

$$x \in \mathcal{S}_F \setminus \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subsetneq F}} T(G, \theta_G);$$

(vii) Setting, for $F \in \mathcal{F}^{<\omega}$,

$$\mathcal{U}_F := \bigcup_{\substack{G\in\mathcal{F}^{<\omega}\\ G\subseteq F}} \bigcup_{S\in\Omega_G} S,$$

we have $2\theta_F \leqslant \operatorname{dist}(\mathcal{S}_F, \mathcal{B}_{\mathcal{Y}} \setminus \mathcal{U}_F)$. In particular, $\mathcal{S}_{\mathcal{Y}} \cap T(F, \theta_F) \subseteq \mathcal{U}_F$.

References

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- P. Hájek and T. Russo, On densely isomorphic normed spaces, *J. Funct. Anal.* **279** (2020), 108667.
- S. Dantas, P. Hájek, and T. Russo, Smooth and polyhedral norms via fundamental biorthogonal systems, *Int. Math. Res. Not. IMRN* **2023** (2023), 13909–13939.
- S. Dantas, P. Hájek, and T. Russo, Smooth norms in dense subspaces of $\ell_p(\Gamma)$ and operator ranges, *Rev. Mat. Complut.* (online first).

Thank you for your attention!